Magnetostatics III

Lecture 25: Electromagnetic Theory

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Vector Potential

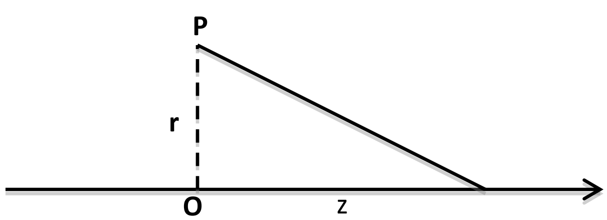
In this lecture we will calculate the vector potential in a few cases. We have seen that the vector potential is not unique and we have a choice of gauge in the matter. The most common gauge in which we work is the Coulomb gauge in which the divergence of the vector potential is chosen to be zero, i.e. $\nabla \cdot \vec{A} = 0$.

We obtained an expression for the vector potential starting with BiotSavart's law and saw that there exists a much stronger relationship between the vector potential than which exists for the magnetic field itself. In many cases where the direction of the current is constant, the vector potential simply points in the direction of the current. We have the following expression for the vector potential,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r'})}{|\vec{r}-\vec{r'}|}$$

Vector Potential for a long straight wire carrying current

Let the current be in the z direction. The vector potential also points the same way.



The current being linear $I\hat{k}$, the vector potential becomes a simple one dimensional integral

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} Ik \int_{-\infty}^{\infty} \frac{dz}{\sqrt{r^2+z^2}} = \frac{\mu_0}{4\pi} I\hat{k}\, \ln(z+\sqrt{r^2+z^2})\Big|_{-\infty}^{\infty}$$

The expression diverges when the limits are evaluated. This is not a very serious issue because we have seen that the vector potential is arbitrary up to a constant which in this case is infinite. For instance, if instead of integrating from $-\infty$ to $+\infty$, we realized that the integrand is even, we could integrate it from zero to infinity and double the result. In that case the integral diverges only in the upper limit leaving us with a finite expression in the lower limit. Discarding the infinite constant, we would then have,

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{2\pi} I \ln r\, \hat{k}$$

In this simple case, we can start from our knowledge of the magnetic field and calculate back. We know that the magnetic field has cylindrical symmetry and is directed along the circumferential direction,

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} = \nabla \times \vec{A}$$

Thus the curl of the vector potential only has ϕ component

$$B_\phi = \left(\nabla \times \vec{A}\right)_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

By symmetry, since the wire is infinite, the derivative with respect to z must be zero and we have

$$-\frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

which gives

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{2\pi} I \ln r\, \hat{k} + \nabla\psi$$

where we have explicitly added gradient of an arbitrary scalar field.

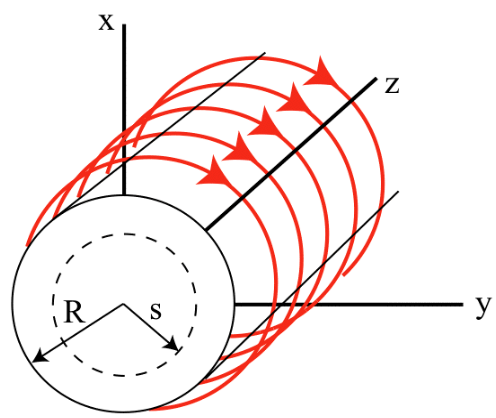
There is another trick which is often used to calculate the vector potential which is to relate the line integral of vector potential to the flux. If we take the line integral of the vector potential along any closed loop, we get, using Stoke's theorem,

$$\oint \vec{A} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{A}\right) \cdot d\vec{S}$$
$$= \int_S \vec{B} \cdot d\vec{S} = \Phi_B$$

We can then use the symmetry of the problem to find the vector potential.

Vector Potential of a solenoid

We know that the field inside a solenoid is along its axis (z direction) and is zero outside. If we take a circular path of radius s centered along its axis, the flux through the circular area is $\Phi_B = \pi s^2 B = \pi\mu_0 n I s^2$. The line integral of the vector potential is $2\pi r s\, A_\Phi$. This is because A is along the direction of current which is circumferential.



The vector potential is thus given by for $s < R$

$$\vec{A} = \mu_0 n I \frac{s}{2}\hat{\phi}$$

Though the field outside is zero, the vector potential does not vanish outside the solenoid. This is because, if we take a circle of radius $s > R$, the flux through the circular area is $\pi R^2 B = \pi R^2 \mu_0 n I$, flux being contributed only from inside the solenoid. Thus for $s > R$

$$\vec{A} = \mu_0 n I \frac{R^2}{2s}\hat{\phi}$$

which falls off as inverse of distance from the axis.

Does vector potential have any physical significance?

AharanovBohm Effect

It may be seen from the above example that the vector potential remains non-zero outside the solenoid even though the magnetic field itself has become zero. It turns out that the vector potential is not just a mathematical artifact but has physical reality. The experiment described below illustrates this though the effect that we are talking about is of quantum mechanical origin. (The source of the picture is William O. Straub (2010))

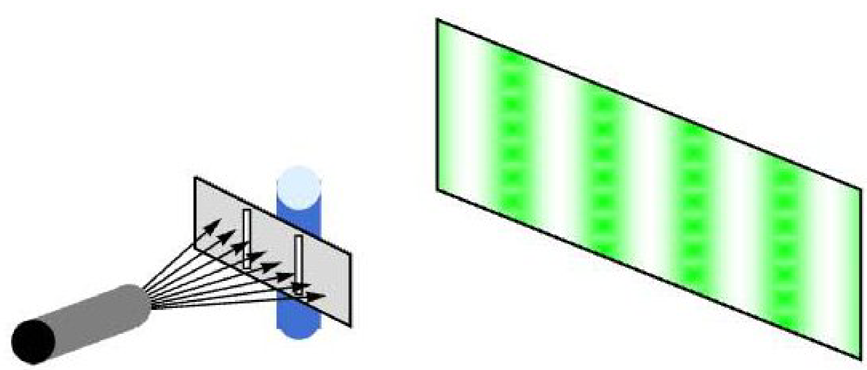


Figure 1. Experimental setup with solenoid turned off.

You are familiar with the Young's double slit experiment done with a beam of coherent light. The experiment is not really restricted to light wave but can be performed with matter wave such as a beam of electrons. We learn from quantum theory that like light exhibits dual behavior, that of wave as well as particles known as photon, a wavelength is associated with material particles as well. This is known as de Broglie wavelength and is given by the ratio of the Planck's constant to the momentum of the particle.

The experiment is performed with an electron beam in place of light. What is done is to put a small solenoid just beyond the slits between the slits and the screen. Initially the solenoid does not carry any current and its dimensions are small enough so that it does not disturb the interference pattern produced on the screen because of the phase difference between electron waves arriving at the screen from the slits.

The current in the solenoid is switched on. The solenoid is of very small dimensions so that most of the electron beam passes outside the solenoid. Since the magnetic field outside is zero, the electron beams does not experience any force due to the magnetic field and should reach the screen undeflected. This should not then affect the interference pattern. However, what is found is that the pattern on the screen shifts suggesting a change in the phase relationship. It can be shown in quantum mechanics that the agent responsible for this phase change is the vector potential which is non-zero outside the solenoid though the magnetic field is zero.

Vector potential for a Uniform Magnetic Field

Uniform magnetic field is of practical importance. Let us take the field to be in an arbitrary direction having a magnitude B. One of the possibilities for the vector potential is

$$\vec{A} = \frac{\vec{B} \times \vec{r}}{2}$$

It can be seen that

$$\begin{aligned} \nabla \times \vec{A} &= \frac{1}{2} \nabla \times (\vec{B} \times \vec{r}) \\ &= \frac{1}{2}\left[\vec{B}(\vec{\nabla} \cdot \vec{r}) - \vec{r}(\vec{\nabla} \cdot \vec{B}) + (\vec{r} \cdot \vec{\nabla})\vec{B} - (\vec{B} \cdot \vec{\nabla})\vec{r}\right] \end{aligned}$$

The first term gives $3\vec{B}$ since the $\nabla \cdot \vec{r} = 3$. The second term is zero because$\vec{\nabla} \cdot \vec{B} = 0$. The thirdand the fourth terms are calculated as follows,

$$(\vec{r} \cdot \vec{\nabla})\vec{B} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(\hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z\right) = 0$$

because the field is uniform.

$$\begin{aligned} (\vec{B} \cdot \vec{\nabla})\vec{r} &= \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)\left(\hat{\imath}x + \hat{\jmath}y + \hat{k}z\right) \\ &= \hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z = \vec{B} \end{aligned}$$

Adding all the four terms, result follows. It can be seen that the Coulomb gauge condition is satisfied.The divergence of the vector potential is given by

$$\nabla \cdot \vec{A} = \frac{1}{2}\nabla \cdot \left(\vec{B} \times \vec{r}\right) = \frac{1}{2}\left[\vec{r} \cdot \left(\nabla \times \vec{B}\right) - \vec{B} \cdot (\nabla \times \vec{r})\right] = 0$$

because the first term vanishes as the field is uniform while the second term vanishes because the curl of position vector is zero. Thus the expression satisfies Coulomb gauge condition. If the magnetic field is along the z direction, we can take either of the following expressions for the vector potential components,

$$\vec{A} = \left(-\frac{By}{2}, \frac{Bx}{2}, 0\right)$$
$$\vec{A} = (-By, 0,0)$$

Both these expressions are valid expressions for the vector potential and they differ by gradient of a scalar field. It can be checked that if we add a term $\nabla\psi$ to the first tem where $\psi = -\frac{Bxy}{2}$, we get the second expression.

Vector Potential of a current sheet

Let the current sheet be in the xy plane with the current flowing along the x direction with a linear current density K,

$$\vec{K} = K\hat{\imath}$$

We have seen that the magnitude of the magnetic field is constant both above the pland and below the plane with a discontinuity at the boundary.

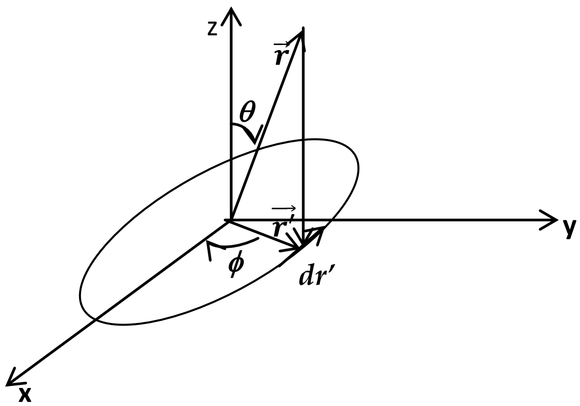
$$\vec{B} = \begin{cases} -\frac{\mu_0 K}{2}\hat{\jmath} & \text{for } z > 0 \\ +\frac{\mu_0 K}{2}\hat{\jmath} & \text{for } z < 0 \end{cases}$$

Since the field is constant, we can use expression for the vector potential from the last section, For$z > 0$

$$\vec{A} = \frac{\vec{B} \times \vec{r}}{2} = -\frac{\mu_0 K}{4}(\hat{\jmath} \times \vec{r})$$
$$= -\frac{\mu_0 K}{4}\left(\hat{\imath}z - \hat{k}x\right)$$

The vecor potential below the plane is obtained by changing the minus sign in the beginning of the expression to a plus. Note that the components of the vector potential (both tangential and normal) are continuous across the boundary, though the magnetic field itself is discontinuous.

Vector Potential of a circular current carrying loop



The current being in the azimuthal direction, the current density vector in this case can be written as

$$\vec{J} = \frac{I}{a}\delta(\cos\theta)\ \delta(r-a)\hat{\phi}$$

(Remember $\hat{\phi}$ not being a constant vector, its direction at the point of observation is not the same as its direction on the current element. As a result this expression cannot be directly substituted in the general expression for vector potential.)

Contribution of a current element $I\overrightarrow{dr'}$ to the vector potential is given by

$$d\vec{A} = \frac{\mu_0 I}{4\pi}\frac{\overrightarrow{dr'}}{\left|\vec{r}-\overrightarrow{r'}\right|}$$

The vector potential due to the loop is given by

$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint\frac{\overrightarrow{dr'}}{\left|\vec{r}-\overrightarrow{r'}\right|}$$

In order to evaluate it, we will expand $\frac{1}{|\vec{r}-\overrightarrow{r'}|}$ in Legendre polynomials, as we did in electrostatic problems.

$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint\sum_{l=0}^{\infty}\oint\frac{1}{r^{l+1}}r'^{l}P_l(\cos\theta)\ \overrightarrow{dr'}$$

For large values of r, we need to keep only leading terms. The lowest order term $l=0$ for which $P_l(\cos\theta)=1$ vanishes because $\oint\overrightarrow{dr'}=0$. Let us keep the next order term with $l=1$ which gives,

$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint \frac{1}{r^2} r' \cos\theta d\overrightarrow{r'}$$
$$= \frac{\mu_0 I}{4\pi r^3}\oint (\vec{r}\cdot\overrightarrow{r'})d\overrightarrow{r'}$$

The scalar product inside the integral can be simplified and expressed as a sum of two quantities ,one of which is a perfect integral ,which on integration would give zero.

$$\vec{r}\times\left(\overrightarrow{r'}\times d\overrightarrow{r'}\right) = \overrightarrow{r'}\left(\vec{r}\cdot d\overrightarrow{r'}\right) - d\overrightarrow{r'}(\vec{r}\cdot\overrightarrow{r'})$$

Remember that in this expression, $\vec{r}$ is a fixed vector (being the observation point) and $d\overrightarrow{r'}$ is change in the value of the vector $\overrightarrow{r'}$.

We can write, using chain rule,

$$d\left[\overrightarrow{r'}(\vec{r}\cdot\overrightarrow{r'})\right] = d\overrightarrow{r'}(\vec{r}\cdot\overrightarrow{r'}) + \overrightarrow{r'}(\vec{r}\cdot d\overrightarrow{r'})$$

Substituting this in the preceding relation, we get,

$$d\overrightarrow{r'}(\vec{r}\cdot\overrightarrow{r'}) = \frac{1}{2}d\left[\overrightarrow{r'}(\vec{r}\cdot\overrightarrow{r'})\right] - \frac{1}{2}\vec{r}\times\left(\overrightarrow{r'}\times d\overrightarrow{r'}\right)$$

The loop integral of the first term vanishes as it is a perfect integral. The second term gives,

$$\vec{A}(\vec{r}) = -\frac{\mu_0 I}{4\pi r^3}\left[\frac{1}{2}\vec{r}\times\oint\overrightarrow{r'}\times d\overrightarrow{r'}\right]$$

Note that the loop integral, along with the factor of ½ is just the area vector of the loop, which, when multiplied by the current I, gives the magnetic moment $\vec{m}$ of the loop. We thus have, interchanging the order of the cross product to take care of the minus sign,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{r}\times\vec{m}}{r^3}$$

This form has the advantage that it can be used to calculate the magnetic field at large distances even away from the axis. If we write this in spherical coordinates, the vector potential clearly has only the azimuthal component, and we can write,

$$A_\phi = \frac{\mu_0}{4\pi r^2}\pi R^2 I \sin\theta$$

This gives for the magnetic field components,

$$B_r = \frac{1}{r^2\sin\theta}\left(\frac{\partial(r\sin\theta A_\phi)}{\partial\theta} - \frac{\partial(rA_\theta)}{\partial\phi}\right)$$
$$= \frac{\mu_0}{2\pi r^3}\pi R^2 I\cos\theta$$

$$B_\theta = -\frac{1}{\text{r}\sin\theta}\left(\frac{\partial}{\partial r}(r\sin\theta A_\phi) - \frac{\partial A_r}{\partial \phi}\right)$$
$$= \frac{\mu_0}{4\pi r^3}\pi R^2 I \sin\theta$$

We can also obtain a coordinate independent form for the magnetic field from the expression for the vector potential,

$$\vec{B} = \frac{\mu_0}{4\pi}\left[3\frac{(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]$$

Proof of this is left as an exericise.

Magnetostatics III

Lecture 25: Electromagnetic Theory

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Tutorial Assignment

1. Show that the coordinate free form of the magnetic field derived from the vector potential for the circular loop is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\left[3\frac{(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]$$

2. A vector potential is given by $\vec{A} = \frac{B_0}{2}(\hat{n}\times\vec{r})$, where $\hat{n}$ is a unit vector along an arbitrary but given direction. Calculate the corresponding magnetic field.
3. A spherical shell of radius R carries a uniform surface charge density σ. The sphere is rotated with a uniform angular velocity ω about an axis (z direction). Find the vector potential and the magnetic field of induction at an arbitrary point both outside and inside the shell.

Solutions to Tutorial Assignment

1. Magnetic field can be written as

$$\vec{B} = \frac{\mu_o}{4\pi}\nabla\times\left(\frac{\vec{r}\times\vec{m}}{r^3}\right)$$
$$= \frac{\mu_0}{4\pi}\left[\vec{m}\left(\nabla\cdot\frac{\vec{r}}{r^3}\right) - (\vec{m}\cdot\vec{\nabla})\frac{\vec{r}}{r^3}\right]$$

$$= \frac{\mu_0}{4\pi}\left[3\frac{(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]$$

(The full expansion of $\nabla \times \left(\frac{\vec{r}\times\vec{m}}{r^3}\right) = \nabla \times \left(\frac{\vec{r}}{r^3}\times\vec{m}\right)$ has four terms of which two vanish because $\vec{m}$ is constant, leaving us with two terms above. The first term gives zero because $\cdot\frac{\vec{r}}{r^3} = 0$, the remaining term is easily expanded to get the desired result).

2. The magnetic field is given by

$$\nabla \times \frac{B_0}{2}(\hat{n}\times\vec{r}) = \frac{B_0}{2}\left[\hat{n}(\nabla\cdot\vec{r}) - (\hat{n}\cdot\vec{\nabla}\vec{r})\right] = B_0\hat{n}$$

The vector potential corresponds to a constant magnetic field in the given direction.

3. Since the charges are on the surface, we can write the current density as follows:

$$\vec{J}(\vec{r}) = \rho\vec{v} = \sigma\delta(r-R)\vec{\omega}\times\vec{r}$$

The vector potential can be written as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\frac{\vec{J}(\vec{r'})}{\left|\vec{r}-\vec{r'}\right|}$$

$$= \frac{\mu_0}{4\pi}\sigma\vec{\omega}\times\int d^3r'\frac{\vec{r'}}{\left|\vec{r}-\vec{r'}\right|}\delta(r'-R)R^2d\Omega'dr'$$

$$= \frac{\mu_0 R^2}{4\pi}\sigma\vec{\omega}\times\int d^3r'\frac{\vec{r'}}{\left|\vec{r}-\vec{r'}\right|}\delta(r'-R)d\Omega'dr'$$

The integral above can only depend on the direction $\vec{r}$. Let us write,

$$\int d^3r'\frac{\vec{r'}}{\left|\vec{r}-\vec{r'}\right|}\delta(r'-R)d\Omega'dr' = f(r)\hat{r}$$

Taking the dot product of both sides with $\hat{r}$, and performing the delta function integration, we get

$$\int d^3r'\frac{\vec{r'}}{\left|\vec{r}-\vec{r'}\right|}\delta(r'-R)d\Omega'dr' = R\int\frac{\cos\theta'}{\left|\vec{r}-\vec{r'}\right|}d\Omega'$$

with $\left|\vec{r'}\right| = R$, where θ' is the angle between the variable vector $\vec{r'}$ and the direction $\hat{r}$, We now, expand $\frac{1}{\left|\vec{r}-\vec{r'}\right|}$ in terms of spherical harmonics and recognize that $\cos\theta' = \sqrt{\frac{4\pi}{3}}Y_{1,0}(\theta',\phi')$

$$\frac{1}{\left|\vec{r}-\vec{r'}\right|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{l,m}^*(\theta',\phi')Y_{l,m}(\theta,\phi)$$

Thus

$$\vec{A}(\vec{r}) = \frac{\mu_0 R^3}{4\pi}\sigma\vec{\omega}\times\hat{r}\int\sqrt{\frac{4\pi}{3}}Y_{1,0}(\theta',\phi')\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{l,m}^*(\theta',\phi')Y_{l,m}(\theta,\phi)d\Omega$$

Using the orthogonality of spherical harmonics, the term that survives is $l = 1, m = 1$ and we are left with

$$\vec{A}(\vec{r}) = \frac{\mu_0 R^3}{4\pi}\sigma\vec{\omega}\times\hat{r}\frac{4\pi}{3}\cos\theta\frac{r_<}{r>^2}$$

Thus the vector potential is given by

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3} r\cos\theta\ (\vec{\omega}\times\hat{r}) & r < R \\ \dfrac{\mu_0}{3}\dfrac{\sigma R^4}{r^2}\cos\theta(\vec{\omega}\times\hat{r}) & r > R \end{cases}$$

[The integral can be done in more direct method. Let us take the point of observation along the z axis and let the sphere be rotating about an axis which lies in the xz plane and which makes an angle ψ with the z axis. If the $\vec{r'}$ have the coordinates (r', θ', ϕ') with respect to a spherical system of coordinates with its origin at the centre of the sphere. We can expand $\vec{\omega}\times\vec{r'}$ in Cartesian coordinates (expending a cross product in spherical coordinates being not a straightforward exercise),

$$\vec{\omega}\times\vec{r'} = (\omega\sin\psi\hat{\imath} + \omega\cos\ \psi\hat{k})\times(R\sin\theta'\cos\phi'\ \hat{\imath} + R\sin\theta'\sin\phi'\hat{\jmath} + R\cos\theta'\hat{k})$$

when these are inserted into the integral, the integration over ϕ' would give zero. We are left with only the contribution coming from the last term of the second term of the cross product. The term which survives is

$$\omega\ \mathrm{R}\sin\psi\cos\theta'\hat{\imath}\times\hat{k} \equiv R\cos\theta'\,\vec{\omega}\times\hat{r}$$

the last relation follows because $\vec{\omega}$ is in xz plane. The rest of the derivation is same as above]

Self Assessment Quiz

1. For the case of vector potential derived for the solenoid show that the expression satisfies Coulomb gauge condition.
2. Find the vector potential inside a cylindrical wire of radius R.
3. A charge q is moving slowly with a uniform velocity $\vec{v}$. Obtain an expression for the vector potential and the magnetic field at a distance r from it.

Solutions to Self Assessment Quiz

1. In cylindrical coordinates (s, ϕ, z)
$$\nabla \cdot \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$
Since $\vec{A} = \left(0, A_\phi, 0\right)$, we only have to consider the second term. As A_ϕ is independent ofϕ, the divergence is zero.
2. The field inside a cylindrical wire carrying a current in the z direction is given by
$$\vec{B} = \mu_0 I \frac{r}{2\pi R^2}\hat{\phi}$$
Since the magnetic field is along the azimuthal direction, we have on equating it with the azimuthal component of the vector potential,
$$\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi R^2} r$$
Since the wire is long there cannot be a variation with respect to z and the only non-zero component of $\vec{A}$ is
$$\frac{\partial A_z}{\partial r} = -\frac{\mu_0 I}{2\pi R^2} r$$
which gives, $\vec{A} = -\frac{\mu_0 I}{4\pi R^2} r^2 \hat{k}$

3. The moving charge can be considered to be equivalent to a current qv with the current density being directed along the direction of the velocity. The vector potential can be written as
$$\vec{A} = \frac{\mu_0}{4\pi}\frac{q\vec{v}}{r}$$
The magnetic field is obtained by taking the curl of this equation. Using,
$$\nabla \times \frac{\vec{v}}{r} = \frac{1}{r}(\nabla \times \vec{v}) + \nabla\left(\frac{1}{r}\right) \times \vec{v}$$
and using the fact that the velocity vector being constant, the first term of above is zero, we get,
$$\vec{B} = -\frac{\mu_0}{4\pi} q \, \frac{\vec{r} \times \vec{v}}{r^3}$$