We first look at the root mean square (rms) fluctuation, \( < \delta N^2 > \), in the number of particles in the BCS condensate before proceeding to show that the number of particles operator \( \hat{N} \) and the phase \( \phi \) are canonically conjugate dynamical variables. The rms in the BCS state is found to be

\[
< \delta N^2 > = \left[ < N^2 > - < N >^2 \right] = \sum_\mathbf{k} 4|\nu_\mathbf{k}|^2 |\mu_\mathbf{k}|^2
\]

(1)

Since

\[
\sum_\mathbf{k} \rightarrow \left[ \frac{V}{(2\pi)^3} d^3 k \right]
\]

(2)

\[
< \delta \mathbf{P}^2 > \propto \frac{T_c}{T_F} \quad \text{if} \quad < N > \frac{T_c}{T_F}
\]

(3)

Hence, the fractional fluctuation is given by

\[
\frac{< \delta N^2 >^{1/2}}{< N >^{1/2}} \sim \frac{(T_c/T_F)^{1/2}}{< N >^{1/2}} \approx 10^{-13}
\]

(6)

(7)

It shows that the fluctuation about \( < N > \) is very small. It also shows that the expansion coefficients around \( < N > \) are strongly peaked.

We have seen that the BCS state can be written as the superposition of all possible \( 2m \)-particle states with each state having the same phase \( \phi \). It turns out that the particle number \( N \) and the phase \( \phi \) are conjugate variables. In order to see that we assume that the particle number can vary continuously from \(-\infty < N < \infty\).

Before we establish the conjugacy relationship between the particle number \( N \) and the phase \( \phi \), we note that the state with the given phase \( \phi \) is written as

\[
| \psi^\phi_{BCS} \rangle = \prod_\mathbf{k} (u_\mathbf{k} + e^{2i\phi} C^+_\mathbf{k} C^-_{-\mathbf{k}}) | 0 >
\]

(8)

while the state with a fixed number of \( 2m \)-particle is

\[
| \psi_{2m} > = \frac{1}{2\pi |A_{2m}|} \int_0^{2\pi} e^{-2i\omega \phi} | \psi^\phi_{BCS} > d\phi
\]

(9)

We contend that operating \(-i\partial/\partial \phi\) on \( | \psi^\phi_{BCS} > \) has the same effect as multiplying \( | \psi_{2m} > \) by \( 2m \). In the following let us use \( \tilde{N} \) for \( 2m \).

Let us operate \(-i\partial/\partial \phi\) on \( | \psi^\phi_{BCS} > \) in the above equation, and integrate by parts the resulting equation.
\[
\frac{1}{2\pi |A_N|} \int_0^{2\pi} e^{-iN\phi} \left(-i \frac{\partial}{\partial \phi} \right) | \psi_{BCS}^\phi > d\phi = -(-i) \frac{1}{2\pi |A_N|} \int_0^{2\pi} \left[(\frac{\partial}{\partial \phi}) e^{-i\Delta \Psi} \right] | \psi_{BCS}^\phi > d\phi
\]
\[
= -(-i)(-iN) \frac{1}{2\pi |A_N|} \int_0^{2\pi} e^{-i\Delta \Psi} | \psi_{BCS}^\phi > d\phi
\]
\[
= N \frac{1}{2\pi |A_N|} \int_0^{2\pi} e^{-i\Delta \Psi} | \psi_{BCS}^\phi > d\phi
\]
\[
= N | \psi_N >
\]

So that we may write
\[
-\frac{i}{\hbar} \frac{\partial}{\partial \phi} \leftrightarrow N
\]

Similarly, we can show that
\[
-\frac{i}{\hbar} \frac{\partial}{\partial N} \leftrightarrow \phi
\]

Thus, \( N \) and \( \phi \) are conjugate dynamical variables. It implies that they also satisfy the Hamilton's equation of motion for a given Hamiltonian \( \mathcal{H} \),
\[
\frac{i\hbar}{\partial t} N = [\mathcal{H}, N]
\]
\[
= \frac{i}{\hbar} \frac{\partial \mathcal{H}}{\partial \phi}
\]

and
\[
\frac{i\hbar}{\partial t} \phi = [\mathcal{H}, \phi]
\]
\[
= -\frac{i}{\hbar} \frac{\partial \mathcal{H}}{\partial N}
\]

These equations are useful in calculating the Josephson supercurrent.