MODULE 2

ONE DIMENSIONAL STEADY STATE HEAT CONDUCTION

2.1 Objectives of conduction analysis:

The primary objective is to determine the temperature field, T(x,y,z,t), in a body (i.e. how temperature varies with position within the body)
T(x,y,z,t) depends on:

- Boundary conditions
- Initial condition
- Material properties (k, cp, ρ)
- Geometry of the body (shape, size)

Why we need T (x, y, z, t)?

- To compute heat flux at any location (using Fourier's eqn.)
- Compute thermal stresses, expansion, deflection due to temp. Etc.
- Design insulation thickness
- Chip temperature calculation
- Heat treatment of metals

2.2 General Conduction Equation

Recognize that heat transfer involves an energy transfer across a system boundary. The analysis for such process begins from the 1st Law of Thermodynamics for a closed system:

$$\left.\frac{dE}{dt}\right|_{system} = \dot{Q}_{in} - \dot{W}_{out}$$

The above equation essentially represents Conservation of Energy. The sign convention on work is such that negative work out is positive work in.

$$\left.\frac{dE}{dt}\right|_{system} = \dot{Q}_{in} + \dot{W}_{in}$$

The work in term could describe an electric current flow across the system boundary and through a resistance inside the system. Alternatively it could describe a shaft turning across the system boundary and overcoming friction within the system. The net effect in either case would cause the internal energy of the system to rise. In heat transfer we generalize all such terms as "heat sources".

$$\left.\frac{dE}{dt}\right|_{system} = \dot{Q}_{in} + \dot{Q}_{gen}$$

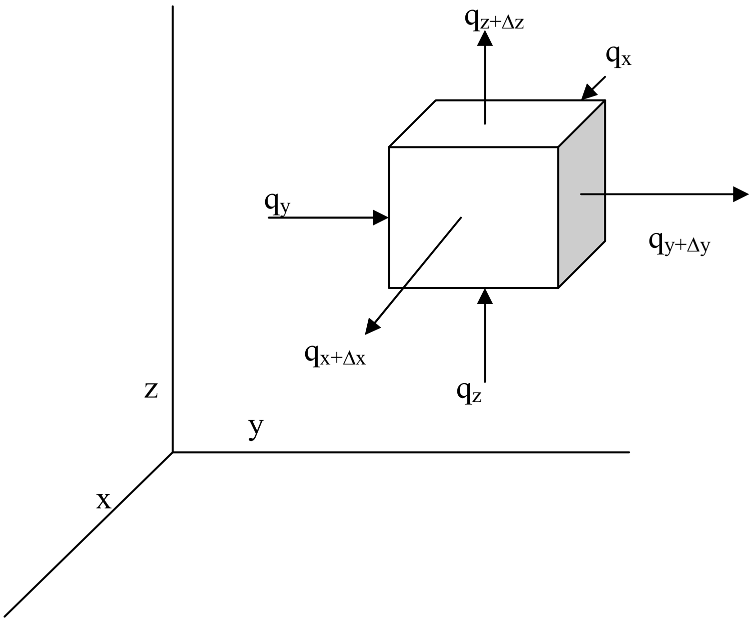
The energy of the system will in general include internal energy, U, potential energy, ½ mgz, or kinetic energy, ½ mv^2. In case of heat transfer problems, the latter two terms could often be neglected. In this case,

$$E = U = m \cdot u = m \cdot c_p \cdot \left(T - T_{ref}\right) = \rho \cdot V \cdot c_p \cdot \left(T - T_{ref}\right)$$

where T_{ref} is the reference temperature at which the energy of the system is defined as zero. When we differentiate the above expression with respect to time, the reference temperature, being constant, disappears:

$$\rho \cdot c_p \cdot V \cdot \left.\frac{dT}{dt}\right|_{system} = \dot{Q}_{in} + \dot{Q}_{gen}$$

Consider the differential control element shown below. Heat is assumed to flow through the element in the positive directions as shown by the 6 heat vectors.



In the equation above we substitute the 6 heat inflows/outflows using the appropriate sign:

$$\rho \cdot c_p \cdot \left(\Delta x \cdot \Delta y \cdot \Delta z\right) \cdot \left.\frac{dT}{dt}\right|_{system} = q_x - q_{x+\Delta x} + q_y - q_{y+\Delta y} + q_z - q_{z+\Delta z} + \dot{Q}_{gen}$$

Substitute for each of the conduction terms using the Fourier Law:

$$\begin{aligned}\rho \cdot c_p \cdot (\Delta x \cdot \Delta y \cdot \Delta z) \cdot \left.\frac{\partial T}{\partial t}\right|_{system} &= \left\{-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} - \left[-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x}\right) \cdot \Delta x\right]\right\} \\ &+ \left\{-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} - \left[-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left(-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y}\right) \cdot \Delta y\right]\right\} \\ &+ \left\{-k \cdot \left(\Delta x \cdot \Delta y\right) \cdot \frac{\partial T}{\partial z} + \left[-k \cdot \left(\Delta x \cdot \Delta y\right) \cdot \frac{\partial T}{\partial z} + \frac{\partial}{\partial z}\left(-k \cdot \left(\Delta x \cdot \Delta y\right) \cdot \frac{\partial T}{\partial z}\right) \cdot \Delta z\right]\right\} \\ &+ \dot{q}\left(\Delta x \cdot \Delta y \cdot \Delta z\right)\end{aligned}$$

where $\dot{q}$ is defined as the internal heat generation per unit volume.
The above equation reduces to:

$$\rho \cdot c_p \cdot \left(\Delta x \cdot \Delta y \cdot \Delta z\right) \cdot \left.\frac{dT}{dt}\right|_{system} = \left\{-\left[\frac{\partial}{\partial x}\left(-k \cdot \left(\Delta y \cdot \Delta z\right) \cdot \frac{\partial T}{\partial x}\right)\right] \cdot \Delta x\right\}$$

$$+\left\{-\left[\frac{\partial}{\partial y}\left(-k\cdot(\Delta x\cdot\Delta z)\cdot\frac{\partial T}{\partial y}\right)\cdot\Delta y\right]\right\}$$

$$+\left\{\left[\frac{\partial}{\partial z}\left(-k\cdot(\Delta x\cdot\Delta y)\cdot\frac{\partial T}{\partial z}\right)\cdot\Delta z\right]\right\}+\dot{q}(\Delta x\cdot\Delta y\cdot\Delta z)$$

Dividing by the volume ($\Delta x\cdot\Delta y\cdot\Delta z$),

$$\rho\cdot c_p\cdot\left.\frac{dT}{dt}\right|_{system}=-\frac{\partial}{\partial x}\left(-k\cdot\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(-k\cdot\frac{\partial T}{\partial y}\right)-\frac{\partial}{\partial z}\left(-k\cdot\frac{\partial T}{\partial z}\right)+\dot{q}$$

which is the **general conduction equation** in three dimensions.

In the case where k is independent of x, y and z then

$$\frac{\rho\cdot c_p}{k}\cdot\left.\frac{dT}{dt}\right|_{system}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}$$

Define the thermodynamic property, α, the thermal diffusivity:

$$\alpha\equiv\frac{k}{\rho\cdot c_p}$$

Then

$$\frac{1}{\alpha}\cdot\left.\frac{dT}{dt}\right|_{system}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}$$

or, :

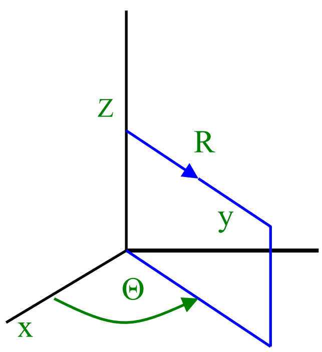
$$\frac{1}{\alpha}\cdot\left.\frac{dT}{dt}\right|_{system}=\nabla^2 T+\frac{\dot{q}}{k}$$

The vector form of this equation is quite compact and is the most general form. However, we often find it convenient to expand the spatial derivative in specific coordinate systems:

Cartesian Coordinates

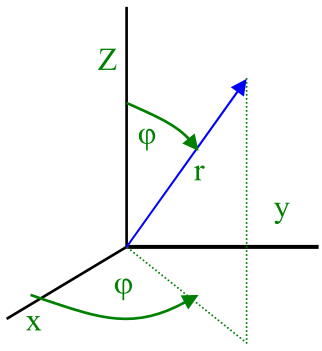
$$\frac{1}{a}\cdot\frac{\partial T}{\partial\tau}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\hat{q}}{k}$$

Circular Coordinates



$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\cdot\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\hat{q}}{k}$$

Spherical Coordinates



$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\cdot\sin^2\theta}\cdot\frac{\partial^2 T}{\partial \phi^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial \theta}\left(\sin\theta\cdot\frac{\partial T}{\partial z}\right)+\frac{\hat{q}}{k}$$

In each equation the dependent variable, T, is a function of 4 independent variables, (x,y,z,τ); (r,θ,z,τ); (r,φ,θ,τ) and is a 2nd order, partial differential equation. The solution of such equations will normally require a numerical solution. For the present, we shall simply look at the simplifications that can be made to the equations to describe specific problems.

- Steady State: Steady state solutions imply that the system conditions are not changing with time. Thus $\partial T/\partial \tau = 0$.
- One dimensional: If heat is flowing in only one coordinate direction, then it follows that there is no temperature gradient in the other two directions. Thus the two partials associated with these directions are equal to zero.
- Two dimensional: If heat is flowing in only two coordinate directions, then it follows that there is no temperature gradient in the third direction. Thus, the partial derivative associated with this third direction is equal to zero.
- No Sources: If there are no volumetric heat sources within the system then the term, $\hat{q} = 0$.

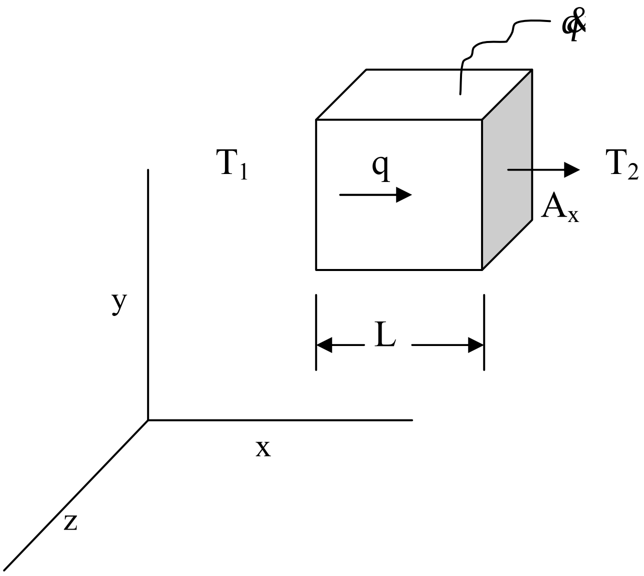
Note that the equation is 2nd order in each coordinate direction so that integration will result in 2 constants of integration. To evaluate these constants two boundary conditions will be required for each coordinate direction.

2.3 Boundary and Initial Conditions

- The objective of deriving the heat diffusion equation is to determine the temperature distribution within the conducting body.

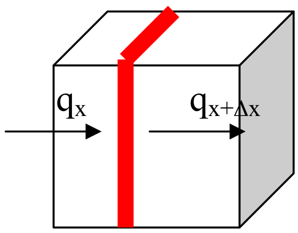
- We have set up a differential equation, with T as the dependent variable. The solution will give us T(x,y,z). Solution depends on boundary conditions (BC) and initial conditions (IC).
- How many BC's and IC's ?
  - Heat equation is second order in spatial coordinate. Hence, 2 BC's needed for each coordinate.
    - * 1D problem: 2 BC in x-direction
    - * 2D problem: 2 BC in x-direction, 2 in y-direction
    - * 3D problem: 2 in x-dir., 2 in y-dir., and 2 in z-dir.
  - Heat equation is first order in time. Hence one IC needed.

2.4 Heat Diffusion Equation for a One Dimensional System



Consider the system shown above. The top, bottom, front and back of the cube are insulated, so that heat can be conducted through the cube only in the x direction. The internal heat generation per unit volume is $\dot{q}$ (W/m^3).

Consider the heat flow through a differential element of the cube.



From the 1st Law we write for the element:

$$(\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} = \dot{E}_{st} \tag{2.1}$$

$$q_x - q_{x+\Delta x} + A_x(\Delta x)\dot{q} = \frac{\partial E}{\partial t} \tag{2.2}$$

$$q_x = -kA_x \frac{\partial T}{\partial x} \tag{2.3}$$

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x}\Delta x \tag{2.4}$$

$$-kA\frac{\partial T}{\partial x} + kA\frac{\partial T}{\partial x} + A\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\Delta x + A\Delta x\dot{q} = \rho Ac\Delta x\frac{\partial T}{\partial t} \tag{2.5}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho c\Delta x\frac{\partial T}{\partial t} \tag{2.6}$$

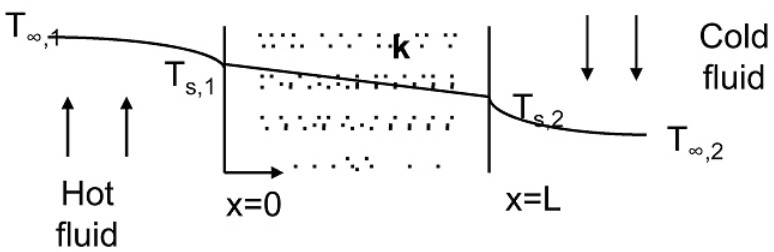
Longitudinal conduction — Internal heat generation — Thermal inertia

If k is a constant, then $\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ (2.7)

- For T to rise, LHS must be positive (heat input is positive)
- For a fixed heat input, T rises faster for higher α
- In this special case, heat flow is 1D. If sides were not insulated, heat flow could be 2D, 3D.

2.5 One Dimensional Steady State Heat Conduction

The plane wall:



The differential equation governing heat diffusion is: $\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$

With constant k, the above equation may be integrated twice to obtain the general solution:

$$T(x)=C_1x+C_2$$

where C_1 and C_2 are constants of integration. To obtain the constants of integration, we apply the boundary conditions at $x = 0$ and $x = L$, in which case

$$T(0)=T_{s,1} \quad \text{and} \quad T(L)=T_{s,2}$$

Once the constants of integration are substituted into the general equation, the temperature distribution is obtained:

$$T(x)=(T_{s,2}-T_{s,1})\frac{x}{L}+T_{s,1}$$

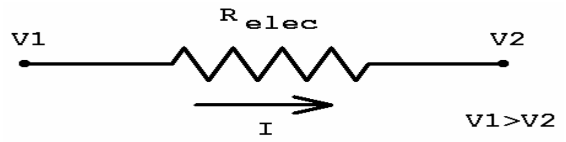
The heat flow rate across the wall is given by:

$$q_x=-kA\frac{dT}{dx}=\frac{kA}{L}\left(T_{s,1}-T_{s,2}\right)=\frac{T_{s,1}-T_{s,2}}{L/kA}$$

Thermal resistance (electrical analogy):

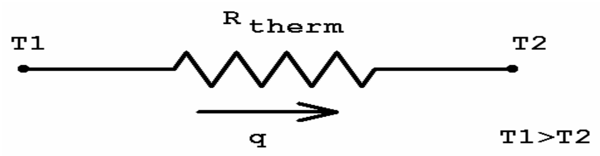
Physical systems are said to be analogous if that obey the same mathematical equation. The above relations can be put into the form of Ohm's law:

$$\mathbf{V=IR_{elec}}$$



Using this terminology it is common to speak of a thermal resistance:

$$\Delta T = qR_{therm}$$



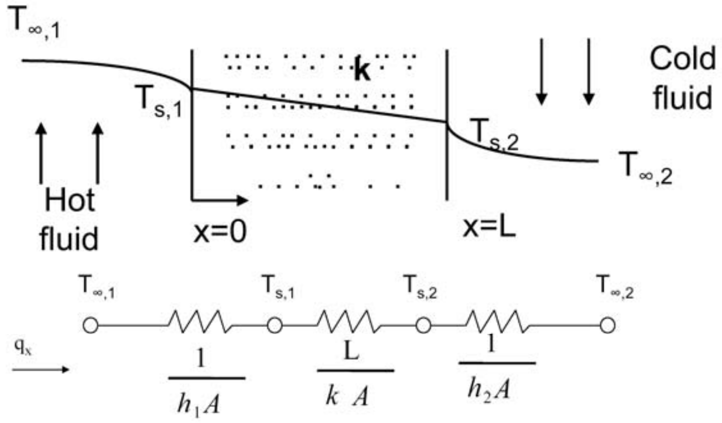
A thermal resistance may also be associated with heat transfer by convection at a surface. From Newton's law of cooling,

$$q=hA(T_s-T_\infty)$$

the thermal resistance for convection is then

$$R_{t,conv}=\frac{T_s-T_\infty}{q}=\frac{1}{hA}$$

Applying thermal resistance concept to the plane wall, the equivalent thermal circuit for the plane wall with convection boundary conditions is shown in the figure below



The heat transfer rate may be determined from separate consideration of each element in the network. Since q_x is constant throughout the network, it follows that

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

In terms of the overall temperature difference $T_{\infty,1} - T_{\infty,2}$, and the total thermal resistance R_{tot}, the heat transfer rate may also be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}}$$

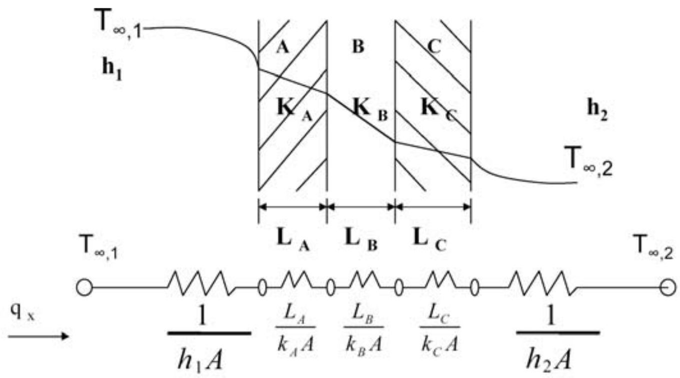
Since the resistance are in series, it follows that

$$R_{tot} = \sum R_t = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

Composite walls:

Thermal Resistances in Series:

Consider three blocks, A, B and C, as shown. They are insulated on top, bottom, front and back. Since the energy will flow first through block A and then through blocks B and C, we say that these blocks are thermally in a series arrangement.



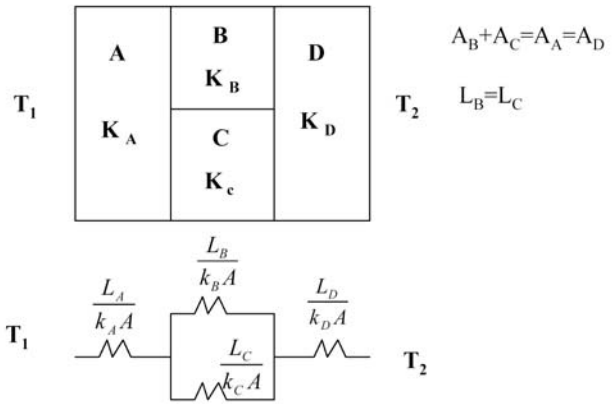
The steady state heat flow rate through the walls is given by:

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{\sum R_t} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2 A}} = UA\Delta T$$

where $U = \frac{1}{R_{tot} A}$ is the overall heat transfer coefficient. In the above case, U is expressed as

$$U = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2}}$$

Series-parallel arrangement:

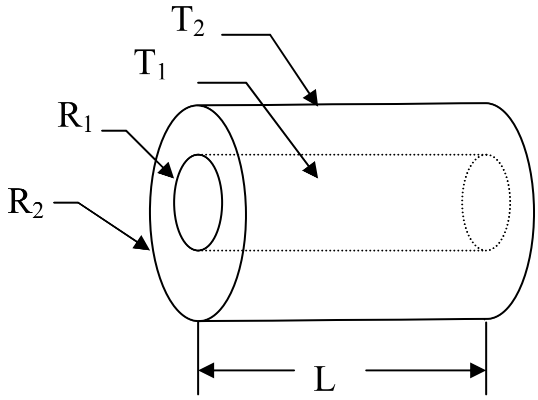


The following assumptions are made with regard to the above thermal resistance model:

1) Face between B and C is insulated.
2) Uniform temperature at any face normal to X.

1-D radial conduction through a cylinder:

One frequently encountered problem is that of heat flow through the walls of a pipe or through the insulation placed around a pipe. Consider the cylinder shown. The pipe is either insulated on the ends or is of sufficient length, L, that heat losses through the ends is negligible. Assume no heat sources within the wall of the tube. If $T_1 > T_2$, heat will flow outward, radially, from the inside radius, R_1, to the outside radius, R_2. The process will be described by the Fourier Law.



The differential equation governing heat diffusion is: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$

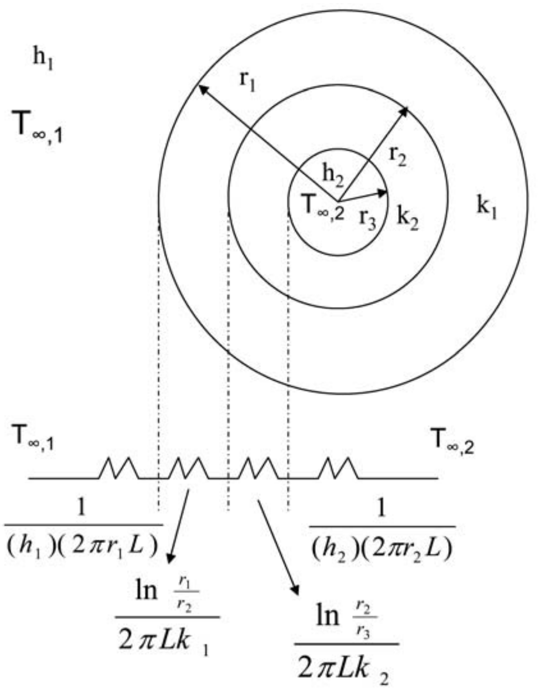
With constant k, the solution is

The heat flow rate across the wall is given by:

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1}-T_{s,2}\right) = \frac{T_{s,1}-T_{s,2}}{L/kA}$$

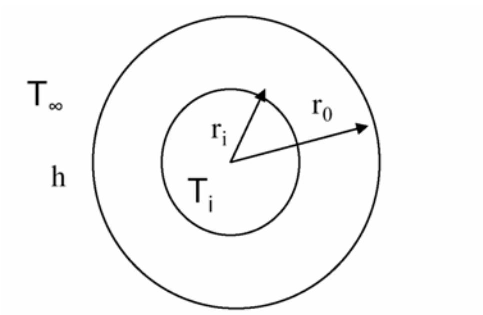
Hence, the thermal resistance in this case can be expressed as: $\frac{\ln\frac{r_1}{r_2}}{2\pi kL}$

Composite cylindrical walls:



$$q_r = \frac{T_{\infty,2}-T_{\infty,1}}{\sum R_t}$$

Critical Insulation Thickness :



$$R_{tot} = \frac{\ln(\frac{r_0}{r_i})}{2\pi kL} + \frac{1}{(2\pi r_0 L)h}$$

Insulation thickness : r_o-r_i

Objective : decrease q , increase R_{tot}

Vary r_o ; as r_o increases, first term increases, second term decreases.

This is a maximum – minimum problem. The point of extrema can be found by setting

$$\frac{dR_{tot}}{dr_0} = 0$$

or, $$\frac{1}{2\pi k r_0 L} - \frac{1}{2\pi h L r_o^2} = 0$$

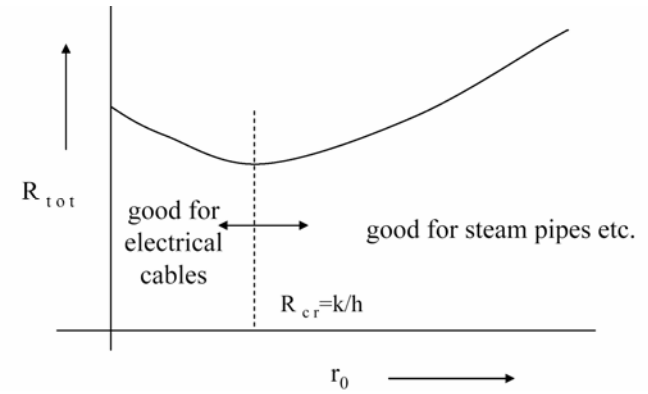
or, $$r_0 = \frac{k}{h}$$

In order to determine if it is a maxima or a minima, we make the second derivative zero:

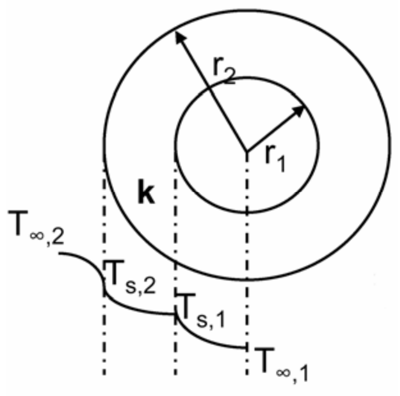
$$\frac{d^2 R_{tot}}{dr_o^2} = 0 \quad \text{at} \quad r_0 = \frac{k}{h}$$

$$\frac{d^2 R_{tot}}{dr_o^2} = \frac{-1}{2\pi k r_o^2 L} + \left.\frac{1}{\pi r_o^2 h L}\right|_{r_0=\frac{k}{h}} \qquad = \frac{h^2}{2\pi L k^3} \rangle 0$$

Minimum q at r_o =(k/h) = r_{cr} (critical radius)



1-D radial conduction in a sphere:



$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right)=0$$

$$\rightarrow T(r)=T_{s,1}-\{T_{s,1}-T_{s,2}\}\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$$

$$\rightarrow q_r=-kA\frac{dT}{dr}=\frac{4\pi k(T_{s,1}-T_{s,2})}{(1/r_1-1/r_2)}$$

$$\rightarrow R_{t,cond}=\frac{1/r_1-1/r_2}{4\pi k}$$

2.6 Summary of Electrical Analogy

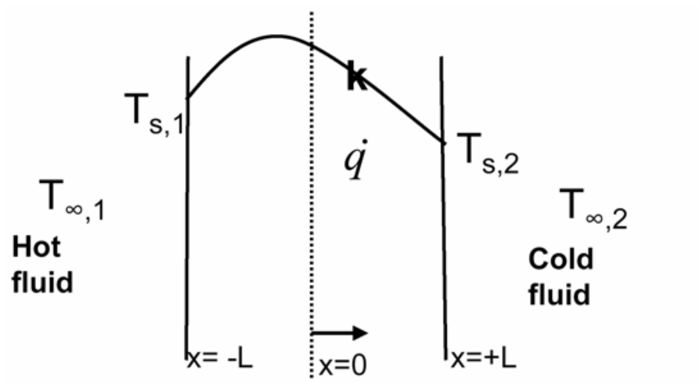
System	Current	Resistance	Potential Difference
Electrical	I	R	ΔV
Cartesian Conduction	q	$\frac{L}{kA}$	ΔT
Cylindrical Conduction	q	$\frac{\ln {r_2}/{r_1}}{2\pi kL}$	ΔT
Conduction through sphere	q	$\frac{1/r_1-1/r_2}{4\pi k}$	ΔT
Convection	q	$\frac{1}{h\cdot A_s}$	ΔT

2.7 One-Dimensional Steady State Conduction with Internal Heat Generation

Applications: current carrying conductor, chemically reacting systems, nuclear reactors.

Energy generated per unit volume is given by $\dot{q}=\frac{\dot{E}}{V}$

<u>Plane wall with heat source:</u> Assumptions: 1D, steady state, constant k, uniform $\dot{q}$



$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$\text{Boundary cond.:} \quad x = -L, \quad T = T_{s,1}$$

$$x = +L, \quad T = T_{s,2}$$

$$\text{Solution:} \quad T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

Use boundary conditions to find C_1 and C_2

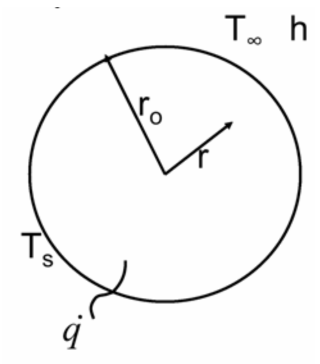
$$\text{Final solution:} \; T = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,2} + T_{s,1}}{2}$$

$$\text{Heat flux:} \qquad q''_x = -k\frac{dT}{dx}$$

<u>Note:</u> From the above expressions, it may be observed that the solution for temperature is no longer linear. As an exercise, show that the expression for heat flux is no longer independent of *x*. *Hence thermal resistance concept is not correct to use when there is internal heat generation.*

<u>Cylinder with heat source:</u> Assumptions: 1D, steady state, constant k, uniform $\dot{q}$

Start with 1D heat equation in cylindrical co-ordinates



$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}}{k} = 0$$

$$\text{Boundary cond.:} \quad r = r_0, \quad T = T_s$$

$$r = 0, \quad \frac{dT}{dr} = 0$$

$$\text{Solution:} \quad T(r) = \frac{\dot{q}}{4k}r_0^2\left(1 - \frac{r^2}{r_0^2}\right) + T_s$$

<u>Exercise:</u> T_s may not be known. Instead, T_∞ and h may be specified. Eliminate T_s, using T_∞ and h.