5.1 Introduction
To this point, we have considered conductive heat transfer problems in which the
temperatures are independent of time. In many applications, however, the temperatures are
varying with time, and we require the understanding of the complete time history of the
temperature variation. For example, in metallurgy, the heat treating process can be controlled
to directly affect the characteristics of the processed materials. Annealing (slow cool) can
soften metals and improve ductility. On the other hand, quenching (rapid cool) can harden
the strain boundary and increase strength. In order to characterize this transient behavior, the
full unsteady equation is needed:

\[ \frac{1}{\alpha} \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q}{k} \] (5.1)

where \( \alpha = \frac{k}{\rho c} \) is the thermal diffusivity. Without any heat generation and considering spatial
variation of temperature only in x-direction, the above equation reduces to:

\[ \frac{1}{\alpha} \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2} \] (5.2)

For the solution of equation (5.2), we need two boundary conditions in x-direction and one
initial condition. Boundary conditions, as the name implies, are frequently specified along the
physical boundary of an object; they can, however, also be internal – e.g. a known
temperature gradient at an internal line of symmetry.

5.2 Biot and Fourier numbers
In some transient problems, the internal temperature gradients in the body may be quite small
and insignificant. Yet the temperature at a given location, or the average temperature of the
object, may be changing quite rapidly with time. From eq. (5.1) we can note that such could
be the case for large thermal diffusivity \( \alpha \).

A more meaningful approach is to consider the general problem of transient cooling of an
object, such as the hollow cylinder shown in figure 5.1.
For very large \( r_i \), the heat transfer rate by conduction through the cylinder wall is approximately

\[
q \approx -k(2\pi r_i l)\left(\frac{T_i - T_s}{r_o - r_i}\right) = k(2\pi r_i l)\left(\frac{T_i - T_s}{L}\right)
\]  

(5.3)

where \( l \) is the length of the cylinder and \( L \) is the material thickness. The rate of heat transfer away from the outer surface by convection is

\[
q = \overline{h}(2\pi r_i l)(T_s - T_\infty)
\]

(5.4)

where \( \overline{h} \) is the average heat transfer coefficient for convection from the entire surface. Equating (5.3) and (5.4) gives

\[
\frac{T_i - T_s}{T_s - T_\infty} = \frac{\overline{h}L}{k} = \text{Biot number}
\]

(5.5)

The Biot number is dimensionless, and it can be thought of as the ratio

\[
\text{Bi} = \frac{\text{resistance to internal heat flow}}{\text{resistance to external heat flow}}
\]

Whenever the Biot number is small, the internal temperature gradients are also small and a transient problem can be treated by the “lumped thermal capacity” approach. The lumped capacity assumption implies that the object for analysis is considered to have a single mass-averaged temperature.

In the derivation shown above, the significant object dimension was the conduction path length, \( L = r_o - r_i \). In general, a characteristic length scale may be obtained by dividing the volume of the solid by its surface area:

\[
L = \frac{V}{A_s}
\]

(5.6)

Using this method to determine the characteristic length scale, the corresponding Biot number may be evaluated for objects of any shape, for example a plate, a cylinder, or a sphere. As a thumb rule, if the Biot number turns out to be less than 0.1, lumped capacity assumption is applied.

In this context, a dimensionless time, known as the Fourier number, can be obtained by multiplying the dimensional time by the thermal diffusivity and dividing by the square of the characteristic length:

\[
\text{dimensionless time} = \frac{\alpha t}{L^2} = \text{Fo}
\]

(5.7)
5.3 Lumped thermal capacity analysis

The simplest situation in an unsteady heat transfer process is to use the lumped capacity assumption, wherein we neglect the temperature distribution inside the solid and only deal with the heat transfer between the solid and the ambient fluids. In other words, we are assuming that the temperature inside the solid is constant and is equal to the surface temperature.

\[ q = hA_s(T - T_\infty) \]

The solid object shown in figure 5.2 is a metal piece which is being cooled in air after hot forming. Thermal energy is leaving the object from all elements of the surface, and this is shown for simplicity by a single arrow. The first law of thermodynamics applied to this problem is

\[
\text{heat out of object during time } dt = \text{decrease of internal thermal energy of object during time } dt
\]

Now, if Biot number is small and temperature of the object can be considered to be uniform, this equation can be written as

\[
\frac{dT}{(T - T_\infty)} = \frac{-hA_s}{\rho c V} dt \quad (5.9)
\]

Integrating and applying the initial condition \( T(0) = T_i \),

\[
\ln \left( \frac{T(t) - T_\infty}{T_i - T_\infty} \right) = \frac{-hA_s}{\rho c V} t \quad (5.10)
\]

Taking the exponents of both sides and rearranging,

\[
\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \quad (5.11)
\]

where

\[
b = \frac{hA_s}{\rho c V} \quad (1/s) \quad (5.12)
\]
Note: In eq. 5.12, \( b \) is a positive quantity having dimension \((\text{time})^{-1}\). The reciprocal of \( b \) is usually called time constant, which has the dimension of time.

Question: What is the significance of \( b \)?

Answer: According to eq. 5.11, the temperature of a body approaches the ambient temperature \( T_\infty \) exponentially. In other words, the temperature changes rapidly in the beginning, and then slowly. A larger value of \( b \) indicates that the body will approach the surrounding temperature in a shorter time. You can visualize this if you note the variables in the numerator and denominator of the expression for \( b \). As an exercise, plot \( T \) vs. \( t \) for various values of \( b \) and note the behaviour.

**Rate of convection heat transfer** at any given time \( t \):

\[
\dot{Q}(t) = hA_i[T(t) - T_\infty]
\]

**Total amount of heat transfer** between the body and the surrounding from \( t=0 \) to \( t \):

\[
Q = mc \left[ T(t) - T_\infty \right]
\]

**Maximum heat transfer** (limit reached when body temperature equals that of the surrounding):

\[
Q = mc \left[ T_\infty - T_i \right]
\]

5.4 Spatial Effects and the Role of Analytical Solutions

If the lumped capacitance approximation can not be made, consideration must be given to spatial, as well as temporal, variations in temperature during the transient process.

The Plane Wall: Solution to the Heat Equation for a Plane Wall with Symmetrical Convection Conditions

- For a plane wall with symmetrical convection conditions and constant properties, the heat equation and initial/boundary conditions are:

\[
\frac{1}{a} \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2}
\]

\[
T(x,0) = T_i, \quad \left. \frac{\partial T}{\partial x} \right|_{x=0} = 0
\]

\[
-k \left. \frac{\partial T}{\partial x} \right|_{x=L} = h\left[ T(L,t) - T_\infty \right]
\]

\[
T(x, 0) = T_i
\]

\[ x^* = \frac{x}{L} \]
The answer is **Non-dimensionalisation.** We first need to understand the physics behind the phenomenon, identify parameters governing the process, and group them into meaningful non-dimensional numbers.

**Note:** Once spatial variability of temperature is included, there is existence of seven different independent variables.

\[ T = T(x,t,T_i,T_\infty,h,k,\alpha) \]

How may the functional dependence be simplified?

- The answer is **Non-dimensionalisation.** We first need to understand the physics behind the phenomenon, identify parameters governing the process, and group them into meaningful non-dimensional numbers.

**Non-dimensionalisation of Heat Equation and Initial/Boundary Conditions:**

The following dimensionless quantities are defined.

- **Dimensionless temperature difference:**
  \[ \theta^* = \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} \]

- **Dimensionless coordinate:**
  \[ x^* = \frac{x}{L} \]

- **Dimensionless time:**
  \[ t^* = \frac{\alpha t}{L^2} = Fo \]

- **The Biot Number:**
  \[ Bi = \frac{hL}{k_{\text{solid}}} \]

The solution for temperature will now be a function of the other non-dimensional quantities

\[ \theta^* = f(x^*, Fo, Bi) \]

**Exact Solution:**

\[ \theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*) \]

\[ C_n = \frac{4 \sin \zeta_n}{2 \zeta_n + \sin(2 \zeta_n)} \]

\[ \zeta_n \tan \zeta_n = Bi \]

The roots (eigenvalues) of the equation can be obtained from tables given in standard textbooks.
The One-Term Approximation $Fo > 0.2$

Variation of mid-plane ($x^* = 0$) temperature with time ($Fo$):

$$\theta_0^* = \frac{T - T_\infty}{T_i - T_\infty} \approx C_1 \exp(-\zeta_1^2 Fo)$$

From tables given in standard textbooks, one can obtain $C_1$ and $\zeta_1$ as a function of $Bi$.

Variation of temperature with location ($x^*$) and time ($Fo$):

$$\theta^* = \theta_0^* = \cos(\zeta_1 x^*)$$

Change in thermal energy storage with time:

$$\Delta E_{sat} = -Q$$

$$Q = Q_0 \left(1 - \frac{\sin \zeta_1}{\zeta_1} \right) \theta_0^*$$

$$Q_0 = \rho c V (T_i - T_\infty)$$

---

Can the foregoing results be used for a plane wall that is well insulated on one side and convectively heated or cooled on the other?

Can the foregoing results be used if an isothermal condition ($T_i \neq T_j$) is instantaneously imposed on both surfaces of a plane wall or on one surface of a wall whose other surface is well insulated?

---

Graphical Representation of the One-Term Approximation:

**The Heisler Charts**

Midplane Temperature:
Temperature Distribution

| ![Graph of Temperature Distribution](image) |

Change in Thermal Energy Storage

| ![Graph of Change in Thermal Energy Storage](image) |

§ Assumptions in using Heisler charts:

1. Constant Ti and thermal properties over the body
2. Constant boundary fluid $T_\infty$ by step change
3. Simple geometry: slab, cylinder or sphere

§ Limitations:

1. Far from edges
2. No heat generation (Q=0)
3. Relatively long after initial times (Fo > 0.2)
Radial Systems
Long Rods or Spheres Heated or Cooled by Convection

\[ Bi = \frac{hr_0}{k} \]
\[ Fo = \frac{\alpha a}{r_0^2} \]

Similar Heisler charts are available for radial systems in standard text books.

**Important tips:** Pay attention to the length scale used in those charts, and calculate your Biot number accordingly.

### 5.5 Numerical methods in transient heat transfer: The Finite Volume Method

Considering the steady convection-diffusion equation:

\[ \frac{\partial (\rho \phi)}{\partial t} + \text{div}(\rho \phi \mathbf{u}) = \text{div}(\Gamma \text{ grad } \phi) + S_\phi \]

- The time and control volume integrations give:

\[ \int_{CV} \left[ \int_t^{t+\Delta t} \frac{\partial (\rho \phi)}{\partial t} \, dt \right] \, dV + \int_{t}^{t+\Delta t} \left[ \int_{A} \mathbf{n} \cdot (\rho \phi \mathbf{u}) \, dA \right] \, dt = \int_{t}^{t+\Delta t} \left[ \int_{A} \mathbf{n} \cdot (\Gamma \text{ grad } \phi) \, dA \right] \, dt + \int_{t}^{t+\Delta t} \left[ \int_{CV} S_\phi \, dV \right] \, dt \]

- Unsteady one-dimensional heat conduction:

\[ \rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + S \]
Consider the one-dimensional control volume. Integration over the control volume and over a time interval gives:

\[
\int_t^{t+\Delta t} \left( \int_{CV} \left( \rho c \frac{\partial T}{\partial t} + k \frac{\partial^2 T}{\partial x^2} \right) \, dV \right) \, dt = \int_t^{t+\Delta t} \left( \int_{CV} \left( k \frac{\partial T}{\partial x} \right) \, dV \right) \, dt + \int_t^{t+\Delta t} \left( \int_{CV} \frac{S}{V} \, dV \right) \, dt
\]

Re-written:

\[
\int_t^{t+\Delta t} \left( \int_{CV} \rho c \frac{\partial T}{\partial t} \, dV \right) \, dt = \int_t^{t+\Delta t} \left( \int_{CV} \left( kA \frac{\partial T}{\partial x} \right) - \left( kA \frac{\partial T}{\partial x} \right)_w \right) \, dV \, dt + \int_t^{t+\Delta t} \left( \int_{SV} \Delta V \right) \, dt
\]

If the temperature at a node is assumed to prevail over the whole control volume, applying the central differencing scheme, we have:

\[
\rho c (T_p - T^0_p) \Delta V = \int_t^{t+\Delta t} \left( \left[ k_e A \frac{T_E - T_p}{\Delta x} - k_w A \frac{T_p - T_w}{\Delta x} \right] \right) \, dV \, dt + \int_t^{t+\Delta t} \left( \int_{SV} \Delta V \right) \, dt
\]

An assumption about the variation of \(T_p, T_E\) and \(T_w\) with time. By generalizing the approach by means of a weighting parameter \(\theta\) between 0 and 1:

\[
I_T = \int_t^{t+\Delta t} T_p \, dt = \left[ \theta T_p + (1 - \theta) T_p^0 \right] \Delta t
\]

Therefore,

\[
\rho c \left( \frac{T_p - T_p^0}{\Delta t} \right) \Delta x = \theta \left[ \left( k_e A \frac{T_E - T_p}{\Delta x} \right) - \left( k_w A \frac{T_p - T_w}{\Delta x} \right) \right] + (1 - \theta) \left[ \left( k_e A \frac{T_E^0 - T_p^0}{\Delta x} \right) - \left( k_w A \frac{T_p^0 - T_w^0}{\Delta x} \right) \right] + \bar{S} \Delta x
\]

Re-arranging:

\[
\left[ \rho c \frac{\Delta x}{\Delta t} + \theta \left( \frac{k_e}{\Delta x_{PE}} + \frac{k_w}{\Delta x_{WP}} \right) \right] T_p = \frac{k_e}{\Delta x_{PE}} \left[ \theta T_E + (1 - \theta) T_E^0 \right] + \frac{k_w}{\Delta x_{WP}} \left[ \theta T_w + (1 - \theta) T_w^0 \right] + \left[ \rho c \frac{\Delta x}{\Delta t} - \theta \frac{k_e}{\Delta x_{PE}} + (1 - \theta) \frac{k_w}{\Delta x_{WP}} \right] T_p^0 + \bar{S} \Delta x
\]

Compared with standard form:

\[
a_p T_p = a_w \left[ \theta T_w + (1 - \theta) T_w^0 \right] + a_E \left[ \theta T_E + (1 - \theta) T_E^0 \right] + \left[ a_p^0 - (1 - \theta) a_w - (1 - \theta) a_E \right] T_p^0 + b
\]

where

\[
a_p = \theta (a_w + a_E) + a_p^0
\]

\[
a_w = \frac{k_w}{\Delta x_{WP}}
\]

\[
a_E = \frac{k_e}{\Delta x_{PE}}
\]

\[
b = \bar{S} \Delta x
\]

✓ When \(\theta = 0\), the resulting scheme is “explicit”.
✓ When \(0 < \theta \leq 1\), the resulting scheme is “implicit”.
✓ When \(\theta = 1\), the resulting scheme is “fully implicit”.

\[
\checkmark \quad \sum_{i=1}^{n} \text{valid equations}
\]

\[
\checkmark \quad \text{total equations}
\]
When $\theta = 1/2$, the resulting scheme is “the Crank-Nicolson”.
• Explicit scheme
  \[ a_p T_p = a_w \left[ \theta T_w + (1 - \theta) T_w^0 \right] + a_E \left[ \theta T_E + (1 - \theta) T_E^0 \right] + \left[ a_p^0 - (1 - \theta) a_w - (1 - \theta) a_E \right] T_p^0 + b \]
  \[ \text{The source term is linearised as } b = S_u + S_p T_p^0 \text{ and set } \theta = 0 \]
  \[ \text{The explicit discretisation:} \]
  \[ a_p T_p = a_w T_w^0 + a_E T_E^0 + \left[ a_p^0 - (a_w + a_E) \right] T_p^0 + S_u \]
  where
  \[ a_p = a_p^0 \]
  \[ a_p^0 = \rho c \frac{\Delta x}{\Delta t} \]
  \[ a_w = \frac{k_w}{\partial x_{wp}} \]
  \[ a_E = \frac{k_i}{\partial x_{pe}} \]
  \[ \text{The scheme is based on backward differencing and its Taylor series truncation error} \]
  \[ \text{accuracy is first-order with respect to times.} \]
  \[ \text{All coefficient must be positive in the discretised equation:} \]
  \[ a_p^0 - (a_w + a_E - S_p) > 0 \]
  or
  \[ \frac{\rho c \Delta x}{\Delta t} - \left( \frac{k_w}{\partial x_{wp}} + \frac{k_i}{\partial x_{pe}} \right) > 0 \]
  or
  \[ \frac{\rho c \Delta x}{\Delta t} > \frac{2k}{\Delta x} \]
  or
  \[ \Delta t < \frac{\rho c (\Delta x)^2}{2k} \]
  \[ \text{It becomes very expensive to improve spatial accuracy. This method is not} \]
  \[ \text{recommended for general transient problems.} \]
  \[ \text{Nevertheless, provided that the time step size is chosen with care, the explicit scheme} \]
  \[ \text{described above is efficient for simple conduction calculations.} \]
• Crank-Nicolson scheme
\[ a_p T_p = a_w \left( \frac{T_{E}^0 + (1-\theta)T_{w}^0}{2} \right) + a_E \left( \frac{T_{E}^0}{2} \right) + \left[ a_p^0 - (1-\theta)a_w - (1-\theta)a_E \right] T_p^0 + b \]

✓ Set \( \theta = 1/2 \)
\[ a_p T_p = a_E \left( \frac{T_{E} + T_{E}^0}{2} \right) + a_w \left( \frac{T_{w} + T_{w}^0}{2} \right) + \left[ a_p - \frac{a_E}{2} - \frac{a_w}{2} \right] T_p^0 + b \]

where
\[ a_p = \frac{1}{2} (a_E + a_w) + a_p^0 - \frac{1}{2} S_p \]
\[ a_p^0 = \rho c \frac{\Delta x}{\Delta t} \]
\[ a_w = \frac{k_w}{\Delta x_{wp}} \]
\[ a_E = \frac{k_e}{\Delta x_{pe}} \]
\[ b = S_u + \frac{1}{2} S_p T_p^0 \]

✓ The method is implicit and simultaneous equations for all node points need to be solved at each time step.

✓ All coefficient must be positive in the discretised equation:
\[ a_p^0 > \frac{a_E + a_w}{2} \]

or
\[ \Delta t < \rho c (\Delta x)^2 k \]

✓ This is only slightly less restrictive than the explicit method. The Crank-Nicolson method is based on central differencing and hence it is second-order accurate in time. So, it is normally used in conjunction with spatial central differencing.
The fully implicit scheme

\[ a_p T_p = a_w \left[ \theta T_w + (1 - \theta) T_w^0 \right] + a_E \left[ \theta T_E + (1 - \theta) T_p^0 \right] + \left[ a_p^0 - (1 - \theta) a_w - (1 - \theta) a_E \right] T_p^0 + b \]

Set \( \theta = 1 \)

\[ a_p T_p = a_E T_E + a_w T_w + a_p^0 T_p^0 \]

where

\[ a_p = a_p^0 + a_E + a_w - S_p \]

\[ a_p^0 = \rho c \Delta x \frac{\Delta t}{\Delta t} \]

\[ a_w = \frac{k_w}{\delta x_{WP}} \]

\[ a_E = \frac{k_E}{\delta x_{PE}} \]

A system of algebraic equations must be solved at each time level. The accuracy of the scheme is first-order in time.

The time marching procedure starts with a given initial field of temperature \( T^0 \). The system is solved after selecting time step \( \Delta t \).

All coefficients are positive, which makes the implicit scheme unconditionally stable for any size of time step.

The implicit method is recommended for general purpose transient calculations because of its robustness and unconditional stability.