Module 17: Surfaces, Surface Area, Surface Integrals, Divergence Theorem and applications

Lecture 50: Surface Integrals [Section 50.1]

Objectives

In this section you will learn the following:

- How to define the integrals of a scalar field over a surface.

50.1 Surface Integrals:

Similar to the integral of a scalar field over a curve, which we called the line integral, we can define the integral of a vector-field over a surface.

Let $S$ be a surface in space with finite surface area. Let $f$ be a continuous scalar-field defined on the surface $S$. We can subdivide $S$ into smaller portions, say $S_1, S_2, ..., S_n$ having areas $\Delta S_1, \Delta S_2, ..., \Delta S_n$, and form the sum

$$\mathcal{S}_k := \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta S_k,$$

where $(x_k, y_k, z_k) \in S_k$, is selected arbitrarily. By refining the patches into more smaller patches such that $\max(\Delta S_k) \to 0$, if $\mathcal{S}_k$ approaches a limit, we call it the surface integral of $f$ over $S$, and denote it by

$$\iint_S f(x, y, z) \, dS.$$
50.1.1 Definition:

Let \( \mathcal{S} \) be a surface with parameterization

\[
r : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (u, \nu) \mapsto r(u, \nu), \quad (u, \nu) \in R.
\]

If \( r(u, \nu) \) is continuous and \( R \) is closed and bounded, then for a continuous function \( f : R \rightarrow \mathbb{R}^3 \), we can define

\[
\iint_{\mathcal{S}} f(x, y, z) \, d\mathcal{S} := \iint_{R} f(x(u, \nu), y(u, \nu), z(u, \nu)) \| r_u \times r_\nu \| \, du \, dv,
\]

called the **surface integral** of \( f \) over the surface \( \mathcal{S} \).

50.1.2 Example:

Let us evaluate the surface integral

\[
\iint_{\mathcal{S}} y^2 \, d\mathcal{S},
\]

where \( \mathcal{S} \) is the sphere

\[
\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.
\]

We give \( \mathcal{S} \) the spherical coordinate parameterization

\[
r(\theta, \phi) = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi + \cos \phi \mathbf{k}, \quad (\theta, \phi) \in [0, 2\pi] \times [0, \pi]
\]

Then

\[
r_\theta = -\sin \theta \sin \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j},
\]

and

\[
r_\phi = \cos \theta \cos \phi \mathbf{i} + \sin \theta \cos \phi \mathbf{j} - \sin \phi \mathbf{k}.
\]

Thus

\[
r_\theta \times r_\phi = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \\
\cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi
\end{vmatrix}
\]

\[
= \left( -\sin^2 \phi \cos \theta \right) \mathbf{i} - \left( +\sin^2 \phi \sin \theta \right) \mathbf{j}
\]

\[
+ \left( -\sin^2 \phi \sin \phi \cos \phi - \cos^2 \theta \sin \phi \cos \theta \right) \mathbf{k}
\]

Hence,

\[
\| r_\theta \times r_\phi \|^2 = \sin^4 \phi (\cos^2 \theta + \sin^2 \theta) + \sin^2 \phi \cos^3 \phi
\]

\[
= \sin^2 \phi (\sin^2 \phi + \cos^2 \phi) = \sin^2 \phi.
\]

Thus,

\[
\| r_\theta \times r_\phi \| = \sin \phi.
\]

This gives, for \( R = [0, 2\pi] \times [0, \pi] \),
50.1.3 Surface Integral for surfaces in explicit form:

For a smooth surface given explicitly as

\[ S = \{(x,y,z) \mid z = h(x,y) \text{ for } (x,y) \in D \}, \]

a parameterization is given by

\[ \mathbf{r}(x,y) = x \mathbf{i} + y \mathbf{j} + h(x,y) \mathbf{k}, \quad (x,y) \in D. \]

Since,

\[ \| \mathbf{r}_x \times \mathbf{r}_y \| = \sqrt{1 + h'^2_x + h'^2_y} \]

we have

\[ \iint_S f(x,y,z) \, dS = \iint_R f(x,y,h(x,y)) \sqrt{1 + h'^2_x + h'^2_y} \, dx \, dy. \]

Similarly, if \( S \) is given by

\[ S = \{(x,y,z) \mid x = g(y,z), \quad (y,z) \in R \}, \]

then

\[ \iint_S f(x,y,z) \, dS = \iint_R f(g(y,z),y,z) \sqrt{1 + g'^2_y + g'^2_z} \, dy \, dz. \]

Finally, if \( S \) is given by

\[ S = \{(x,y,z) \mid y = k(x,z), \quad (x,z) \in R \}, \]

then

\[ \iint_S f(x,y,z) \, dS = \iint_R f(x,k(x,z),z) \sqrt{1 + k'^2_x + k'^2_z} \, dx \, dz. \]

50.1.4 Example:

Let us evaluate the integral

\[ \iint_R f \, dS, \]
where \( f(x,y,z) = z^2 \) and \( S \) is the surface of the cone \( z^2 = x^2 + y^2 \) between the planes \( z = 1 \) and \( z = 2 \).

We can give the surface the following parameterization:

\[
\mathbf{r}(x,y) = x \mathbf{i} + y \mathbf{j} + \left(\sqrt{x^2 + y^2}\right) \mathbf{k}, \quad (x,y) \in \mathbb{R},
\]

where \( \mathbb{R} \) is the projection of the surface on the \( xy \)-plane, 

\[ R := \{(x,y) | 1 \leq x^2 + y^2 \leq 2\} \]

Since \( \mathbf{r}_x \times \mathbf{r}_y \parallel \sqrt{2} \), we have

\[
\iiint_S f \, dS = \iint_R (x^2 + y^2) \sqrt{2} \, dx \, dy
= \int_{r=1}^{2} \int_{\theta=0}^{2\pi} \sqrt{2} \, r^2 \, dr \, d\theta \quad \text{(Using polar coordinates)}
= 2\sqrt{2} \, \pi \int_{1}^{2} r^2 \, dr
= 2\sqrt{2} \, \pi \left[ \frac{16}{4} - \frac{1}{4} \right]
= \frac{15\sqrt{2} \, \pi}{2}.
\]

**50.1.5 Note:**

Recall that, for a surface \( S \) given explicitly by \( z = h(x,y), (x,y) \in D \), the surface integral of a scalar field \( f \) over \( S \) is given by

\[
\iint_S f \, dS = \iint_D f(x,y,h(x,y)) \sqrt{1 + h_x^2 + h_y^2} \, dx \, dy,
\]

where \( D \) is the projection of \( S \) onto the \( xy \)-plane. Thus,

\[
\iint_S f(x,y,z) \left(\frac{1}{\sqrt{1 + h_x^2 + h_y^2}}\right) \, dS = \iint_D f(x,y,h(x,y)) \, dx \, dy. \quad \text{-------(64)}
\]

Since \( S \) has parameterization

\[
\mathbf{r}(x,y) = x \mathbf{i} + y \mathbf{j} + h(x,y) \mathbf{k}, \quad (x,y) \in D,
\]
and 

\[ \mathbf{r}_x \times \mathbf{r}_y = -h_y \mathbf{i} - h_x \mathbf{j} + \mathbf{k}, \]  

----------(65)

this gives, 

\[ \sqrt{1 + h_x^2 + h_y^2} = \| \mathbf{r}_x \times \mathbf{r}_y \|. \]  

----------(66)

Using (65), we have 

\[ 1 = (\mathbf{r}_x \times \mathbf{r}_y) \cdot \mathbf{k} = \| \mathbf{r}_x \times \mathbf{r}_y \| \cos \gamma, \]  

----------(67)

where \( \gamma \) is the acute angle between \( \mathbf{r}_x \times \mathbf{r}_y \), the normal to \( \mathcal{S} \), and \( \mathbf{k} \). From (66) and (67), we have 

\[ \cos \gamma = \frac{1}{\sqrt{1 + h_x^2 + h_y^2}}. \]

Hence, (64) gives us the relation 

\[ \iint_{\mathcal{S}} f \cos \gamma \, dS = \iint_{\mathcal{D}} f(x, y, h(x, y)) \, dx \, dy. \]

**Practice Exercises**

1. Evaluate the surface integral 

\[ \iint_{\mathcal{S}} (y^2 + 2yz) \, dS, \]  

where \( \mathcal{S} \) is portion of the plane \( 2x + y + 2z = 6 \) in the first octant 

**Answer:** \( \frac{24z}{2} \)

2. Evaluate 

\[ \iint_{\mathcal{S}} (x + z) \, dS \]  

where \( \mathcal{S} \) is the portion of the cylinder \( y^2 + z^2 = 9 \) in the first octant between the planes \( x = 0 \) and \( x = 4 \). 

**Answer:** \( 12\pi + 36 \)

3. Evaluate 

\[ \iint_{\mathcal{S}} x\sqrt{y^2 + 4} \, dS, \]  

where \( \mathcal{S} \) is the portion of the cylinder \( y^2 + 4z = 16 \) cut by the planes \( x = 0, x = 1 \) and \( z = 0 \). 

**Answer:** \( \frac{56}{3} \)
Recap:
In this section you have learnt the following

- How to define the integrals of a scalar field over a surface.

**Section 50.2**

**Objectives**
In this section you will learn the following:

- Some application of the surface integrals.

### 50.2 Applications of surface integrals:

#### 50.2.1 Mass and center of mass of a surface.

Consider a surface \( S \) of density (mass per unit area) \( \rho(x,y,z), (x,y,z) \in S \). Then the mass of \( S \) can be defined to be

\[
M := \iint_S \rho(x,y,z) \, dS,
\]

The moments of \( S \) about the three axes planes is defined by

\[
\iint_S x \, \rho(x,y,z) \, dS, \quad \iint_S y \, \rho(x,y,z) \, dS, \quad \iint_S z \, \rho(x,y,z) \, dS.
\]

Further the point \((\bar{x}, \bar{y}, \bar{z})\) is called the **center of mass** of \( S \), where

\[
\bar{x} = \frac{\iint_S x \, \rho(x,y,z) \, dS}{\iint_S \rho(x,y,z) \, dS},
\]

\[
\bar{y} = \frac{\iint_S y \, \rho(x,y,z) \, dS}{\iint_S \rho(x,y,z) \, dS},
\]

\[
\bar{z} = \frac{\iint_S z \, \rho(x,y,z) \, dS}{\iint_S \rho(x,y,z) \, dS}.
\]

#### 50.2.2 Flux of a fluid across a surface

Let \( \mathbf{V}(x,y,z) \) represent the velocity field of a fluid flow in space at a point \((x,y,z)\). Let \( \rho(x,y,z) \) be its
density at \((x, y, z)\). Then, \(\mathbf{F}(x, y, z) = \rho(x, y, z) \mathbf{V}(x, y, z)\),

represents the **flux-density** (mass per unit area per unit time) of the flow. Consider a surface \(S\) in the flow. If \(S\) is smooth, then the flux-density across a small patch \(\Delta S\) of the surface at a point \((x, y, z) \in S\) is given by the normal component of \(\mathbf{F}\), i.e., \(\mathbf{F} \cdot \mathbf{n}\). Thus, the mass of the fluid flow across \(\Delta S\) can be taken to be \((\mathbf{F} \cdot \mathbf{n}) \Delta S\), where \(\mathbf{n}\) is the unit normal at \((x, y, z)\). Thus, the total mass of the fluid crossing across the surface \(S\) can be defined to be

\[
\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS. \tag{68}
\]

In order to be able to do so, it becomes necessary to ensure that the function

\[
(\mathbf{F} \cdot \mathbf{n})(x, y, z), (x, y, z) \in S
\]

is integrable over \(S\). For example, this will be so if \((x, y, z) \mapsto (\mathbf{F} \cdot \mathbf{n})(x, y, z)\) is continuous. For this, we can assume that \(\mathbf{F}\) is continuous. Thus, to be able to define (68), we should be able to say that our surface \(S\) is such that at every point \((x, y, z) \in S\), three exist unit normal \(\mathbf{n}(x, y, z)\) which varies continuously as \((x, y, z)\) very over \(S\). This motivates our next definition:

**50.2.3 Definition:**

A surface \(S\) is said to be **orientable** if there exists a continuous vector-field

\[(x, y, z) \mapsto \mathbf{n}(x, y, z), (x, y, z) \in S\]

such that \(\mathbf{n}(x, y, z)\) is the unit normal vector to \(S\) at \((x, y, z) \in S\).

Orientability of a surface essentially means that there are two sides of the surface.

**50.2.4 Examples:**

1. Every simple closed surface is orientable, we can have a continuous inward or an outward normal to the surface.
   For example, surfaces like sphere, ellipsoid, etc, are all orientable, with a continuous normal pointing in the region enclosed or pointing away from the region enclosed.
2. If $\mathcal{S}$ is the boundary of an annulus region in space, it is orientable. For example, the surface enclosing two concentric spheres is orientable (note, it is not connected).

3. **Möbius strip**: The surface as shown below is not orientable. It is not possible to define a continuous normal along, say, the curve $C$.

50.2.5 **Definition**

Let $\mathcal{S}$ be an oriented surface with the continuous unit normal $n(x,y,z), (x,y,z) \in \mathcal{S}$. Let $\mathbf{F}$ be a continuous vector field on $\mathcal{S}$. Then the integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} := \iint_{\mathcal{S}} (\mathbf{F} \cdot n) \, dS$$

is called the **flux-integral** of $\mathbf{F}$ over the surface $\mathcal{S}$.

Physically, $\iint_{\mathcal{S}} (\mathbf{F} \cdot n) \, dS$ represents the flux of the fluid with flux density $\mathbf{F}$ across the surface $\mathcal{S}$ in the direction of the chosen normal.
50.2.6 Example:

Let 
\[ \mathbf{F} = xz \mathbf{i} + yz \mathbf{j} + x^2 \mathbf{k} \]
and 
\[ S = \{(x,y,z) \mid x^2 + y^2 + z^2 = a^2\}, \]
oriented with outward unit normal. We want to compute 
\[ \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS. \]

We can write \( S = S_1 \cup S_2 \), where \( S_1 \) is the upper hemisphere and \( S_2 \) is the lower hemisphere. The upper part \( S_1 \), parameterized as 
\[ \mathbf{r}(x,y) = x \mathbf{i} + y \mathbf{j} + (\sqrt{a^2 - x^2 - y^2}) \mathbf{k}, (x,y) \in \mathbb{R} \{ x^2 + y^2 \leq a^2 \}. \]

Thus, for \( S_1 \),
\[ \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial}{\partial x} \left( \sqrt{a^2 - x^2 - y^2} \right) \mathbf{i} - \frac{\partial}{\partial y} \left( \sqrt{a^2 - x^2 - y^2} \right) \mathbf{j} + \mathbf{k} \]
\[ = \frac{x}{\sqrt{a^2 - x^2 - y^2}} \mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}} \mathbf{j} + \mathbf{k}, \]
and this is the outward normal for the upper hemisphere as the \( \mathbf{k} \) component is positive. Similarly, the surface \( S_2 \), has parameterization 
\[ \mathbf{r}(x,y) = x \mathbf{i} + y \mathbf{j} - (\sqrt{a^2 - x^2 - y^2}) \mathbf{k}, (x,y) \in \mathbb{R} \{ x^2 + y^2 \leq a^2 \}. \]

and 
\[ \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial}{\partial x} \left( -\sqrt{a^2 - x^2 - y^2} \right) \mathbf{i} - \frac{\partial}{\partial y} \left( -\sqrt{a^2 - x^2 - y^2} \right) \mathbf{j} + \mathbf{k} \]

But, this is not the outward normal, as the \( \mathbf{k} \) component is positive. In fact, the outward normal for \( S_2 \) is given by 
\[ -\mathbf{r}_x \times \mathbf{r}_y = -\left( \frac{x}{\sqrt{a^2 - x^2 - y^2}} \mathbf{i} - \frac{y}{\sqrt{a^2 - x^2 - y^2}} \mathbf{j} + \mathbf{k} \right). \]

Thus,
\[ \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS. \quad \text{----------(69)} \]

The integrand of first integral in the right hard side of (69) is
Similarly, the integrand of the second integral in (69) is
\[
\frac{x}{\sqrt{a^2 - x^2 - y^2}} \left\{ \frac{\sqrt{a^2 - x^2 - y^2}}{a} \right\} y + \frac{y}{\sqrt{a^2 - x^2 - y^2}} \left\{ \frac{\sqrt{a^2 - x^2 - y^2}}{a} \right\} x = -x^2 = -2 \left( x^2 + y^2 \right)
\]
Thus, from (69)
\[
\int_S \mathbf{F} \cdot \mathbf{n} \, dS = 0.
\]
An alternate way of analyzing the above problem is the following. First of all, the surface $S$ can also be described by the implicit equation
\[
f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0.
\]
Since
\[
\nabla f = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k},
\]
a unit normal to $S$ is given by
\[
\mathbf{n} = \pm \frac{\nabla f}{\| \nabla f \|} = \pm \frac{2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}}{2 \sqrt{x^2 + y^2 + z^2}} = \pm \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \pm (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}).
\]
Clearly, $\mathbf{n}$ with the positive sign is the unit outward normal to $S$. Thus
\[
\int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_S \left( \frac{xz \mathbf{i} + yz \mathbf{j} + x^2 \mathbf{k}}{a} \right) \cdot (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \, dS
\]
\[
= \frac{1}{a} \int_S (x^2 z + y^2 z + x^2 z) \, dS
\]
\[
= \frac{1}{a} \int_S (x^2 + y^2 + z^2) \, dS
\]
\[
= \frac{\alpha^2}{a} \int_S \, dS
\]
\[
= \left( \int_S 1 \, dS - \int_{S_2} 1 \, dS \right)
\]
\[
= 0.
\]

**50.2.7 Example:**

Consider the surface $S$ to be the boundary of the region
\[
\{(x, y, z) | 1 \leq x^2 + y^2 + z^2 \leq 4\}.
\]
Let us evaluate

$$\iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS,$$

for $\mathbf{n}$ to be the outward normal on $S$ and

$$\mathbf{F}(x,y,z) = \frac{-x}{r^3} \mathbf{i} - \frac{y}{r^3} \mathbf{j} - \frac{z}{r^3} \mathbf{k},$$

where

$$r = \sqrt{x^2 + y^2 + z^2}.$$

The surface $S_1$ is the outer sphere of radius 1 and the inner sphere $S_2$ of radius 2.

![Figure: The surface $S$](image)

As in previous example, the outward unit normal for $S_2$ given by

$$\mathbf{n} = \frac{\nabla f}{\| \nabla f \|} = \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{r}.$$

Thus, for $S_2$ we have

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_2} \left( -\frac{x}{r^3} \mathbf{i} - \frac{y}{r^3} \mathbf{j} - \frac{z}{r^3} \mathbf{k} \right) \cdot \left( \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{r} \right) \, dS$$

$$= \iint_{S_2} \left( \frac{x^2}{r^4} + \frac{y^2}{r^4} + \frac{z^2}{r^4} \right) \, dS$$

$$= -\frac{1}{r^3} \iint_{S_1} \, dS$$

$$= -4 \pi.$$

Similarly for $S_1$, the outward unit normal is

$$\mathbf{n} = \frac{\nabla f}{\| \nabla f \|} = \frac{-x \mathbf{i} - y \mathbf{j} - z \mathbf{k}}{r}.$$
Thus,
\[ \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} \left( -\frac{x}{r^3} \mathbf{i} - \frac{y}{r^3} \mathbf{j} - \frac{z}{r^3} \mathbf{k} \right) \left( \frac{-x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{r} \right) \, dS = 4 \pi. \]

Hence
\[ \iint_{S} (\mathbf{F} \cdot \mathbf{n}) \, dS = 0. \]

50.2.8 Note:

1. Note that for an orientable surface, if \( \mathbf{n}(x,y,z) \) is one choice of continuous unit normal vector to \( S \), then \( -\mathbf{n}(x,y,z) \) is also another choice of continuous unit normal to \( S \). The flux integral changes sign if we change one selection to other. When we select positive sign, we call \( \mathbf{n} \) as the positive unit normal, and \( -\mathbf{n} \) will be called the negative-unit normal. Thus, for an orientable surface \( S \) with parametrization \( \mathbf{r}(u,v), (u,v) \in D \), we have
\[ \iint_{S} (\mathbf{F} \cdot \mathbf{n}) \, ds = \pm \iint_{D} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dudv, \]
depending upon one choice of the unit normal.

2. Special forms of flux-integral

Let us look at the special cases of \( S \). Suppose \( S \) is given explicitly by \( z = g(x,y), (x,y) \in D \). Then, a parametrization of \( S \) is given by
\[ \mathbf{r}(x,y) = x \mathbf{i} + y \mathbf{j} + g(x,y) \mathbf{k}. \]

Thus,
\[ \mathbf{r}_x \times \mathbf{r}_y = -g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k}, \]
and hence for the positive orientation of \( S \)
\[ \iint_{S} (\mathbf{F} \cdot \mathbf{n}) \, ds = \iint_{D} \mathbf{F} \cdot \left( -g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k} \right) \, dxdy. \]

Thus, if
\[ \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}, \]
then
\[ \iint_{S} (\mathbf{F} \cdot \mathbf{n}) \, ds = \iint_{D} \left( -P \frac{g_x}{g} - Q \frac{g_y}{g} + R \right) \, dxdy, \]
where \( \mathbf{n} \) is the positive unit normal. If we write \( G(x,y,z) = z - g(x,y) \), then
\[ \mathbf{r}_x \times \mathbf{r}_y = -g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k} = \nabla G. \]

Hence, for the choice of positive oriented normal on \( S \), given by \( z = g(x,y) \) and \( G(x,y,z) = z - g(x,y) \),
\[ \int_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \int_S (\mathbf{F} \cdot \nabla G) \, dS \]

Similar formula holds if \( S \) is represented as \( y = h(x,z) \) or \( x = k(y,z) \).

3. There is another representation possible for the flux-integral
\[ \int_S \mathbf{F} \cdot d\mathbf{S}. \]

Let the continuous normal \( \mathbf{n} \) have direction cosines \( \cos \alpha, \cos \beta, \cos \gamma \), i.e.,
\[ \mathbf{n} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k} \]

Then, for \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \)
\[ \int_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \int_S P \cos \alpha \, ds + \int_S Q \cos \beta \, ds + \int_S R \cos \gamma \, dS. \]

While evaluating, care must be taken the integrals on right hand side since \( S \) is oriented. Suppose, we select the positive orientation for the normal. Then for
\( S: z = g(x,y), (x,y) \in D, \)
\[ \int_S R \cos \gamma \, dS = \begin{cases} \int_D R(x,y,g(x,y)) \, dx \, dy & \text{if } \cos \gamma > 0 \\ -\int_D R(x,y,g(x,y)) \, dx \, dy & \text{if } \cos \gamma < 0 \end{cases} \]

(iv) If \( \mathbf{F} \) and \( \mathbf{r} \) are expressed in terms of their components:
\( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}, \)
\( \mathbf{r} (u,v) = x(u,v) \mathbf{i} + y(u,v) \mathbf{j} + z(u,v) \mathbf{k}, (u,v) \in D, \)
then
\[ \mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k}, \]
\[ \mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}, \]
and hence
\[ \mathbf{r}_u \times \mathbf{r}_v = \left( \frac{\partial y \partial z}{\partial u \partial v} - \frac{\partial z \partial y}{\partial u \partial v} \right) \mathbf{i} + \left( \frac{\partial z \partial x}{\partial u \partial v} - \frac{\partial x \partial z}{\partial u \partial v} \right) \mathbf{j} + \left( \frac{\partial x \partial y}{\partial u \partial v} - \frac{\partial y \partial x}{\partial u \partial v} \right) \mathbf{k}. \]

Thus, for positive orientation of the surface,
\[ \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \iint_D^P \left( \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y}{\partial v} \right) \, dudv \]
+ \[ \iint_D^Q \left( \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} \right) \, dudv \]
+ \[ \iint_D^R \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right) \, dudv. \]

The three integrals on the right hand side are represented as follows

\[ \iint_S^P (x,y,z) \, dy \wedge dz := \iint_D^P (r(u,v)) \left( \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y}{\partial v} \right) \, dudv, \]

\[ \iint_S^Q (x,y,z) \, dz \wedge dx := \iint_D^Q (r(u,v)) \left( \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} \right) \, dudv, \]

and

\[ \iint_S^R (x,y,z) \, dx \wedge dy := \iint_D^R (r(u,v)) \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right) \, dudv, \]

Note that the order of the notation \( dx \wedge dy \), etc, is important. Thus, in the above notations, the flux integral of \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \),
is written as

\[ \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \iint_S (P \, dy \wedge dz + Q \, dz \wedge dx + P \, dx \wedge dy). \]

50.2.9 Examples:

1. Let us find the flux of \( \mathbf{F}(x,y,z) = xi + yj + zk \), outward across \( S \), the portion of the cone \( z = 1 - x^2 - y^2 \), that lies above the \( xy \)-plane. The surface \( S \) is given by \( G(x,y,z) = z + x^2 + y^2 - 1 = 0 \). Thus, the normal vector is

![Figure: Cone above the \( xy \)-plane](image-url)
\[ \pm \nabla G = (2x \mathbf{i} + 2y \mathbf{j} + \mathbf{k}) \]

Note that for the outward normal, the \( z \) component is always positive. So, we choose \( \nabla G = 2x \mathbf{i} + 2y \mathbf{j} + \mathbf{k} \) for \( S \). Hence,

flux across \( S \) is
\[
\int_R (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \cdot (2x \mathbf{i} + 2y \mathbf{j} + \mathbf{k}) \, dx \, dy
= \int_R \left( x^2 + y^2 + 1 \right) \, dx \, dy
= \int_0^1 \int_0^{2\pi} (1 + r^2) \, r \, dr \, d\theta
= \frac{3\pi}{2}
\]

2. Let us compute the flux of the vector field
\[ \mathbf{F}(x, y, z) = 3z^2 \mathbf{i} + 6 \mathbf{j} + 6xz \mathbf{k} \]
across parabolic cylinder \( S \) given by
\[ y = x^2, \ 0 \leq x \leq 2, \ 0 \leq z \leq 3 \]

We parameterize the surface as
\[ \mathbf{r}(x, z) = x \mathbf{i} + x^2 \mathbf{j} + z \mathbf{k}, \ (x, z) \in D = [0, 2] \times [0, 3] \]

Then,
\[ \mathbf{r}_x \times \mathbf{r}_y = (\mathbf{i} + 2x \mathbf{j}) \times (\mathbf{k}) = 2x \mathbf{i} - \mathbf{j} \]

Thus the positive oriented normal is
\[ \mathbf{n} = \frac{2x \mathbf{i} - \mathbf{j}}{\sqrt{5}} \]

The flux integral along this orientation is
Let us evaluate the same flux integral using the formula
\[ \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \iint_D \left( (P \cos \alpha) \, dS + (Q \cos \beta) \, dS + (R \cos \gamma) \, dS \right) \]
In this case,
\[ \mathbf{n} = \frac{1}{\sqrt{5}} (2 \mathbf{i} - \mathbf{j}) = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}. \]

Thus, \( \cos \alpha > 0 \), while \( \cos \beta < 0 \). Hence, the required flux integral along the positive orientation is
\[
\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \iint_D P \, dy \, dz - \iint_D 6 \, dz \, dx
= \iint_D 3 \, z^2 \, dy \, dz - \iint_D 6 \, dz \, dx
\]
Since, the surface \( S \) is \( \sqrt{y} \mathbf{i} + y \mathbf{j} + z \mathbf{k}, (y, z) \in D' \), where \( D' = \{(y, z) | 0 \leq y \leq 4, 0 \leq z \leq 3\} \),
we have
\[
\iint_D 3 \, z^2 \, dy \, dz = 3 \int_0^4 \left( \int_0^3 z^2 \, dz \right) \, dy = 3 \int_0^4 \frac{27}{3} \, dy = 108,
\]
and
\[
\iint_D 6 \, dz \, dx = 3 \int_0^3 6 \, dz \, dx = 6 \times 3 \times 2 = 36.
\]
Hence, the required flux is given by
\[ \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS = 108 - 36 = 72. \]

**Practice Exercises**

1. Evaluate the surface integral
\[ \iint_S \mathbf{F} \cdot d\mathbf{S}, \]
where \( S \) is the surface given by
\[ r(\varphi, \theta) = \cos \varphi \sin \theta \mathbf{i} + \sin \varphi \mathbf{j} + \cos \varphi \mathbf{k}, 0 \leq \varphi \leq 2\pi, 0 \leq \theta \leq \pi, \]
and
\[ \mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \]

**Answer:** $-4\pi$

2. Compute the flux of the vector field

\[ \mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \]

across the surface \( S \) that is the portion of the paraboloid

\[ z = 4 - x^2 - y^2, \]

lying above the \( xy \)-plane, oriented by the upward unit normal.

**Answer:** $24\pi$

3. Show that the flux of the universe square vector field

\[ \mathbf{F}(x,y,z) = \frac{\mathbf{r}}{||\mathbf{r}||^3}, \mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \]

across the sphere \( R \)

\[ x^2 + y^2 + z^2 = 4 \]

towards the outward unit normal is given by \( 4\pi \)

4. Find the coordinates of the center of mass of the surface out from the cylinder

\[ y^2 + z^2 = 9, z \geq 0, \] by the planes \( x = 0 \) and \( x = 3. \)

**Answer:**

\[ \bar{x} = \frac{3}{2}, \bar{y} = 0, \bar{z} = \frac{5}{\pi} \]

5. Let \( \mathbf{F} \) be a vector field such that \( \mathbf{F} \cdot \mathbf{r} = 1 \) for all \( (x,y,z) \) on the unit sphere \( S \).

Show that

\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = 2\pi^2 \]

**Recap**

In this section you have learnt the following

- Some application of the surface integrals.