Module 4

Signal Representation and Baseband Processing
Lesson 15

Orthogonality

Version 2 ECE IIT, Kharagpur
After reading this lesson, you will learn about

- Basic concept of orthogonality and orthonormality;
- Strum - Lion;
- Slope overload distortion;
- Granular Noise;
- Condition for avoiding slope overloading;

The Issue of Orthogonality

Let $f_m(x)$ and $f_n(x)$ be two real valued functions defined over the interval $a \leq x \leq b$. If the product $[f_m(x) \times f_n(x)]$ exists over the interval, the two functions are called orthogonal to each other in the interval $a \leq x \leq b$ when the following condition holds:

$$\int_a^b f_m(x) f_n(x) dx = 0 , \quad m \neq n \quad 4.15.1$$

A set of real valued functions $f_1(x), f_2(x) \ldots f_N(x)$ is called an orthogonal set over an interval $a \leq x \leq b$ if

(i) all the functions exist in that interval and

(ii) all distinct pairs of the functions are orthogonal to each other over the interval, i.e.

$$\int_a^b f_i(x) f_j(x) dx = 0 \quad , \quad i = 1, 2, \ldots; \quad j = 1, 2, \ldots \text{ and } i \neq j \quad 4.15.2$$

The norm $\|f_m(x)\|$ of the function $f_m(x)$ is defined as,

$$\|f_m(x)\| = \sqrt{\int_a^b f_m^2(x) dx} \quad 4.15.3$$

An orthogonal set of functions $f_1(x), f_2(x) \ldots f_N(x)$ is called an orthonormal set if,

$$\int_a^b f_m(x). f_n(x) dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \quad 4.15.4$$

An orthonormal set can be obtained from a corresponding orthogonal set of functions by dividing each function by its norm. Now, let us consider a set of real functions $f_1(x), f_2(x) \ldots f_N(x)$ such that, for some non-negative weight function $w(x)$ over the interval $a \leq x \leq b$

$$\int_a^b f_m(x). f_n(x) . w(x) dx = 0 , \quad m \neq n \quad 4.15.5$$

Do $f_i$ -s form an orthogonal set? We say that the $f_i$-s form an orthogonal set with respect to the weight function $w(x)$ over the interval $a \leq x \leq b$ by defining the norm as,
\[ \| f_m(x) \| = \sqrt{\int_a^b f_m^2(x)w(x)dx} . \] 4.15.6

The set of \( f_i \)-s is orthonormal with respect to \( w(x) \) if the norm of each function is 1. The above extension of the idea of orthogonal set makes perfect sense. To see this, let

\[ g_m(x) = \sqrt{w(x)} f_m(x) , \text{ where } w(x) \text{ is a non-negative function.} \] 4.15.7

It is now easy to verify that,

\[ \int_a^b f_m(x) f_n(x) w(x) dx = \int_a^b g_m(x) g_n(x) dx = 0 . \] 4.15.8

This implies that if we have orthogonal \( f_i \)-s over \( a \leq x \leq b \), with respect to a non-negative weight function \( w(x) \), then we can form an usual orthogonal set of \( f_i \)-s over the same interval \( a \leq x \leq b \) by using the substitution,

\[ g_m = \sqrt{w(x)} f_m(x) \]

Alternatively, an orthogonal set of \( g_i \)-s can be used to get an orthogonal set of \( f_i \)-s with respect to a specific non-negative weight function \( w(x) \) over \( a \leq x \leq b \) by the following substitution (provided \( \sqrt{w(x)} \neq 0, a \leq x \leq b \)):

\[ f_m(x) = \frac{g_m(x)}{\sqrt{w(x)}} . \] 4.15.9

A real orthogonal set can be generated by using the concepts of Strum-Liouville (S-L) equation. The S-L problem is a boundary value problem in the form of a second order differential equation with boundary conditions. The differential equation is of the following form:

\[ \frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] + \left[ q(x) + \lambda \omega(x) \right] y = 0 , \text{ for } a \leq x \leq b; \] 4.15.10

It satisfies the following boundary conditions:

i) \( c_2 \frac{dy}{dx} + c_1 y = 0 \) ; at \( x = a \);

ii) \( d_2 \frac{dy}{dx} + d_1 y = 0 \) ; at \( x = b \);

Here \( c_1, c_2, d_1 \) and \( d_2 \) are real constants such that at least one of \( c_1 \) and \( c_2 \) is non zero and at least one of \( d_1 \) and \( d_2 \) is non zero.

The solution \( y = 0 \) is a trivial solution. All other solutions of the above equation subject to specific boundary conditions are known as characteristic functions or eigenfunctions of the S-L problem. The values of the parameter ‘\( \lambda \)’ for the non trivial solutions are known as characteristic values or eigen values. A very important property of the eigen-functions is that they are orthogonal.
Orthogonality Theorem:

Let the functions $p(x)$, $q(x)$ and $\omega(x)$ in the S-L equation (4.15.10) be real valued and continuous in the interval $a \leq x \leq b$. Let $y_m(x)$ and $y_n(x)$ be eigen functions of the S-L problem corresponding to distinct eigenvalues $\lambda_m$ and $\lambda_n$ respectively. Then, $y_m(x)$ and $y_n(x)$ are orthogonal over $a \leq x \leq b$ with respect to the weight function $w(x)$.

Further, if $p(x = a) = 0$, then the boundary condition (i) may be omitted and if $p(x = b) = 0$, then boundary condition (ii) may be omitted from the problem. If $p(x = a) = p(x = b)$, then the boundary condition can be simplified as,

$$y'(a) = y'(b) \quad \text{and} \quad \frac{dy}{dx}igg|_{x=a} = y'(a) = y'(b) = \frac{dy}{dx}igg|_{x=b}$$

Another useful feature is that, the eigenvalues in the S-L problem, which in general may be complex based on the forms of $p(x)$, $q(x)$ and $w(x)$, are real valued when the weight function $\omega(x)$ is positive in the interval $a \leq x \leq b$ or always negative in the interval $a \leq x \leq b$.

Examples of orthogonal sets:

**Ex#1:** We know that, for integer ‘$m$’ and ‘$n$’,

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad \text{E4.15.1}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad \text{E4.15.2}$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0 \quad \text{E4.15.3}$$

Let us consider equation E4.15.1 and rewrite it as:

$$\int_{-\frac{1}{2}f}^{\frac{1}{2}f} (\cos 2\pi mft). (\cos 2\pi nft) dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad \text{E4.15.4}$$

by substituting $x = 2\pi ft = \omega t$ and $dx = 2\pi f dt = \omega dt$

Note that the functions ‘$\cos mx$’ and ‘$\cos nx$’ are orthogonal over the range $2\pi$ of the independent variable $x$ and its integral multiple, i.e. $M \cdot 2\pi$, in general, where ‘$M$’ is an integer. This implies that equation (E4.15.4) is orthogonal in terms of the independent
variable ‘t’ over the fundamental range \(\frac{1}{f}\) and, in general, over \(M\frac{1}{f} = M T_0\), where ‘\(T_0\)’ indicates the fundamental time interval over which \(\cos 2\pi mf t\) and \(\cos 2\pi nf t\) are orthogonal to each other. Now ‘\(m\)’ and ‘\(n\)’ can have a minimum difference ‘1’ if

\[
\int_{-T_0}^{T_0} (\cos 2\pi mf t)(\cos 2\pi nf t) dt = 0 \quad \text{E4.15.5}
\]

i.e., \(mf - nf = f = \frac{1}{T_0}\)

So, if two cosine signals have a frequency difference ‘\(f\)’, then we may say,

\[
\int_{-T_0}^{T_0} \cos 2\pi (f_c + \frac{f}{2}) t \cos 2\pi (f_c - \frac{f}{2}) t dt = 0 \quad \text{E4.15.6}
\]

Re-writing equation (E4.15.6)

\[
\int_{-T_0/2}^{T_0/2} \cos 2\pi (f_c + \frac{f}{2}) t \cos 2\pi (f_c - \frac{f}{2}) t dt = 0 \quad \text{where, } T_0 = \frac{1}{f} \quad \text{E4.15.7}
\]

Looking back at equation E4.15.5, we may write a general form for equation (E4.15.6):

\[
\int_{-T_0/2}^{T_0/2} \cos 2\pi (f_c + \frac{f}{2}) t \cos 2\pi (f_c - \frac{f}{2}) t dt = 0 \quad \text{where, } mf = (n+p)f \quad \text{and ‘p’ is an integer.}
\]

Following similar observations on equation E4.15.2, one can say,

\[
\int_{-T_0/2}^{T_0/2} \sin 2\pi (f_c + \frac{f}{2}) t \sin 2\pi (f_c - \frac{f}{2}) t dt = 0 \quad \text{E4.15.8}
\]

Equation E4.15.3 may also be expressed as,

\[
\int_{-T_0/2}^{T_0/2} \cos 2\pi (f_c + \frac{f}{2}) t \sin 2\pi (f_c - \frac{f}{2}) t dt
\]

\[
= \int_{-T_0/2}^{T_0/2} \sin 2\pi (f_c + \frac{f}{2}) t \cos 2\pi (f_c - \frac{f}{2}) t dt = 0 \quad \text{E4.15.9}
\]

Let us define \(s_1 = \cos 2\pi \left( f_c + \frac{f}{2} \right) t \), \(s_2 = \cos 2\pi \left( f_c - \frac{f}{2} \right) t \), \(s_3 = \sin 2\pi \left( f_c + \frac{f}{2} \right) t \) and \(s_4 = \sin 2\pi \left( f_c - \frac{f}{2} \right) t \). Can we use the above observations on orthogonality to distinguish among ‘\(s_i\)’s’ over a decision interval of \(T_5 = T_0 = \frac{1}{f}\)?

**Ex#2:** \(x_1(t) = 1.0 \text{ for } 0 \leq t \leq T/2 \text{ and zero elsewhere,} \)

\(x_2(t) = 1.0 \text{ for } T/2 \leq t \leq T \text{ and zero elsewhere,} \)

Version 2 ECE IIT, Kharagpur
Ex#3: \[ x_1(t) = 1.0 \quad \text{for} \quad 0 \leq t \leq T/2 \quad \text{and} \quad x_1(t) = -1.0 \quad \text{for} \quad T/2 < t \leq T, \quad \text{while} \]
\[ x_2(t) = -1.0 \quad \text{for} \quad 0 \leq t \leq T \]

Importance of the concepts of Orthogonality in Digital Communications

a. In the design and selection of information bearing pulses, orthogonality over a symbol duration may be used to advantage for deriving efficient symbol-by-symbol demodulation scheme.

b. Performance analysis of several modulation demodulation schemes can be carried out if the information-bearing signal waveforms are known to be orthogonal to each other.

c. The concepts of orthogonality can be used to advantage in the design and selection of single and multiple carriers for modulation, transmission and reception.

Orthogonality in a complex domain

Let, \( z_1(t) = x_1(t) + jy_1(t) \) and \( z_2(t) = x_2(t) + jy_2(t) \)

Now, \( x_1(t) = \frac{z_1(t) + z_1^*(t)}{2} \) and \( x_2(t) = \frac{z_2(t) + z_2^*(t)}{2} \)

If \( x_1 \) and \( x_2 \) are orthogonal to each other over \( a \leq t \leq b \),
\[
\int_{a}^{b} x_1(t).x_2(t) \, dt = 0
\]

i.e.,
\[
\int_{a}^{b} [z_1(t) + z_1^*(t)][z_2(t) + z_2^*(t)] \, dt = 0
\]

or,
\[
\int_{a}^{b} [z_1(t).z_2(t) + z_1(t).z_2^*(t) + z_1^*(t).z_2(t) + z_1^*(t).z_2^*(t)] \, dt = 0
\]

Let us consider a complex function
\[ z_1(t) = x(t) + jy(t), \quad a \leq t \leq b \]
\[ = r(t) \left[ \cos \Phi(t) + j \sin \Phi(t) \right] \]

where, \( r(t) = |z(t)| \), a non-negative function of ‘t’.

\[ \therefore x(t) = r(t) \cos \Phi(t) \quad \text{and} \quad y(t) = r(t) \sin \Phi(t) \]

Now, \[
\int_{a}^{b} x(t).y(t) \, dt = \int_{a}^{b} r^2(t).\cos \Phi(t).\sin \Phi(t) \, dt
\]

We know that \( \cos \theta \) & \( \sin \theta \) are orthogonal to each other over \(-\pi \leq \theta < \pi\), i.e.,
\[ \int_{-\pi}^{\pi} \cos \theta \sin \theta d\theta = 0 \]

So, using a constant weight function \( w = r \), which is non-negative, we may say
\[ \int_{-\pi}^{\pi} r^2 \cos \theta \sin \theta d\theta = 0 \]

Now, \( x = r \cos \theta \) and \( y = r \sin \theta \) are also orthogonal over \( -\pi \leq \theta < \pi \).

Now, let \( \theta \) be a continuous function of \( t \) over \( -\pi \leq \theta < \pi \). And,
\[
\theta \bigg|_{t=a} = \theta_a = -\pi \quad \text{and} \quad \theta \bigg|_{t=b} = \theta_b = \pi
\]

Assuming a linear relationship, let, \( \theta(t) = 2\pi ft \)
\[
\therefore d\theta(t) = 2\pi f dt
\]

Under these conditions, we see,
\[
\int_{a}^{b} r^2(t) \cos \Phi(t) \sin \Phi(t) dt = \frac{1}{2\pi f} \int_{-\pi}^{\pi} r^2(t) \cos \Phi(t) \sin \Phi(t) d\Phi
\]
\[
= \frac{1}{2\pi f} \int_{-\pi}^{\pi} r^2 \cos \Phi(t) \sin \Phi(t) d\Phi = 0
\]

i.e., \( x(t) \) and \( y(t) \) are orthogonal over the interval \( -\frac{1}{2f} \leq t \leq \frac{1}{2f} \) or \( \frac{T}{2} \leq t \leq \frac{T}{2} \)

So, if \( \tilde{z}(t) = x(t) + jy(t) \) represents a phasor in the complex plane rotating at a
uniform frequency of \( f \), then \( x(t) \) and \( y(t) \) are orthogonal to each other over the interval
\( -\frac{T}{2} \leq t \leq \frac{T}{2} \) or, equivalently \( \frac{T}{2} \leq t \leq \frac{T}{2} \) where \( T = \frac{1}{f} \), i.e., \( \int_{-T/2}^{T/2} x(t) y(t) dt = 0 \)

Now, let us consider two complex functions:
\[ \tilde{z}_1(t) = x_1(t) + jy_1(t) = \left| \tilde{z}_1(t) \right| e^{j\phi_1(t)} \]
and \[ \tilde{z}_2(t) = x_2(t) + jy_2(t) = \left| \tilde{z}_2(t) \right| e^{j\phi_2(t)} \]

\( [x_1(t), y_1(t)] \) and \( [x_2(t), y_2(t)] \) are orthogonal pairs over the interval \( -\frac{T}{2} \leq t \leq \frac{T}{2} \). So, \( \tilde{z}_1(t) \)
and \( \tilde{z}_2(t) \) may be viewed as two phasors rotating with equal speed.

Now, two static phasors are orthogonal to each other if their dot or scalar product is zero,
\( i.e., \left| A \right| \left| B \right| \cos \gamma = A_x B_x + A_y B_y = 0 \), where \( \gamma \) is the angle between \( \vec{A} \) and \( \vec{B} \)

In general, two complex functions \( \tilde{z}_1(t) \) and \( \tilde{z}_2(t) \) with finite energy are said to be
orthogonal to each other over an interval \( a \leq t \leq b \), if
\[
\int_{a}^{b} z_1(t) z_2^*(t) dt = 0
\]

**Problems**

Q4.15.1) Verify whether two signals are orthogonal over one time period of the signal with smallest frequency signal.

i) \( X_1(t) = \cos 2\pi ft \) and \( X_2(t) = \sin 2\pi ft \)

ii) \( X_1(t) = \cos 2\pi ft \) and \( X_2(t) = \cos (2\pi ft + \frac{\pi}{3}) \)

iii) \( X_1(t) = \cos 2\pi ft \) and \( X_2(t) = \cos (4\pi ft + \frac{\pi}{4}) \)

iv) \( X_1(t) = \sin 4\pi ft \) and \( X_2(t) = -\cos (\pi ft - \frac{\pi}{6}) \)