STiCM Lecture 02: Unit 1 Equations of Motion (i)
Unit 1: Equations of Motion


Lagrangian/Hamiltonian formulation. Application to SHO.
Learning goals:

‘Mechanical system’ is described by (position, velocity) or (position, momentum) or some well defined function thereof.

How is an ‘inertial frame of reference’ identified?

What is the meaning of ‘equilibrium’?

What causes departure from equilibrium?

*Can we ‘derive’ I law from the II by putting* 

\[ F = 0 \text{ in } F = ma. \]
....... Learning goals:

Learn about the ‘principle of variation’ and how it provides us an alternative and more powerful approach to solve mechanical problems.

Introduction to Lagrangian and Hamiltonian methods which illustrate this relationship and apply the technique to solve the problem of the simple harmonic oscillator.
Central problem in ‘Mechanics’: How is the ‘mechanical state’ of a system described, and how does this ‘state’ evolve with time? Formulations due to Galileo/Newton, Lagrange and Hamilton.

Why ‘position’ and ‘velocity’ are both needed to specify the mechanical state of a system?

They are independent parameters that specify the ‘state’.

The mechanical state of a system is characterized by its position and velocity, \((q, \dot{q})\) or, position and momentum, \((q, p)\)

Or, equivalently by their well-defined functions:

\[ L(q, \dot{q}): \text{Lagrangian} \]
\[ H(q, p): \text{Hamiltonian} \]
A two-dimensional space spanned by the two orthogonal (thus independent) degrees of freedom.

The physical dimensions of a piece of area in this space will not be $L^2$.

Instead, the dimension of area in such a space would be $L^2T^{-1}$. 
position-momentum phase space

Dimensions?

Depends on [q], [p]  Accordingly, dimensions of the position-velocity phase will also not always be $L^2 T^{-1}$

[area] : angular momentum
The mechanical state of a system is described by a point in phase space.

Coordinates: \((q, \dot{q})\) or \((q, p)\)

Evolution: \((\dot{q}, \ddot{q})\) or \((\dot{q}, \dot{p})\)

What is meant by ‘Equation of Motion’?

- Rigorous mathematical relationship between position, velocity and acceleration.
Galileo’s experiments that led him to the law of inertia.

Galileo Galilei
1564 - 1642
What is ‘equilibrium’?
What causes departure from ‘equilibrium’?

Galileo Galilei
1564 - 1642

Isaac Newton
(1642-1727)

I Law

II Law

Causality & Determinism

\[ \vec{F} = ma \quad \text{Effect is proportional to the Cause.} \]

Linear Response.

Principle of causality.

PCD, STICM
\[ \vec{F} = m\vec{a} \quad \text{Effect is proportional to the Cause.} \]

Linear Response. Principle of causality.

Now, we already need calculus!
From Carl Sagan’s ‘Cosmos’:

“Like Kepler, he (Newton) was not immune to the superstitions of his day… in 1663, at his age of twenty, he purchased a book on astrology, …. he read it until he came to an illustration which he could not understand, because he was ignorant of trigonometry. So he purchased a book on trigonometry but soon found himself unable to follow the geometrical arguments, So he found a copy of Euclid’s *Elements of Geometry*, and began to read. Two years later he invented the differential calculus.”
Differential of a function. Derivative of a function Slope of the curve.

It is a quantitative measure of how sensitively the function $f(x)$ responds to changes in the independent variable $x$.

The sensitivity may change from point to point and hence the derivative of a function must be determined at each value of $x$ in the domain of $x$. 
$$\left[ \frac{df}{dx} \right]_{x_0} = \lim_{\delta x \to 0} \left[ \frac{\delta f}{\delta x} \right]_{x_0} = \lim_{\delta x \to 0} \frac{f(x_0 + \frac{\delta x}{2}) - f(x_0 - \frac{\delta x}{2})}{\delta x}$$

Tangent to the curve.
Dimensions of the derivative of the function: $[f][x]^{-1}$
Functions of many variables: eg. height above a flat horizontal surface of a handkerchief that is stretched out in a warped surface, ……

Or, the temperature on a flat surface that has various heat sources spread under it – such as a tiny hot filament here, a hotter there, a tiny ice cube here, a tiny beaker of liquid helium there, etc.

One must then define ‘Partial derivatives’ of a function of more than one variable.
\[
\left[ \frac{\partial h}{\partial x} \right]_{(x_0, y_0)} = \lim_{\delta x \to 0} \frac{h(x_0 + \frac{\delta x}{2}, y_0) - h(x_0 - \frac{\delta x}{2}, y_0)}{\delta x} = \lim_{\delta x \to 0} \left[ \frac{\delta h}{\delta x} \right]_{y_0}
\]

\[
\left[ \frac{\partial h}{\partial y} \right]_{(x_0, y_0)} = \lim_{\delta y \to 0} \frac{h(x_0, y_0 + \frac{\delta y}{2}) - h(x_0, y_0 - \frac{\delta y}{2})}{\delta y} = \lim_{\delta y \to 0} \left[ \frac{\delta h}{\delta y} \right]_{x_0}
\]

**Partial derivatives of a function of more than one variable.**

**h**: dependent variable

**x & y**: independent variables

**PCD - STiCM**
\[ \delta t = t_2 - t_1 \]

Time-Derivatives of position & velocity.

\[ \vec{v} = \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{r}(t_2) - \vec{r}(t_1)}{\delta t} \]

\[ a = \lim_{\delta t \to 0} \frac{\delta \vec{v}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{v}(t_2) - \vec{v}(t_1)}{\delta t} \]

‘Equation of Motion’
Rigorous relationship between position, velocity and acceleration.

PCD_STiCM
What is ‘equilibrium’?
Relative to whom?
- frame of reference

Equilibrium means ‘state of rest’, or of uniform motion along a straight line.

Equilibrium sustains itself, needs no cause; determined entirely by initial conditions.

Effect $\vec{a}$ is proportional to the Cause $\vec{F}$.
Proportionality: Mass/Inertia.
Linear Response.
Principle of causality/determinism.

Galileo; Newton

$\vec{F} = ma$
Force: Physical agency that changes the state of equilibrium of the object on which it acts.

\[
m \frac{d^2 \vec{r}}{dt^2} = m \frac{d \vec{v}}{dt} = m \vec{a} = \vec{F} = \frac{d(m \vec{v})}{dt} = \frac{dp}{dt}.
\]

\[
t \rightarrow -t \quad \frac{d}{dt} \rightarrow \left(-\frac{d}{dt}\right) \quad \vec{r} \rightarrow -\vec{v} \text{ and } \vec{v} \rightarrow \left(-\frac{F}{m}\right)
\]

Direction of velocity and acceleration both reverse.

System’s trajectory would be only reversed along essentially the same path.

Newton’s laws are therefore symmetric under time-reversal.

Did we derive Newton’s laws from anything?

We got the first law from Galileo’s brilliant experiments.

We got the second law from Newton’s explanation of departure from equilibrium in terms of the linear-response cause-effect relationship.

No violation of these predictions was ever found.

- especially elevated status: ‘laws of nature’;
- universal in character since no mechanical phenomenon seemed to be in its range of applicability.
We have introduced the first two laws of Newton as fundamental principles. Newton’s III law makes a qualitative and quantitative statement about each pair of interacting objects, which exert a mechanical force on each other.

\[ \vec{F}_{12} = -\vec{F}_{21} \]

In Newtonian scheme of mechanics, this is introduced as a ‘fundamental’ principle –i.e., as a law of nature.
Newton’s III law:
‘Action and Reaction are Equal and Opposite’

\[ \vec{F}_{12} = -\vec{F}_{21} \]

\[ \frac{dp_1}{dt} = - \frac{dp_2}{dt} \]

\[ \frac{d}{dt} (p_1 + p_2) = 0 \]

*Newton's III Law as statement of conservation of linear momentum*

We have obtained a conservation principle from ‘law of nature’
Are the conservation principles consequences of the laws of nature? Or, are the laws of nature the consequences of the symmetry principles that govern them?

Until Einstein's special theory of relativity, it was believed that conservation principles are the result of the laws of nature.

Since Einstein's work, however, physicists began to analyze the conservation principles as consequences of certain underlying symmetry considerations, enabling the laws of nature to be revealed from this analysis.
Instead of introducing Newton’s III law as a fundamental principle, we shall now deduce it from symmetry / invariance.

This approach places SYMMETRY ahead of LAWS OF NATURE.

It is this approach that is of greatest value to contemporary physics. This approach has its origins in the works of Albert Einstein, Emmily Noether and Eugene Wigner.
Next Class:
STiCM Lecture 03: Unit 1 Equations of Motion (ii)
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Select / Special Topics in Classical Mechanics

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STiCM Lecture 03: Unit 1 Equations of Motion (ii)
Newton’s III law:
‘Action and Reaction are Equal and Opposite’

We have obtained a conservation principle from ‘law of nature’

\[ \vec{F}_{12} = -\vec{F}_{21} \]
\[ \frac{d \vec{p}_1}{dt} = -\frac{d \vec{p}_2}{dt} \]
\[ \frac{d}{dt} (\vec{p}_1 + \vec{p}_2) = 0 \]

Newton's III Law as a statement of conservation of linear momentum
Are the conservation principles consequences of the laws of nature?
Or, are the laws of nature the consequences of the symmetry principles that govern them?

Until Einstein's special theory of relativity, it was believed that conservation principles are the result of the laws of nature.

Since Einstein's work, however, physicists began to analyze the conservation principles as consequences of certain underlying symmetry considerations, enabling the laws of nature to be revealed from this analysis.
Before we proceed, we remind ourselves of another illustration of the connection between symmetry and conservation law.

Examine the ANGULAR MOMENTUM
\[ \vec{l} = \vec{r} \times \vec{p} \]
of a system subjected to a central force.

\[ \vec{\tau} = \frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{r} \times \vec{F}, \]

since \[ \frac{d\vec{r}}{dt} \times \vec{p} = \vec{0} \] and \[ \frac{d\vec{p}}{dt} = \vec{F}. \]

\[ \vec{F} = \left| \vec{F} \right| \hat{e}_r \]

\[ \Rightarrow \vec{\tau} = \vec{0} \]

\[ \vec{l}: \text{constant} \]

SYMMETRY

CONSERVED QUANTITY
The connection between ‘symmetry’ and ‘conservation law’ is so intimate, that we can actually derive Newton’s III law using ‘symmetry’.

*Begin with a symmetry principle:*

*translational invariance in homogenous space.*

Consider a system of N particles in a medium that is homogenous.

A displacement of the entire N-particle system through $\delta S$ in this medium would result in a new configuration that would find itself in an environment that is completely indistinguishable from the previous one.

*This invariance of the environment of the entire N particle system following a translational displacement is a result of translational symmetry in homogenous space.*
FOUR essential considerations:

1. Each particle of the system is under the influence *only* of all the remaining particles; the system is isolated: no external forces act on any of its particles.

2. The entire N-particle system is deemed to have undergone simultaneous identical displacement; all inter-particle separations and relative orientations remain invariant.

3. Entire medium: essentially homogeneous;
   - spanning the entire system both before and after the displacement.

4. The displacement the entire system under consideration is deemed to take place at a certain **instant of time**.
The implication of these FOUR considerations:

“Displacement considered is only a *virtual displacement*.”

- only a mental thought process;

- real physical displacements would require a certain time interval over which the displacements would occur

- virtual displacements can be thought of to occur *at an instant of time* and subject to the specific four features mentioned.
Displacement being virtual, it is redundant to ask “what agency has caused it”
Now, the internal forces do **no work** in this virtual displacement.

Therefore ‘work done’ by the internal forces in the ‘virtual displacement’ must be zero.

This work done (**rather not done**) is called ‘virtual work’.

\[ 0 = \delta W = \left\{ \sum_{k=1}^{N} \vec{F}_k \right\} \bullet \vec{\delta s} = \left\{ \sum_{k=1}^{N} \sum_{i=1, i \neq k}^{N} \vec{F}_{ik} \right\} \bullet \vec{\delta s}, \]

where \( \vec{F}_{ik} \): force on \( k^{th} \) particle by the \( i^{th} \),

and

\( \vec{F}_k \) force on the \( k^{th} \) particle due to the **remaining** \( N - 1 \) particles.
0 = \delta W = \left\{ \sum_{k=1}^{N} \vec{F}_k \right\} \cdot \delta \vec{s} = \left\{ \sum_{k=1}^{N} \sum_{i=1, i \neq k}^{N} \vec{F}_{ik} \right\} \cdot \delta \vec{s}

The mathematical techniques: Jean le Rond d'Alembert
(1717 – 1783)

Under what conditions can the above relation hold for an ARBITRARY displacement \( \delta \vec{s} \)?

\[
\vec{0} = \left\{ \sum_{k=1}^{N} \vec{F}_k \right\} = \sum_{k=1}^{N} \frac{d\vec{p}_k}{dt} = \frac{d}{dt} \sum_{k=1}^{N} \vec{p}_k = \frac{d\vec{P}}{dt}
\]

Newton's I,II laws used; not the III.
The relation
\[ \vec{0} = \left\{ \sum_{k=1}^{N} \vec{F}_k \right\} = \sum_{k=1}^{N} \frac{d\vec{p}_k}{dt} = \frac{d}{dt} \sum_{k=1}^{N} \vec{p}_k = \frac{d\vec{P}}{dt} \]
is obtained from the properties of **translational symmetry** in **homogenous space**.

**Amazing!**
- since it suggests a path to discover the laws of physics by exploiting the connection between symmetry and conservation laws!

For just two particles:
\[ \frac{d\vec{P}}{dt} = \vec{0}, \]
i.e., \[ \frac{d\vec{p}_2}{dt} = -\frac{d\vec{p}_1}{dt}, \]
which gives \( \vec{F}_{12} = -\vec{F}_{21} \), the **III law of Newton**.

PCD_STiCM
Newton’s III law need not be introduced as a fundamental principle/law; we deduced it from symmetry / invariance.

SYMMETRY placed ahead of LAWS OF NATURE.

Albert Einstein, Emmily Noether and Eugene Wigner.
Emilly Noether: Symmetry ↔ Conservation Laws

Eugene Wigner's profound impact on physics: symmetry considerations using `group theory' resulted in a change in the very perception of just what is most fundamental.

`symmetry': the most fundamental entity whose form would govern the physical laws.
The connection between SYMMETRY and CONSERVATION PRINCIPLES brought out in the previous example, becomes even more transparent in an alternative scheme of MECHANICS.

While Newtonian scheme rests on the principle of causality (effect is linearly proportional to the cause), this alternative principle does not invoke the notion of ‘force’ as the ‘cause’ that must be invoked to explain a system’s evolution.
This alternative principle also begins by the fact that the mechanical system is characterized by its position and velocity (or equivalently by position and momentum),

...... but with a very slight difference!
The mechanical system is determined by a well-defined function of position and velocity/momentum.

The functions that are employed are the Lagrangian $L(q, \dot{q})$ and the Hamiltonian $H(q, p)$.

The primary principle on which this alternative formulation rests is known as the *Principle of Variation*.
We shall now introduce the ‘Principle of Variation’, often referred to as the ‘Principle of Least (extremum) Action’.
The principle of least (extremum) action: *in its various incarnations applies to all of physics.*

- explains why things happen the way they do!

- Explains trajectories of mechanical systems subject to certain initial conditions.
The principle of least action has for its precursor what is known as Fermat's principle - explains why light takes the path it does when it meets a boundary of a medium.

Common knowledge: when a ray of light meets the edge of a medium, it usually does not travel along the direction of incidence - gets reflected and refracted.

Fermat's principle explains this by stating that light travels from one point to another along a path over which it would need the ‘least’ time.
Pierre de Fermat (1601(?)-1665) : French lawyer who pursued mathematics as an active hobby.

Best known for what has come to be known as Fermat's last theorem, namely that the equation $x^n+y^n=z^n$ has no non-zero integer solutions for $x,y$ and $z$ for any value of $n>2$.

"To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it."

-- Pierre de Fermat

It took about 350 years for this theorem to be proved (by Andrew J. Wiles, in 1993).
Actually, the time taken by light is not necessarily a minimum.

More correctly, the principle that we are talking about is stated in terms of an ‘extremum’,

and even more correctly as

‘The actual ray path between two points is the one for which the optical path length is stationary with respect to variations of the path’.

…………………. Usually it is a minimum:
Light travels along a path that takes the least time.
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Select / Special Topics in Classical Mechanics

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STiCM Lecture 04: Unit 1 Equations of Motion (iii)
We shall now introduce the ‘Principle of Variation’, often referred to as the ‘Principle of Least (extremum) Action’.
Hamilton's principle of least action thus has an interesting development, beginning with Fermat's principle about how light travels between two points, and rich contributions made by Pierre Louis Maupertuis (1698 -- 1759), Leonhard Paul Euler (1707 - 1783), and Lagrange himself.

**Hamilton’s principle**

‘principle of least (rather, extremum) action’
Hamilton’s principle

‘principle of least *(rather, extremum)* action’

Mechanical state of a system 'evolves' (along a 'world line') in such a way that 'action', \( S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \) is an extremum.

...and now, we need ‘action’,
- ‘integral’ of the ‘Lagrangian’!
Integral of a function of time.

**Definite integral of the function**

\[ \int_{t_1}^{t_2} f(t) \, dt \]

**Area under the curve**

Dimensions of this ‘area’: \[\left[ f(t) \right] \times T \]

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Path Integral. 'action', \( S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \)

Dependence of L on time is not explicit. It is implicit through dependence on position and velocity which depend on t.

\( 'action', \ S = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt \)

The system evolution cannot be shown on a two-dimensional surface.

The system then evolves along a path in the ‘phase space’.

The additive property of ‘action’ as area under the L vs. time curve remains applicable.

Thus, the dimensions of ‘action’ are equal to dimensions of the Lagrangian multiplied by \( T \).

We shall soon discover what L is!
Consider alternative paths along which the mechanical state of the system may evolve in the phase space:

$q(t_2), \dot{q}(t_2)$

$q(t_1), \dot{q}(t_1)$

$q$ changed by $\delta q$, and $\dot{q}$ changed by $\delta \dot{q}$.
Alternative paths:
$q$ changed by $\delta q$, and $\dot{q}$ changed by $\delta \dot{q}$

The mechanical system evolves in such a way that

'\textit{action}', $S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$ is an extremum

Note! ‘Force’, ‘Cause-Effect Relationship’ is NOT invoked!

$S$ would be an extremum when the variation in $S$ is zero;

i.e. $\delta S = 0$

$$\delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0$$
\[ \delta f = \frac{df}{dx} \delta x \]

\[ \delta h(x, y) = \frac{\partial h}{\partial x} \delta x + \frac{\partial h}{\partial y} \delta y \]
\[
\left[ \frac{\partial h}{\partial x} \right]_{(x_0, y_0)} = \lim_{\delta x \to 0} \frac{h(x_0 + \frac{\delta x}{2}, y_0) - h(x_0 - \frac{\delta x}{2}, y_0)}{\delta x} = \lim_{\delta x \to 0} \left[ \frac{\delta h}{\delta x} \right]_{y_0}
\]

\[
\left[ \frac{\partial h}{\partial y} \right]_{(x_0, y_0)} = \lim_{\delta y \to 0} \frac{h(x_0, y_0 + \frac{\delta y}{2}) - h(x_0, y_0 - \frac{\delta y}{2})}{\delta y} = \lim_{\delta y \to 0} \left[ \frac{\delta h}{\delta y} \right]_{x_0}
\]

Partial derivatives of a function of more than one variable.

h: dependent variable

x & y: independent variables
\[ \delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0 \]

i.e., \[ 0 = \delta S = \int_{t_1}^{t_2} \delta L(q, \dot{q}, t) dt \]

\[ 0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right\} dt \]

\[ 0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} (\delta q) \right\} dt \]

We need: Integration of product of two functions
differential and integral of a product of two functions.

\[
\frac{d}{dx} \{ f(x) g(x) \} = \left\{ \frac{df}{dx} \right\} g(x) + f(x) \left\{ \frac{dg}{dx} \right\}
\]

\[
\therefore \quad f(x) \left\{ \frac{dg}{dx} \right\} = \frac{d}{dx} \{ f(x) g(x) \} - \left\{ \frac{df}{dx} \right\} g(x)
\]
\[ f(x) \left\{ \frac{dg}{dx} \right\} = \frac{d}{dx} \{ f(x)g(x) \} - \left\{ \frac{df}{dx} \right\} g(x) \]

Integrating both sides:
\[
\int_{x_1}^{x_2} f(x) \left\{ \frac{dg}{dx} \right\} dx = \int \frac{d}{dx} \{ f(x)g(x) \} dx - \int \left\{ \frac{df}{dx} \right\} g(x) dx
\]

\[
\int_{x_1}^{x_2} f(x) \left\{ \frac{dg}{dx} \right\} dx = f(x_2)g(x_2) - f(x_1)g(x_1) - \int_{x_1}^{x_2} \left\{ \frac{df}{dx} \right\} g(x) dx
\]
\[
0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right\} dt = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} (\delta q) \right\} dt
\]

i.e. \[
0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} (\delta q) \right\} dt
\]

Integration of product of two functions

\[
\int_{x_1}^{x_2} f(x) \left\{ \frac{dg}{dx} \right\} dx = f(x) g(x) \mid_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{df}{dx} \right\} g(x) dx
\]

\[
0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt +
\]

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\[ 0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right\} dt = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \frac{d}{dt} (\delta q) \right\} dt \]

\[ i.e. \quad 0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right\} dt + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \frac{d}{dt} (\delta q) \right\} dt \]

Integration of product of two functions

\[ \int_{x_1}^{x_2} f(x) \left\{ \frac{dg}{dx} \right\} dx = f(x) g(x) \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{df}{dx} \right\} g(x) dx \]

\[ 0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right\} dt + \left[ \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} dt \]

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\( \delta q(t) \) at time \( t \)
\( t_1 < t < t_2 \)

\( q(t_2), \dot{q}(t_2) \)

\( \delta q(t_1) = \delta q(t_2) = 0 \)

\[
0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \left[ \frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} dt
\]
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STiCM Lecture 05: Unit 1 Equations of Motion (iv)
Mechanical state of a system 'evolves' (along a 'world line') in such a way that 'action', \( S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \) is an extremum.

\[
\delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0
\]

i.e., \( 0 = \delta S = \int_{t_1}^{t_2} \delta L(q, \dot{q}, t) dt \)

\[
0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \left[ \frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} dt
\]
$\delta q(t)$ at time $t$
$t_1 < t < t_2$

$q(t_2), \dot{q}(t_2)$

$\delta q(t_1) = \delta q(t_2) = 0$

$0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \left[ \frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} dt$
\[ 0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial \dot{q}} \delta q \right\} dt - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right\} \delta q dt \]

i.e. \[ 0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right\} \delta q dt \]

Hence, \[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0 \]

\textbf{Lagrange's Equation}

We have not, as yet, provided a recipe to construct the Lagrangian!

\( L = L(q, \dot{q}) \) is all the we know about it as yet!
\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0
\]

**Lagrange's Equation**

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}
\]

**Homogeneity & Isotropy of space**

\( L \) can only be quadratic function of the velocity.

\[
L(q, \dot{q}, t) = f_1(\dot{q}^2) + f_2(q)
\]

\[
L(q, \dot{q}, t) = \frac{m}{2} \dot{q}^2 - V(q)
\]

\[
= T - V
\]
Lagrange's Equation
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} \]

\[ L(q, \dot{q}, t) = \frac{m}{2} \dot{q}^2 - V(q) = T - V \]

\[ \frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q} = F, \text{ the force} \]
\[ \frac{\partial L}{\partial \dot{q}} = m\dot{q} = p, \text{ the momentum} \]

i.e., \[ \frac{dp}{dt} = F : \text{ in 3D: } \frac{d\vec{P}}{dt} = \vec{F} \]

Newton's II Law

\[ \text{Newton's II Law} \]
Interpretation of $L$ as $T-V$ gives equivalent correspondence with Newtonian formulation.

$L(q, \dot{q}, t) = \frac{m}{2} \dot{q}^2 - V(q) = T - V$

$p = \frac{\partial L}{\partial \dot{q}}$, generalized momentum

$\frac{\partial L}{\partial q} = - \frac{\partial V}{\partial q} = F$, the force

$\frac{\partial L}{\partial \dot{q}} = m\dot{q} = p$, the momentum
\[ L = L(q, \dot{q}, t) \]

\[
\frac{dL}{dt} = \left( \frac{\partial L}{\partial q} \right) \dot{q} + \left( \frac{\partial L}{\partial \dot{q}} \right) \ddot{q} + \frac{\partial L}{\partial t}
\]

\[
0 = \left( \frac{\partial L}{\partial q} \right) - \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right)
\]

\[
\left( \frac{\partial L}{\partial \dot{q}} \right) = \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right)
\]

\[
\frac{d}{dt} \left( \left\{ \frac{\partial L}{\partial \dot{q}} \right\} \dot{q} - L \right) = -\frac{\partial L}{\partial t}
\]

\[\text{What if } \frac{\partial L}{\partial t} = 0?\]
\[ \left\{ \frac{\partial L}{\partial \dot{q}} \right\} \dot{q} - L \text{ is CONSTANT} \leftrightarrow \frac{\partial L}{\partial t} = 0 \]

Hamiltonian's Principal Function

\[ H = \left[ \left\{ \frac{\partial L}{\partial \dot{q}} \right\} \dot{q} - L \right] = p\dot{q} - L \]

\[ H = m\dot{v}^2 - L \]

\[ = m\dot{v}^2 - \left( \frac{1}{2} m\dot{v}^2 - V \right) \]

\[ H = 2T - L = 2T - (T - V) = T + V \]

\textbf{TOTAL ENERGY}
When there are

\[ N \] degrees of freedom,

\[ H = \sum_{i=1}^{N} p_i \dot{q}_i - L \]
\[ p = \frac{\partial L}{\partial \dot{q}} \]

\( q \) : Generalized Coordinate

\( \dot{q} \) : Generalized Velocity

\( p \) : Generalized Momentum
Symmetry $\leftrightarrow$ Conservation Laws

(Noether)
Lagrangian of a closed system does not depend explicitly on time.

\[ \frac{\partial L}{\partial t} = 0 \]

\[ \dot{q} \frac{\partial L}{\partial \dot{q}} - L \]

is a CONSTANT

Hamiltonian: “ENERGY”

Conservation of Energy is thus connected with the symmetry principle regarding invariance with respect to temporal translations.

Hamiltonian / Hamilton’s Principal Function
In an inertial frame, Time: homogeneous;

Space is homogenous and isotropic

the condition for homogeneity of space: \( \delta L(x, y, z) = 0 \)

i.e.,

\[
\delta L = \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial z} \delta z = 0
\]

which implies

\[
\frac{\partial L}{\partial q} = 0
\]

where \( q = x, y, z \)

since

\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0,
\]

this means i.e.

\[
\frac{\partial L}{\partial \dot{q}} = p
\]

is conserved.

i.e., is independent of time, is a constant of motion
\[ \frac{\partial L}{\partial q} = 0 \]

since \[ \frac{\partial L}{\partial q} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \]

= 0, this means \textit{i.e.} \( \frac{\partial L}{\partial \dot{q}} = p \) is conserved.

\textit{i.e.,} \( p \) is independent of time, it is a constant of motion

\textbf{Law of conservation of momentum, arises from the homogeneity of space.}

\textbf{Symmetry} \iff \textbf{Conservation Laws}

\textbf{Momentum that is canonically conjugate to a cyclic coordinate is conserved.}
Hamiltonian (Hamilton’s Principal Function) of a system

Many degrees of freedom: \( H = \sum_k \left[ \dot{q}_k p_k - L(q_k, \dot{q}_k) \right] \)

\[
dH = \sum_k p_k \, dq_k + \sum_k \dot{q}_k \, dp_k - \sum_k \frac{\partial L}{\partial q_k} \, dq_k - \sum_k \frac{\partial L}{\partial \dot{q}_k} \, d\dot{q}_k
\]

\[
dH = \sum_k \dot{q}_k \, dp_k - \sum_k \frac{\partial L}{\partial q_k} \, dq_k
\]

\[
= \sum_k \dot{q}_k \, dp_k - \sum_k \dot{p}_k \, dq_k
\]
\[ dH = \sum_k \dot{q}_k \, dp_k - \sum_k \dot{p}_k \, dq_k \]

But, \( H = H(p_k, q_k) \)

so \( dH = \sum_k \frac{\partial H}{\partial p_k} \, dp_k + \sum_k \frac{\partial H}{\partial q_k} \, dq_k \)

Hence \( \forall \ k: \frac{\partial H}{\partial p_k} = \dot{q}_k \) and \( \frac{\partial H}{\partial q_k} = -\dot{p}_k \)

Hamilton’s equations of motion
Hamilton’s Equations

\[ H = \sum_k \left( q_k p_k - L(q_k, \dot{q}_k) \right) \]

\[ H = H(p_k, q_k) \]

\[ \forall k : \frac{\partial H}{\partial p_k} = \dot{q}_k \quad \text{and} \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k \]

Hamilton’s Equations of Motion:

Describe how a mechanical state of a system characterized by \((q, p)\) ‘evolves’ with time.
On the extremality of the action integral

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Received 14 February 1983

Abstract. Some necessary and sufficient conditions for a critical point of the action integral to be locally or globally extremal are proved. Applications to systems with finite or infinite number of degrees of freedom are discussed.
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On virtual displacement and virtual work in Lagrangian dynamics

`Getting the most action out of least action: A proposal'
Am. J. Phys. 72:4 p522-527

`Deriving Lagrange's equations using elementary calculus'
Am. J. Phys. 72:4, p510-513

`From conservation of energy to the principle of least action: A story line'
Am. J. Phys. 72:4, p.514-521

NEXT CLASS: STiCM Lecture 06: Unit 1 Equations of Motion (v)
Applications of Lagrange’s/Hamilton’s Equations

- Entire domain of Classical Mechanics

- Enables emergence of ‘Conservation of Energy’ and ‘Conservation of Momentum’ on the basis of a single principle.

- Symmetry $\leftrightarrow$ Conservation Laws

- Governing principle: Variational principle – *Principle of Least Action*

*These methods have a charm of their own and very many applications*....
Applications of Lagrange’s/Hamilton’s Equations

• Constraints / Degrees of Freedom
  - offers great convenience!

• ‘Action’: dimensions

  ‘angular momentum’: 

  \[ h \text{ : Max Planck : fundamental quantity in Quantum Mechanics} \]

Illustrations: use of Lagrange’s / Hamilton’s equations to solve simple problems in Mechanics.
Manifestation of simple phenomena in different unrelated situations

**Dynamics of**

- spring–mass systems,
- pendulum,
- oscillatory electromagnetic circuits,
- bio rhythms,
- share market fluctuations ...

**radiation oscillators,**
**molecular vibrations,**
**atomic, molecular, solid state, nuclear physics,**
**electrical engineering,**
**mechanical engineering ...**

**Musical instruments**
SMALL OSCILLATIONS

1581:
Observations on the swaying chandeliers at the Pisa cathedral.

Galileo (when only 17 years old) recognized the constancy of the periodic time for small oscillations.

http://www.daviddarling.info/images/Pisa_cathedral_chandelier.jpg
http://roselli.org/tour/10_2000/102.html
Use of Lagrange’s / Hamilton’s equations to solve the problem of Simple Harmonic Oscillator.

Generalized Coordinate

Generalized Velocity

Generalized Momentum

\[ p = \left( \frac{\partial L}{\partial \dot{q}} \right) \]

\[ H = H(q, p, t) \]
\[ L = L(q, \dot{q}, t) \quad \text{Lagrangian} \]

\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0 \quad \text{Lagrange's Equation}
\]

\[ H = \sum_{k} \dot{q}_k p_k - L \]

\[ H = H(q, p, t) \quad \text{Hamiltonian} \]

\[ \forall \ k: \quad \frac{\partial H}{\partial p_k} = \dot{q}_k \quad \text{and} \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k \]

\[ \text{Hamilton's Equations} \]

2nd order differential equation

TWO

1st order differential equations
Mass-Spring Simple Harmonic Oscillator

\[ L = L(q, \dot{q}, t) \quad \text{Lagrangian} \]

\[ L = T - V = \frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2 \]

\[ -kq - m\ddot{q} = 0 \]

\[ m\ddot{q} = -kq \]

Newton’s Lagrange’s differential equation

2nd order
\[ m\ddot{q} = -kq \quad \ddot{q} = -\frac{k}{m}q \]

Linear relation between restoring force and displacement for spring-mass system:

Robert Hooke (1635-1703), (contemporary of Newton), empirically discovered this relation for several elastic materials in 1678.

[Image of Robert Hooke]

[Image showing \( \ddot{x} = -\frac{k}{m}x \)]

[Link to Hooke's historical page]
HAMILTONIAN approach

Note! Begin

Always with the LAGRANGIAN.

\[ L = L(q, \dot{q}, t) \]

\[ H = H(q, p, t) \]

VERY IMPORTANT!

\[ L = T - V = \frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2 \]

\[ p = \left( \frac{\partial L}{\partial \dot{q}} \right) = m \dot{q} \]
Mass-Spring

Lagrangian: \( L = T - V = \frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2 \)

\[
p = \left( \frac{\partial L}{\partial \dot{q}} \right) = m \dot{q}
\]

\[
H = p \dot{q} - L = m \dot{q}^2 - \frac{m}{2} \dot{q}^2 + \frac{k}{2} q^2
\]

\[
H = \frac{p^2}{2m} + \frac{k}{2} q^2
\]
\[ H = \frac{p^2}{2m} + \frac{k}{2} q^2 \]

\[ \dot{q} = \frac{\partial H}{\partial p} = \frac{2p}{2m} = \frac{p}{m} \]

and \[ \dot{p} = -\frac{\partial H}{\partial q} = -\frac{k}{2} 2q \]

(i.e. \( f = -kq \))

**Hamilton's Equations**

**TWO first order equations**
Be careful about how you write the Lagrangian and the Hamiltonian for the Harmonic oscillator!

\[ L = T - V = \frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2 \]

\[ p = \left( \frac{\partial L}{\partial \dot{q}} \right) = m \dot{q} \]

\[ H = \frac{p^2}{2m} + \frac{k}{2} q^2 \]

**Generalized Momentum** is interpreted only as \( p = \left( \frac{\partial L}{\partial \dot{q}} \right) \), and not a product of mass with velocity.
Remember this!
First: set-up the Lagrangian

\[ L = L(r, \theta, \dot{r}, \dot{\theta}) \]
\[ L = T - V \]
\[ L = \frac{1}{2} m(r^2 + r^2 \dot{\theta}^2) - mg \ell (1 - \cos \theta) \]

\[ r = l: \text{ constant} \]

\[ L = \frac{1}{2} m \ell^2 \dot{\theta}^2 - mg \ell (1 - \cos \theta) \]
\[ L = \frac{1}{2} m \ell^2 \dot{\theta}^2 - mg \ell + mg \ell \cos \theta \]
Now, we can find the generalized momentum for each degree of freedom.

\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0
\]

\[
\frac{\partial L}{\partial \dot{r}} = p_r = 0
\]

\[
\frac{\partial L}{\partial \dot{\theta}} = p_\theta = ml^2 \dot{\theta}
\]
Simple pendulum

\[ L = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl + mg l \cos \theta \]

\[ \frac{\partial L}{\partial r} = 0. \]
\[ \frac{\partial L}{\partial \theta} = -mgl \sin \theta \approx -mgl \theta \]

\[ \frac{\partial L}{\partial \dot{r}} = p_r = 0 \]
\[ \frac{\partial L}{\partial \dot{\theta}} = p_{\dot{\theta}} = ml^2 \dot{\theta} \]

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0 \]
\[ -mgl \theta - \frac{d}{dt} (ml^2 \dot{\theta}) = 0 \]
\[ -mgl \theta = ml^2 \ddot{\theta} \]
\[ \ddot{\theta} = -\frac{g}{l} \theta \]
\[ \ddot{\theta} = -\frac{g}{l} \theta \]

(1) \[ \ddot{q} = -\alpha q \]

(2) Solution: \[ q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \]

Substitute (2) in (1) \[ \Rightarrow \omega_0 = \sqrt{\alpha} \]

\[ \omega_0 = \sqrt{\frac{g}{l}} \]
(1) Newtonian
(2) Lagrangian

\[ \ddot{\theta} = -\frac{g}{l}\theta \]

\[ \omega_0 = \sqrt{\frac{g}{l}} \]

Note! We have not used 'force', 'tension in the string' etc. in the Lagrangian and Hamiltonian approach!
Hamilton’s equations: simple pendulum

\[ L = L(r, \theta, \dot{r}, \dot{\theta}) = T - V \]

\[ L = \left( \frac{1}{2} ml^2 \dot{\theta}^2 \right) - (mg\ell - mg\ell \cos \theta) \]

\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \]

\[ p_r = \frac{\partial L}{\partial \dot{r}} = 0 \]

\[ H = \left[ \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right] = \left[ \sum \dot{q}_i p_i - L \right] \]
\[ H = \left[ \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right] \]

\[ = \left[ \sum \dot{q}_i p_i - L \right] \]

\[ L = \left( \frac{1}{2} ml^2 \dot{\theta}^2 \right) - (mg \ell - mg \ell \cos \theta) \]

\[ H = \dot{\theta} p_\theta + \dot{r} p_r - \frac{1}{2} ml^2 \dot{\theta}^2 + mg \ell + mg \ell \cos \theta \]

\[ \frac{\partial H}{\partial \theta} = mgl(-\sin \theta) \approx -mgl\theta \]

\[ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mgl\theta \]
**Simple Pendulum**

\[
\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mg l \theta
\]

\[-ml^2 \ddot{\theta} = \dot{p}_\theta = mg l \theta
\]

\[\ddot{\theta} = -\frac{g}{l} \theta
\]

\[p_\theta = \frac{\partial L}{\partial \dot{\theta}} = -ml^2 \dot{\theta}
\]

\[p_r = \frac{\partial L}{\partial \dot{r}} = 0
\]

(1) \[\ddot{q} = -\alpha q \]

(2) Solution: \[q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\]

Substitute (2) in (1) \[\Rightarrow \omega_0 = \sqrt{\alpha}\]

\[\omega_0 = \sqrt{\frac{g}{l}}\]
(1) Newtonian
(2) Lagrangian
(3) Hamiltonian

\[ \ddot{\theta} = -\frac{g}{l} \theta \]

\[ \omega_0 = \sqrt{\frac{g}{l}} \]

We have NOT used ‘force’, causality, linear-response
Lagrangian and Hamiltonian Mechanics has very many applications.

All problems in ‘classical mechanics’ can be addressed using these techniques.
However, they do depend on the premise that mechanical system is characterized by position and velocity/momentum, simultaneously and accurately.
Central problem in ‘Mechanics’:

How is the ‘mechanical state’ of a system described, and how does this ‘state’ evolve with time?

- Formulations due to Galileo/Newton,
- Lagrange and Hamilton.
(q,p) : How do we get these?

Heisenberg’s principle of uncertainty

New approach required!
‘New approach’ is not required on account of the Heisenberg principle!

Rather,

the measurements of $q$ and $p$ are not compatible…. 

….. so how could one describe the mechanical state of a system by $(q,p)$?
Heisenberg principle comes into play as a result of the fact that simultaneous measurements of \( q \) and \( p \) do not provide consistent accurate values on repeated measurements. ….

….. so how could one describe the mechanical state of a system by \((q,p)\) ?
Mechanical State:
State vectors in Hilbert Space

Characterize? Labels?
“Good” quantum numbers/labels

Measurement: C.S.C.O.

Complete Set of Commuting Operators
Complete Set of Compatible Observables

\[ i\hbar \frac{\partial}{\partial t} \left| \psi(t) \right\rangle = H \left| \psi(t) \right\rangle \]

Schrödinger Equation

Evolution of the Mechanical State of the system
Linear Response.
Principle of causality.

Principle of Variation

\[ L(q, \dot{q}) \]
\[ H(q, p) \]

\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0
\]

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q_k}
\]