Module 5: Plane Waves at Media Interface
Lecture 32: Plane Wave in Arbitrary Direction

Objectives
In this course you will learn the following

- Wave Vector at Arbitrary Direction.
- Electric & Magnetic fields for a wave moving in direction $\hat{n}$.
- Phase Velocity and Wave length.
Wave Vector at Arbitrary Direction

Let the wave be moving in direction making angles $\phi_x$, $\phi_y$, $\phi_z$ respectively with three axis $x, y, z$ as shown in fig.

The unit vector in the direction of the wave propagation is

$$\hat{n} = \cos \phi_x \mathbf{\hat{x}} + \cos \phi_y \mathbf{\hat{y}} + \cos \phi_z \mathbf{\hat{z}}$$

where $\cos \phi_x$, $\cos \phi_y$, $\cos \phi_z$ are called the direction cosines of the vector $\hat{n}$.

The equation of a constant phase plane (the phase front) is given as

$$\hat{n} \cdot \mathbf{OP} = \hat{n} \cdot \mathbf{r} = \text{constant}$$

Therefore, the phase of this constant phase plane is

$$\beta |OA| = \beta \hat{n} \cdot \mathbf{r}$$
Electric & Magnetic fields for a wave moving in direction \( \hat{n} \)

The electric field of a plane wave travelling in direction \( \hat{n} \) can then be written as

\[
E = E_0 e^{-j\hat{n} \cdot r}
\]

Where \( E_0 \) is a vector perpendicular to the unit vector \( \hat{n} \).

Let us define the wave vector as

\[
k = \beta \hat{n} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}
\]

The electric field then is

\[
\vec{E} = E_0 e^{-j\vec{k} \cdot r} = E_0 e^{-j(k_x x + k_y y + k_z z)}
\]

We therefore get

\[
\frac{\partial}{\partial x} = -jk_x \\
\frac{\partial}{\partial y} = -jk_y \\
\frac{\partial}{\partial z} = -jk_z
\]

The magnetic field is then obtained as

\[
\vec{H} = -\frac{1}{j\omega \mu} \nabla \times \vec{E} \\
= -\frac{1}{j\omega \mu} \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
-jk_x & -jk_y & -jk_z \\
E_x & E_y & E_z 
\end{vmatrix} \\
= -\frac{1}{j\omega \mu} \{-jk \times E\} = \frac{1}{\omega \mu} k \times E
\]

\[
= \frac{(\hat{n} \times \vec{E}_0) e^{-j\hat{k} \cdot r}}{\eta} = \vec{H}_0 e^{-j\hat{k} \cdot r}
\]

The \( \vec{E}_0, \vec{H}_0 \) and \( \hat{n} \) vectors are perpendicular to each other and

\[
\left| \frac{\vec{E}_0}{\vec{H}_0} \right| = \text{ Electric impedance of the medium } \eta
\]
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Phase Velocity and Wave length

The electric field of a uniform plane wave travelling in a direction which makes angles \( \phi_x \), \( \phi_y \), and \( \phi_z \) with three axis \( x \), \( y \) and \( z \) respectively is written as

\[
E = E_0 e^{-j\mathbf{k}\cdot\mathbf{r}}
\]

\[
= E_0 e^{-j\beta x \cos \phi_x} e^{-j\beta y \cos \phi_y} e^{-j\beta z \cos \phi_z}
\]

Separating out the \( z \) variation, we can write the electric field as

\[
E = E_0 e^{-j(\beta x \cos \phi_x + y \cos \phi_y)} e^{-j\beta z \cos \phi_z}
\]

**NOTE:**

In the \( xy \) plane (plane perpendicular to \( z \)-direction) the phase is not constant. So \( xy \)-plane is not a constant phase plane.

The phase constant along \( z \)-direction is \( k_z = \beta \cos \phi_z \).

The phase velocity in the \( z \)-direction therefore is

\[
\nu_{pz} = \frac{\omega}{k_z} = \frac{\omega}{\beta \cos \phi_z} = \frac{\nu_0}{\cos \phi_z}
\]

Similarly we can get the phase velocities along the \( x \) and \( y \) directions as

\[
\nu_{px} = \frac{\omega}{k_x} = \frac{\omega}{\beta \cos \phi_x} = \frac{\nu_0}{\cos \phi_x}
\]

\[
\nu_{py} = \frac{\omega}{k_y} = \frac{\omega}{\beta \cos \phi_y} = \frac{\nu_0}{\cos \phi_y}
\]

Since \( |\cos \phi_x|, |\cos \phi_y|, |\cos \phi_z| \leq 1 \), the velocities \( \nu_{px}, \nu_{py}, \nu_{pz} \) are always greater than or equal to \( \nu_0 \). In fact when any of the angles \( \phi_x, \phi_y, \phi_z \to \pi/2 \), the cosines of these angles tend to 0 and the corresponding velocities approach infinity. The bounds of the phase velocity therefore are

\[
\nu_0 \leq \nu_{px}, \nu_{py}, \nu_{pz} \leq \infty
\]

The wavelength of the wave in \( x, y, z \) directions respectively are

\[
\lambda_x = \frac{\nu_{px}}{f} = \frac{\lambda_0}{\cos \phi_x}
\]

\[
\lambda_y = \frac{\nu_{py}}{f} = \frac{\lambda_0}{\cos \phi_y}
\]

\[
\lambda_z = \frac{\nu_{pz}}{f} = \frac{\lambda_0}{\cos \phi_z}
\]

Where \( \lambda_0 = \nu_0/f \)

**Interesting observation**

If we consider the unbound medium as the free-space, the phase velocity of the wave is \( \nu_0 = c \) (velocity of light in vacuum), we get

\[
c \leq \nu_{px}, \nu_{py}, \nu_{pz} \leq \infty
\]
Recap

In this course you have learnt the following

- Wave Vector at Arbitrary Direction.
- Electric & Magnetic fields for a wave moving in direction $\hat{n}$.
- Phase Velocity and Wave length.