Module 18 : Stokes's theorem and applications

Lecture 54 : Application of Stokes' theorem [Section 54.1]

Objectives

In this section you will learn the following:

- Computational applications of Stokes' theorem.
- Physical applications of Stokes' theorem.
- Sufficient conditions for a vector field to be conservative.

54.1 Applications of Stokes' theorem

Stokes' theorem gives a relation between line integrals and surface integrals. Depending upon the convenience, one integral can be computed in terms of the other.

54.1.1 Example (computation of line integral):

We want to compute

\[ \oint_C \mathbf{F} \cdot d\mathbf{r}, \quad \text{where} \quad \mathbf{F} = yi + xz^2j - zy^2k \]

and \( C \) is the circle in the plane \( z = -3, x^2 + y^2 = 4 \), oriented anti-clockwise. To apply Stokes' theorem, let us find a convenient surface \( S \) whose boundary is \( C \). The most natural surface in this case is the circular disc \( x^2 + y^2 \leq 4, z = -3 \).

For \( S \), if we choose the normal vector \( \mathbf{n} \) to be \( \mathbf{k} \), then \( \partial(S) = C \) will have anti-clockwise orientation, and by Stokes' theorem we will have

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl} \mathbf{F} \cdot \mathbf{n}) \, dS. \]

Since,

\[ \text{curl}(\mathbf{F}) \cdot \mathbf{n} = \left[ \frac{\partial}{\partial x} (xz^2) - \frac{\partial}{\partial y} (yi) \right] = z^3 - 1. \]

we have

\[ \oint_C (\mathbf{F} \cdot d\mathbf{r}) = \iint_S (z^3 - 1) \, dS = \iint_S (-28) \, dS = -28(4\pi). \]
54.1.2 Example:

Let us try to apply the above technique to evaluate

\[ \oint_C (\mathbf{F} \cdot d\mathbf{r}) \text{ where } \mathbf{F} = -\frac{y}{x^2 + y^2} \mathbf{i} + \frac{y}{x^2 + y^2} \mathbf{j}, \]

and \( C \) is the circle \( x^2 + y^2 = 1, z = 0 \) oriented clockwise. We note that for \( \mathbf{F} \), \( \text{curl}(\mathbf{F}) = 0 \), and hence

\[ \iint_S (\text{curl} \mathbf{F}) \cdot \mathbf{n} dS = 0, \]

whatever be the surface \( S \), as long as \( \partial(S) = C \). We also know (see example 47.2.8) that

\[ \oint_C (\mathbf{F} \cdot d\mathbf{r}) = -2\pi. \]

Thus,

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} \neq \iint_S (\text{curl} \mathbf{F} \cdot \mathbf{n}) dS. \]

Does this contradict Stokes' theorem? The answer is no. The reason for this is that Stokes' theorem is not applicable for the given \( \mathbf{F} \) because

\[ \mathbf{F}(x, y, z) = -\frac{y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j} \]

is defined in \( D := \{(x, y, z) \in \mathbb{R}^3 \mid (0, 0, z)\} \). Thus, there does not exist any surface \( S \subseteq D \) whose boundary is the unit circle \( C : x^2 + y^2 = 1 \) in the \( xy \)-plane. However, we can still evaluate the integral

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} \]

as follows: Consider the vector field

\[ \mathbf{\tilde{F}}(x, y, z) := -\frac{y}{x^2 + y^2 + z^2} \mathbf{i} + \frac{x}{x^2 + y^2 + z^2} \mathbf{k}, \]

\((x, y, z) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} \). Note that \( \mathbf{\tilde{F}} = \mathbf{F} \) on \( C \), and if we consider the upper hemisphere \( S \) given by \( x^2 + y^2 + z^2 = 1, z \geq 0 \), then \( \partial(S) = C \) and \( S \cup \mathbb{C} \mathbb{R}^3(\{0, 0, 0\}) \). If we select the outward unit normal

\[ \mathbf{n} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}, \]

\( x^2 + y^2 + z^2 = 1 \) on \( S \), then \( \partial(S) = C \) gets the anti-clockwise orientation and we have by Stokes' theorem

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{\tilde{F}} \cdot d\mathbf{r} = \iint_S (\text{curl} \mathbf{\tilde{F}}) \cdot \mathbf{n} dS. \]
Let $S$ be given the parameterizations:

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (\sqrt{1-x^2-y^2})\mathbf{k}, (x,y) \in D.$$ 

where $D$ is the unit disc in $xy$-plane. Then

$$\mathbf{r}_x \times \mathbf{r}_y = -f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k},$$

for $f(x,y) = \sqrt{1-x^2-y^2}$. This gives

$$\mathbf{r}_x \times \mathbf{r}_y = \frac{x}{z}\mathbf{i} + \frac{y}{z}\mathbf{k} + \mathbf{k}.$$ 

Also

$$\text{curl}(\mathbf{F}) = -2zx \mathbf{i} - 2zy \mathbf{k} - z^2 \mathbf{k}.$$ 

Hence

$$\iint_S (\text{curl} \mathbf{F} \cdot \mathbf{n}) \, dS = \iint_D \left( \frac{x}{z}\mathbf{i} + \frac{y}{z}\mathbf{k} + \mathbf{k} \right) (-2zx \mathbf{i} - 2zy \mathbf{k} - z^2 \mathbf{k}) \, dxdy$$

$$= \iint_D -(2x^2 + 2y^2 + 2z^2) \, dxdy$$

$$= -2 \iint_D dxdy$$

$$= -2\pi.$$

### 54.1.3 Example (Calculation of surface integral):

In the formula

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} (\mathbf{F}) \cdot \mathbf{n} \, dS$$

the only relation between $C$ and $S$ is that $\partial(S) = C$. Thus, if $S_1$ and $S_2$ are two surfaces such that $\partial(S_1) = \partial(S_2) = C$ and $\mathbf{F}$ is defined in a region including have

$$\iint_{S_1} \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dS$$

This is useful in computations. We consider an example. Let us evaluate

$$\iint_S (\nabla u \times \nabla v) \cdot \mathbf{n} \, dS$$
where
\[ u = x^3 - y^3 + z^2, \text{ and } v = x + y + z, \]
\( \mathcal{S} \) is the upper hemispherical sheet
\[ x^2 + y^2 + z^2 = 1, \ z \geq 0, \]
\( \mathbf{n} \) is the unit normal to \( \mathcal{S} \) with non-negative \( z \)-component. Since
\[ (\nabla u \times \nabla v) = \text{curl} \ (u \ \nabla v), \]
we have by Stokes’ theorem
\[
\iint_{\mathcal{S}} (\nabla u \times \nabla v) \cdot \mathbf{n} \ dS = \oint_{\partial \mathcal{S}} (u \ \nabla v) \cdot d\mathbf{r} = \iint_{\mathcal{S}} (\nabla u \times \nabla v) \cdot \mathbf{n} \ dS',
\]
where we select \( \mathcal{S}_1 \) be the circular disc
\[ D = \{(x, y, z) \mid x^2 + y^2 = a^2, z = 0 \}. \]
Thus
\[
\iint_{\mathcal{S}} (\nabla u \times \nabla v) \cdot \mathbf{n} \ dS = \iint_{D} (\nabla u \times \nabla v) \cdot \mathbf{k} \ dxdy.
\]
Since, \( (\nabla u \times \nabla v) \cdot \mathbf{k} = 3x^2 + 3y^2 \), we have
\[
\iint_{\mathcal{S}} (\nabla u \times \nabla v) \cdot \mathbf{n} dS = \int_{x^2 + y^2 \leq a^2} 3 (x^2 + y^2) \ dxdy
\]
\[ = 3 \int_{0}^{a} \int_{0}^{2\pi} r^3 \ d\theta \ dr
\]
\[ = \frac{3\pi}{2}. \]

54.1.4 Note (Green’s theorem, a particular case of Stokes’ theorem):

Consider a planer vector-field.
\[ \mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}. \]
Let \( D \) be a region in the \( xy \)-plane with \( \partial(D) = \mathcal{C} \) a simple closed curve. If we treat \( D \) as a flat surface, oriented \( \mathbf{k} \) as the unit normal, then by Stokes’ theorem, treating \( \mathbf{F} \) as a vector field in \( z \)-space,
\[ \mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} + 0 \mathbf{k} \]
\[ \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{D} \text{curl}(\mathbf{F}) \cdot \mathbf{k} \ dxdy, \]
which is Green’s theorem, since
\[ \text{curl}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}. \]
We showed in section 47.2, that given points \( A, B \in D \subseteq \mathbb{R}^3 \) and \( \mathbf{F} \), a continuously differentiable vector field on \( D \), if the line integral
\[ \int_{\mathcal{C}(A, B)} \mathbf{F} \cdot d\mathbf{r} \]
is independent of the path \( C \) going \( A \) to \( B \), then \( \text{curl}(\mathbf{F}) = 0 \). We also stated that this condition is also sufficient if the domain \( D \) is simply connected. We prove this as an application of Stokes' theorem.

### 54.1.5 Theorem (Conditions for conservativeness):

Let \( D \) be a simply connected domain in \( \mathbb{R}^3 \) \( : D \to \mathbb{R}^3 \) be a continuously differentiable vector field. Then the following are equivalent:

1. \( \mathbf{F} \) is conservative.
2. \( \int_{C(A,B)} \mathbf{F} \cdot dr \) is independent of the path \( C(A,B) \) joining \( A \) to \( B \).
3. \( \text{curl}(\mathbf{F}) = 0 \).

**Proof**

In view of the statements above, we only have to show that \((iii) \Rightarrow (ii)\). Let \( C \) be any simple closed curve in \( D \). Since \( D \) is simply connected, we can find an orientable piecewise smooth surface \( S \subseteq D \) such that \( C = \partial S \). Then by Stokes' theorem

\[
\oint_C \mathbf{F} \cdot dr = \iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dS = 0.
\]

Hence, by theorem 47.2.5, \((ii)\) words.

### 54.1.6 Physical interpretation of Curl:

Stokes' theorem provides a way of interpreting the curl of a vector-field \( \mathbf{F} \) in the context of fluid-flows. Consider a small circular disc \( S_a \) of radius \( a \) at a point \( P \) in the domain of \( \mathbf{F} \). Let \( \mathbf{n} \) be the unit normal to the disc at \( P_0 \). Then by Stokes' theorem

\[
\phi_{C_a} (\mathbf{F} \cdot \mathbf{T}) d\mathbf{s} = \iint_{C_a} \mathbf{F} \cdot dr
= \iint_{S_d} \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, d\mathbf{s}
\equiv \left( \text{curl} \mathbf{F}(P_0) \cdot \mathbf{n}(P_0) \right) \iint_{S_d} \, d\mathbf{s}
= \left( \text{curl} \mathbf{F}(P_0) \cdot \mathbf{n} \right) A(S_a)
\]

**Figure: Flux along \( C_a \)**

Thus,
\[
\left(\text{curl } \mathbf{F}(P_0)\right) \cdot \mathbf{n}(P_0) = \frac{1}{A(S_d)} \oint_{C_a} (\mathbf{F} \cdot \mathbf{T}) ds
\]

Note that \(\mathbf{F} \cdot \mathbf{T}\) gives the component of \(\mathbf{F}\) in the direction of the tangent and hence gives the rotational component of \(\mathbf{F}\) along \(ds\). Then \(\Phi_{C_{\alpha}} (\mathbf{F} \cdot \mathbf{T}) ds\) is called the circulation density of \(\mathbf{F}\) around \(C_{\alpha}\). If we let \(\alpha \to 0\) in (92), then we will have (as error in approximation goes to zero)

\[
\left(\text{curl } \mathbf{F}(P_0)\right) \cdot \mathbf{n}(P_0) = \lim_{\alpha \to 0} \frac{1}{A(S_d)} \oint_{C_{\alpha}} (\mathbf{F} \cdot \mathbf{T}) ds
\]

For this reason, the normal component of \(\text{curl}(\mathbf{F})\), also called the specific circulation of the fluid at the point \(P_0\).

Note that the specific circulation is maximum when \(\text{curl}\mathbf{F}(P_0)\) and \(n(P_0)\) have the same direction we can interpret Stokes' theorem.

\[
\int_S (\text{curl } \mathbf{F} \cdot \mathbf{n}) dS = \Phi_C \mathbf{F} \cdot d\mathbf{r}
\]

as follows: the collective measure of rotational tendency is equal to the tendency of the fluid to circulate around its boundary. Thus, if \(\text{curl}\mathbf{F} = 0\) in \(S\), then

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = 0
\]

i.e., there is no circulation tendency, or one says the fluid is irrotational.

**Practice Exercises**

1. Using Stokes' theorem, evaluate the integral

   \[
   \int_C \mathbf{F} \cdot d\mathbf{r},
   \]

   where

   1. \(\mathbf{F}(x, y, z) = 2y \mathbf{i} + 3z \mathbf{j} + x \mathbf{k}\), and \(C\) is the triangle with vertices \(A(0, 0, 0), B(0, 2, 0), C(1, 1, 1)\) oriented for \(A\) to \(B\) to \(C\) to \(A\).

   2. \(\mathbf{F}(x, y, z) = z^2 \mathbf{i} + y \mathbf{j} + x \mathbf{k}\), and \(C\) is the boundary of the upper hemisphere \(S: z = \sqrt{4 - x^2 - y^2}\), oriented counter clockwise.

   3. \(\mathbf{F}(x, y, z) = x^2 \mathbf{i} + 4y^3 \mathbf{j} + y^2 x \mathbf{k}\) and \(C\) is the rectangle going \(A(0, 0, 0), B(1, 0, 0), C(1, 3, 2), D(0, 3, 2)\), oriented from \(A\) to \(B\) to \(C\) to \(D\) to \(A\).

   4. \(\mathbf{F}(x, y, z) = 3y \mathbf{i} + 4x \mathbf{j} + 2y \mathbf{k}\), and \(C\) is the boundary if the paraboloid \(z = 4 - x^2 - y^2, z \geq 0\) oriented counter clockwise.

   Answer:

   (i) 1

   (ii) 0
2. Let \( \mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - x)\mathbf{j} + (z - xy)\mathbf{k} \), and \( \mathcal{C} \) be the boundary of the triangle \( \mathcal{S} \) with vertices \( A(1, 0, 0), B(0, 2, 0), C(0, 0, 1) \). Find the following.

1. Circulation of \( \mathbf{F} \) around the triangle when \( \mathcal{C} \) is oriented counterclockwise.

2. Circulation density of \( \mathbf{F} \) at \((0, 0, 0)\) in the direction \( \mathbf{k} \).

3. Find the direction of \( \mathbf{n} \) along which the circulation density of \( \mathbf{F} \) at \((0, 0, 0)\) is maximum.

Answer:

(i) \( \frac{3}{2} \)

(ii) \(-1\)

(iii) \(-\frac{1}{\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}\)

3. Let \( S \) be the upper half of the ellipsoid \( \frac{x^2}{9} + y^2 + \frac{z^2}{9} = 1 \), oriented so that \( \mathbf{n} \) is upward. For \( \mathbf{F}(x, y, z) = x^2\mathbf{i} + y^4\mathbf{j} + z^2 \sin xy \mathbf{k} \)

Evaluate

\[
\iint_S (\text{curl} \, \mathbf{F}) \, \mathbf{n} \, dS,
\]

Replacing \( S \) by a suitable simpler surface with the same boundary as that of \( S \).

Answer: 0

Recap

In this section you have learnt the following

- Computational applications of Strokes' theorem.
- Physical applications of Strokes' theorem.
- Sufficient conditions for a vector field to be conservative