Finite Element for structures with Piezo electric material

Lecture 30
Smart and Micro systems
Introduction

- Modeling of systems with smart material patches is very similar to conventional structures. However, additional complexities will arise due to the presence of coupling terms in the constitutive law of these smart materials.
- These coupling will introduce additional matrices in the finite element formulations.
- Piezoelectric materials have two constitutive laws, one of which is used for sensing and the other for actuation purposes.
- For 2-D problems, the constitutive model for the piezoelectric material is of the form

\[
\{\sigma\}_{3x1} = [C]^{(E)}_{3x3} \{\varepsilon\}_{3x1} - [e]_{3x2} \{E\}_{2x1} \Rightarrow \text{Actuation Law} \quad (a)
\]

\[
\{D\}_{2x1} = [e]_{2x3}^T \{\varepsilon\}_{3x1} + [\mu]^{(\sigma)}_{2x2} \{E\}_{2x1} \Rightarrow \text{Sensing Law} \quad (b)
\]
Here, $\{\sigma\}^T = \{\sigma_{xx}, \sigma_{yy}, \tau_{xy}\}$ is the stress vector, $\{\varepsilon\}^T = \{\varepsilon_{xx}, \varepsilon_{yy}\}$ is the strain vector, $[e]$ is the matrix of piezoelectric coefficients, which has units of Newton/Volts-mm and is of size $3 \times 2$.

$\{E\}^T = \{E_x, E_y\} = \{V_x/t, V_y/t\}$ is the applied field in two coordinate directions, where $V_x$ and $V_y$ are the applied voltages in the two coordinate directions, and $t$ is the thickness parameter. The units of $E_x$ (and $E_y$) is $V/mm$.

$[\mu]$ is the Permittivity matrix of size $2 \times 2$, measured at constant stress and has a unit of $N/V/V$.

$\{D\}^T = \{D_x, D_y\}$ is the vector of electric displacement in the two coordinate directions. This has a units of $N/V-mm$. 
Introduction (Cont)

- \([C]\) (in Eqn (a)) is the mechanical constitutive matrix measure (or Hooke’s law) at constant electric field. The above Constitutive law can also be written in the form
  \[\{\varepsilon\} = [S]\{\sigma\} + [d]\{E\}\]

- In the above expression, \([S]\) is the compliance matrix, which is the inverse of the mechanical material matrix \([C]\) and \([d] = [C]^{-1}\{e\}\) is the electromechanical coupling matrix, where the elements of this matrix have a unit \(\text{mm} / \text{V}\) and the elements of this matrix are direction dependent.

- In the most analysis, it is assumed that the mechanical properties will change very little with the change in the electric field and as a result, the actuation law (Eqn (a)) can be assumed to behave linearly with the electric field, while the sensing law (Eqn (b)) can be assumed to behave linearly with the stress.
Introduction (cont)

- The first part of Eqn. (a) represents the stresses developed due to mechanical load, while the second part of the same equation gives the stresses due to voltage input.
  \[
  \mathbf{\varepsilon} = \mathbf{\sigma} + \mathbf{d} \varepsilon
  \]

- From Eqns. (a & b), it is clear that the structure will be stressed due to the application of electric field even in the absence of mechanical load.

- Alternatively, when the mechanical structure is loaded, it generates an electric field.

- In other words, the above constitutive law demonstrates the electromechanical coupling, which is exploited for variety of structural applications involving sensing and actuation such as vibration control, noise control, shape control or Structural Health Monitoring.
Actuation Through Piezo Electric material

- The actuation using piezoelectric materials can be demonstrated using a plate of dimensions $L \times W \times t$, where $L$ and $W$ are the length and width of the plate and $t$ is its thickness.

- Thin piezoelectric electrodes are placed on the top and bottom surface of the plate as shown. Such a plate is called the *Bimorph* plate.

- When the voltage is passed between the electrodes as shown in figure (which is normally referred to as poling direction), the deformation in the length, width and thickness directions are given by

\[
\delta L = d_{31}E_1L = \frac{d_{31}VL}{t}, \quad \delta W = d_{31}E_2W = \frac{d_{31}WW}{t}, \quad \delta t = d_{33}V
\]
Actuation Through Piezo Electric material (Cont)

- Here, $d_{31}$ and $d_{33}$ are the electro-mechanical coupling coefficients in the directions 1 and 3, respectively. Conversely, if a force $F$ is applied in any of the length, width or thickness directions, the voltage $V$ developed across the electrodes in the thickness direction is given by

$$V = \frac{d_{31}F}{\mu L} \text{ or } \frac{d_{31}F}{\mu W} \text{ or } \frac{d_{33}F}{\mu LW}$$

- Here is the dielectric permitivity of the material. The reversibility between the strain and voltages makes piezoelectric materials ideal for both sensing and actuation.
Piezoelectric material Types

- There are different types of piezoelectric materials that are used for many structural applications. The most commonly used material is the PZT (Lead-Zirconate-Titanate) material, which is extensively used as bulk actuator material as they have high electromechanical coupling coefficient. $d_{31}, d_{33}$

- Due to the low electromechanical coupling coefficient, piezo polymers (PVDF) are extensively used only as sensor material.

- With the advent of smart composite structures, a new brand of material called *Piezo Fiber Composite (PFC)* is found to be very effective actuator material for use in vibration/noise control applications.
Finite element modeling of the mechanical part is very similar to what was discussed in the previously except that the coupling terms introduce additional energy terms in the variational statements, which results in additional coupling matrices in the FE formulation.

Introduction of Piezoelectric material, introduces an additional degree of freedom in the FE formulation. This additional DOF can be the electrical potential (Normally referred to as \( \Phi \), which is related to the Electrical field vector \( \vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k} = \nabla \Phi \) ) or electrical field itself.

Alternatively, the analysis can be performed by using conventional beam, or plane stress element derived earlier and the effects of coupling terms can be translated as equivalent concentrated nodal loads. Such an approach is possible when we assume that the sensing and actuation law are not coupled, which is mostly true in piezoelectric material. We will demonstrate both approaches in this lecture.
The bimorph beam consists of two identical PVDF beams laminated together with opposite polarities. The schematic diagram of the bimorph beam is shown.

The PVDF patches are poled in such a way that the strains are produced in the axial $x$ direction due to an applied electric field in the $z$ direction.

The dimensions of the beam are taken as $100\text{mm} \times 5.0\text{mm} \times 0.5\text{mm}$. The aim of this example is to see how the cantilever deforms due to the applied voltage.
Exact Solution- Through Strength of Materials

- This problem can be statically reduced to a problem of a cantilever beam with an end moment \( M \) shown.
- The moment \( M \) needs to be determined from the constitutive law of the PVDF material.
- The beam is under a 1-D state of stress with the stress acting in the \( x \) direction.
- From Eqn (a), we have

\[ \{\sigma\} = [S]^{-1}\{\varepsilon\} - [S]^{-1}[d]\{E\} \]

- The inverse of the compliance matrix is the constitutive matrix \([C]\) and representing \([e]= [S]^{-1}[d]\) the above equation becomes

\[ \{\sigma\} = [C]\{\varepsilon\} - [e]\{E_{z}\} \]
Exact Solution (Cont)

- The first part of the above equation is due to mechanical load, which is zero in the present case and hence is not relevant to the present problem.
- Since the beam is in 1-D state of stress, only $\sigma_{xx}$, bending stress in the axial direction, exists.
- The only material property of relevance here is the Young’s Modulus $Y$ and the relevant piezoelectric coefficient is $e_{31}$, which is first element of third row of the matrix $[\varepsilon]$ given in the above Equation. Hence the constitutive law can be written as
  \[ \sigma_{xx} = -e_{31} E_z = -e_{31} V / t \]

- From the elementary beam theory, we have
  \[ M / I = \sigma_{xx} / z \]

- where $M$ is the moment acting on the cross section, $I$ is the area Moment of Inertia of the cross section, and $z$ is the coordinate in the thickness direction.
Exact Solution (Cont)

- Substituting for $M$ from Eqn. (5.109) in the elementary beam equation, we can express the moment developed due to electrical excitation is given by

$$M = -\frac{2 e_{31} V I}{t^2}$$

- From the theory of deflection of beams, we can show that the transverse displacement $w(x)$ of a cantilever beam with a tip moment $M$ is given by

$$w(x) = -\frac{M x^2}{2EI}$$

- Using the value of Moment $M$, we can write the displacement variation in a bi-morph piezoelectric cantilever beam as

$$w(x) = \frac{e_{31} V}{E} \left( \frac{x}{t} \right)^2$$

$$w(L)|_{\text{elec}} = \frac{e_{31} V}{E} \left( \frac{L}{t} \right)^2$$

At $x=L$
Exact Solution (Cont)

- Next, in addition to electric field, we will introduce a mechanical load $P$ applied at the tip.
- Deflection of a cantilever beam subjected to tip concentrated load $P$ is
  $$w(x) = \frac{P}{EI} \left(\frac{x^3}{6} - \frac{x^2L}{2}\right)$$
- The tip deflection in this case is got by substituting $x=L$ in the above equation, which is equal to
  $$w(L)_{\text{mech}} = -\frac{PL^3}{3EI}$$
- Now when both mechanical and electrical loads are applied, the total deflection will be sum of the deflections due to mechanical load and electrical load, which given by
  $$w(L)_{\text{total}} = -\frac{PL^3}{3EI} + \frac{e_{31}}{E} \left(\frac{L}{t}\right)$$
• From this expression, we see that as the voltage is increased, it reduces the net downward deflection due to the mechanical load

• The above expression can also give the voltage necessary to force the total deflection equal to zero, which is given by

\[ V = \frac{P L}{e_3 b t} \]

• It is clear that the presence of electrical load helps to totally eliminate the deflection of the cantilever beam due to mechanical load. This, in essence, is the main principle of actuation, which can be exploited for a variety of applications such as vibration control, noise control or shape control in structures.
Finite Element Solution

• In this lumped approach, we do not need any smart or electrical d.o.f.

• Here, the forces generated due to electrical field are needed to be lumped corresponding to FE degrees of freedom, which in this case is a moment ( ).

\[ M = - \frac{2 e_{31} VI}{t^2} \]

• Since, our objective here is to see how the deflection caused by pure mechanical load is negated by the electrical field generated through the PVDF patches, we need to retain the degrees of freedom corresponding to transverse mechanical force (shear force) at the tip of the cantilever beam.
• The FE equation of the beam derived in lecture 28, will be used here, which is given by

\[
\begin{bmatrix}
P_1 \\
M_1 \\
P_2 \\
M_2
\end{bmatrix}
= \frac{YI}{L^3}
\begin{bmatrix}
12 & -6L & -12 & -6L \\
-6L & 4L^2 & 6L & 2L^2 \\
-12 & 6L & 12 & 6L \\
-6L & 2L^2 & 6L & 4L^2
\end{bmatrix}
\begin{bmatrix}
w_1 \\
\theta_1 \\
w_2 \\
\theta_2
\end{bmatrix}
\]

• We will model the beam only through one element.

• We need to first enforce the boundary condition. The left end of the beam, which is designated here as node 1, has both deflection and slope equal to zero. This amounts to eliminating first two rows and column of the stiffness matrix.

• The reduced stiffness matrix after the enforcement of boundary conditions then becomes

\[
\begin{bmatrix}
P_2 \\
M_2
\end{bmatrix}
= \frac{EI}{L^3}
\begin{bmatrix}
12 & 6L \\
6L & 4L^2
\end{bmatrix}
\begin{bmatrix}
w_2 \\
\theta_2
\end{bmatrix}
= \frac{EI}{L^3}[\bar{K}]{w}
\]
The displacements and rotation can be obtained by inverting the reduced stiffness matrix, which is given by

\[
\begin{align*}
\begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix} &= \frac{L}{12EI} \begin{bmatrix} 4L^2 & -6L \\ -6L & 12 \end{bmatrix} \begin{bmatrix} P_2 \\ M_2 \end{bmatrix} \\
\end{align*}
\]

For purely electrical load, \( P_2 = 0 \). As mentioned earlier, pure electrical load causes a moment in the beam \( M_2 \). Substituting this, we get the tip deflection as

\[
\begin{align*}
\frac{w_2}{\theta_2} &= \frac{L}{12EI} (-M_2 6L) = -\frac{M_2 L^2}{2EI} = \frac{e_{31} V}{E} \left( \frac{L}{t} \right)^2
\end{align*}
\]

This is the same as the exact solution. If we now apply a tip vertical load, in addition to the moment \( M_2 \), we can get the tip deflection as

\[
\begin{align*}
\frac{w_2}{\theta_2} &= \frac{L}{12EI} (4L^2 P_2 - 6LM_2) = \frac{L}{12EI} \left( -4L^2 P + \frac{6e_{31} VI}{t^2} \right) = \frac{PL^3}{3EI} \frac{e_{31} V}{Y} \left( \frac{L}{t} \right)^2
\end{align*}
\]

This result is same as the exact solution.
2-D Plane Stress -4-noded isoparametric FE Formulation

- This element will have 2 mechanical degrees of freedom, namely the two displacement components \( u(x,y,t) \) and \( w(x,y,t) \), respectively and a single electrical degree of freedom \( E_z(x,y,t) \) in the \( z \) direction.
- We have apriori assumed that stresses to be in the horizontal \( x \)-direction.
- Thus, this element will have a total of 12 degrees of freedom.

- Here, we will use isoparametric formulation outlined in the earlier lecture.
Since the proposed element is four noded, we will use the bilinear shape functions for the mechanical displacements will be required, which can be written as

\[ u(x, y, t) = \sum_{i=1}^{4} N_i(\xi, \eta)u_i(t), \quad w(x, y, t) = \sum_{i=1}^{4} N_i(\xi, \eta)w_i(t) \]

where \( \xi \) and \( \eta \) are the isoparametric coordinates and \( u_i \) and \( w_i \) are the nodal mechanical degrees of freedom. The four bilinear shape functions are given by

\[ N_1 = \frac{1}{4}(1 - \xi)(1 - \eta), \quad N_2 = \frac{1}{4}(1 + \xi)(1 - \eta) \]
\[ N_3 = \frac{1}{4}(1 + \xi)(1 + \eta), \quad N_4 = \frac{1}{4}(1 - \xi)(1 + \eta) \]

The choice of electrical dof variation is obvious. The element has to support four electrical dof and hence bilinear shape functions are the minimum order required, which can be written as

\[ E_{\xi}(x, y, t) = \sum_{i=1}^{4} N_i(\xi, \eta)E_{\xi i}(t) \]
• Here, the same shape functions used for mechanical dof is also used here, and \( E_{zi} \) are the nodal electrical degrees of freedom at the four nodes.

• In isoparametric formulation, we map the actual geometry of the element to a square of size 2 defined in the generalized coordinate system through a Jacobian transformation.

• This requires the variation of the coordinate system in the generalized coordinates in terms of the nodal coordinates of the actual element geometry. Hence, one can use the same displacement shape functions to describe this variation and can be written as

\[
x(x, y) = \sum_{i=1}^{4} N_i(\xi, \eta)x_i, \quad z(x, y) = \sum_{i=1}^{4} N_i(\xi, \eta)z_i
\]
• Jacobian is computed using the chain rule. This was explained in lecture 29.

• The strains are evaluated by using strain-displacement relationship. That is,

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
2\varepsilon_{xy} \\
E_z
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix} u \\ w \\ D_z \end{bmatrix}
\]

• Using the assumed variation of field variables in the above equation enables expressing the strains in terms of nodal displacement vector with

\[
\{u\}_e = \{u_1 \ w_1 \ u_2 \ w_2 \ u_3 \ w_3 \ u_4 \ w_4\}^T
\]

and electric field vector

\[
\{E_z\}_e = \{E_{z1} \ E_{z2} \ E_{z3} \ E_{z3}\}^T
\]
That is strain can be written as

\[ \{\varepsilon\} = [B]\{u\} = \begin{bmatrix} [B]_{u(3 \times 8)} & 0 \\ 0 & [B]_{E(1 \times 4)} \end{bmatrix} \]

where \([B]\) matrix, is given by

\[ [B] = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} & 0 & \frac{\partial N_4}{\partial z} & 0 & 0 & 0 & 0 \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial z} & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4
\end{bmatrix} \]
The weak form of the governing equation for this problem is given by taking the variation of the total energy, which is given by

\[
\delta \left( \frac{1}{2} \int_{t_1}^{t_2} \{ \delta u \}^T \mathbf{\rho} \{ \delta u \} dV dt + \frac{1}{2} \int_{t_1}^{t_2} \{ \delta \sigma \}^T \{ \varepsilon \} dV dt + \frac{1}{2} \int_{t_1}^{t_2} E_z D_z dV dt + \int_{t_1}^{t_2} \{ u \}^T \{ F_c \} dt + \int_{S_1} \{ u \}^T \{ F_s \} dS_1 dt + \int_{S_2} E_z D_s dS_2 dt \right)
\]

where \( S_1 \) and \( S_2 \) are the surfaces in the structure where the surface forces, and residual displacements act. Using the constitutive model, we can re-write the weak form of the governing equation as

\[
\begin{align*}
\mathbf{u} &= \mathbf{N} \mathbf{u}^0 \\
W &= \mathbf{N}^T \mathbf{u}^0 \\
\varepsilon &= \mathbf{N} \varepsilon^0 \\
\end{align*}
\]

\[
\begin{align*}
\int_{t_1}^{t_2} \{ \delta u \}^T \mathbf{\rho} \{ \delta u \} dV dt + \int_{t_1}^{t_2} \{ \delta \varepsilon \}^T [\mathbf{C}] \{ \varepsilon \} dV dt - \int_{t_1}^{t_2} \{ \delta \varepsilon \}^T \mathbf{\varepsilon} dV dt \\
+ \int_{t_1}^{t_2} \{ \delta E_z \}^T \{ \varepsilon \} dV dt + \int_{t_1}^{t_2} \{ \delta E_z \}^T \mathbf{\varepsilon} dV dt + \int_{t_1}^{t_2} \{ \delta u \}^T \{ F_c \} dt \\
+ \int_{S_1} \{ \delta u \}^T \{ F_s \} dS_1 dt + \int_{S_2} \varepsilon E_z D_s dS_2 dt = 0
\end{align*}
\]
Substitution of assumed variation and the strain displacement matrix, we can re-write the weak form as

\[
\begin{align*}
\{\delta u\}_e^T & \left( \int V [N]^T \rho [N] dV \right) \{\ddot{u}\}_e + \{\delta u\}_e^T \left( \int V [B_u]^T \hat{C} [B_u] dV \right) \{u\}_e - \{\delta u\}_e^T \left( \int V [B_u]^T \hat{k}_{ue} [B_u] dV \right) \{\ddot{u}\}_e = \\
\{\delta u\}_e^T & \left( \int V [B_u]^T \hat{\mu}_{33} [B_u] dV \right) \{E_{z}\}_e - \{\delta E_z\}_e^T \left( \int V [B_E]^T \hat{c} [B_E] dV \right) \{u\}_e \\
\{\delta E_z\}_e^T & \left( \int V [B_E]^T \hat{\mu}_{33} [B_E] dV \right) \{E_{z}\}_e - \{\delta u\}_e^T \{F_c\}_e - \{\delta u\}_e^T \{F\}_e dS_1 - \\
\{\delta E_z\}_e^T & \int_{S_2} [B_E]^T D_s dS_2 = 0
\end{align*}
\]

Since \(\{\delta u\}_e\) and \(\{\delta E_z\}_e\) are arbitrary, the above expression can be written in a concise matrix form as

\[
\begin{bmatrix}
[M_{uu}] & [0] \\
[0] & \otimes [0]
\end{bmatrix}
\begin{bmatrix}
\{\dddot{u}\}_e \\
\{\dddot{E}_{z}\}_e
\end{bmatrix}
+ 
\begin{bmatrix}
[K_{uu}] & \otimes [K_{ue}] \\
[K_{ue}^T] & [K_{EE}]
\end{bmatrix}
\begin{bmatrix}
\{u\}_e \\
\{E_{z}\}_e
\end{bmatrix}
= 
\begin{bmatrix}
\{F\}_e \\
\{q\}_e
\end{bmatrix}
\]
The above equation is the elemental equilibrium in the discretized form, where $[K_{uE}]$ is the mass matrix, $[M_{uu}]$ is the stiffness matrix corresponding to mechanical degrees of freedom, $[K_{u\mu}]$ is the stiffness matrix due to electro-mechanical coupling and $[K_{EE}]$ is the stiffness matrix due to electrical degrees of freedom alone. Note that all these matrices require the volume integral to be evaluated. Since the exact integration of these is most difficult to achieve, we resort to numerical integration. Here, $\{F\}_e$ is the elemental nodal vector and $\{q\}_e$ is the elemental charge vector. These matrices are given by

$$[M_{uu}] = \frac{1}{t} \int_{-1}^{1} \int_{-1}^{1} [N]^T \rho[N] J |d\xi d\eta|, \quad [K_{uu}] = \frac{1}{t} \int_{-1}^{1} \int_{-1}^{1} [B_u]^T \hat{\mathcal{C}} [B_u] J |d\xi d\eta|, \quad [K_{EE}] = \frac{1}{t} \int_{-1}^{1} \int_{-1}^{1} [B_E]^T \hat{\mu}_{33} [B_E] J |d\xi d\eta|$$
The elemental load and charge vectors are given by

\[
\{F\}_e = \{F\}_c + \int [N]^T \{F\}_s \, dS_1, \quad \{q\}_e = -\int [N]^T D_s \, dS_2
\]

The matrices in FE Equations are assembled to obtain their global counterparts and solved for obtaining solutions for displacements and electric field.

Note that it has a zero diagonal block in the mass matrix, which requires special solution schemes.

The method of solution for sensing and actuation problem is quite different.

For sensing problem, for a given mechanical loading, we need to determine the voltage developed across the smart patch.

This is done by first obtaining the mechanical displacement due to the given mechanical load, which is then used to obtain the electric field and hence the voltage developed in the sensor patch.
In order to solve this, the global matrix equation can be expanded and written as

\[ [M_{uu}]\{\ddot{u}\} + [K_{uu}]\{u\} + [K_{uE}]\{E_z\} = \{F\} \]

\[ [K_{uE}]^T\{u\} + [K_{EE}]\{E_z\} = \{q\} \]  \quad (A)

We can write the second of the above equation as

\[ \{E_z\} = [K_{EE}]^{-1}\{q\} - [K_{EE}]^{-1}[K_{uE}]^T\{u\} \]  \quad (a)

Using the above equation in the first equation and simplifying, we get

\[ [M_{uu}]\{\ddot{u}\} + [\bar{K}_{uu}] = \{\bar{F}\} \]  \quad (b)

\[ [\bar{K}_{uu}] = [K_{uu}] - [K_{uE}][K_{EE}]^{-1}[K_{uE}]^T, \quad \{\bar{F}\} = \{F\} - [K_{EE}]^{-1}\{q\} \]

Note that Equation (b) is only in terms of mechanical displacements, which can be solved using conventional solution techniques. Using this solution, electrical fields are obtained using Equation (a), from which voltages can be obtained.
For actuation problem, the voltages and hence the electric field goes as input. That is, the second of the Equation (A) is not required. Hence the equation that requires solution becomes

\[ [M_{uu}]\{\ddot{u}\} + [K_{uu}]\{u\} = (F) - [K_{uE}]\{E_z\} = \{F^*\} \]

If an arbitrary value of \{E_z\} is specified, the problem comes under the category of open-loop control. If the value of \{E_z\} comes from the sensor input that is fed back to the controller, then the control scheme is referred to as closed-loop control.
Numerical Examples

- The same bi-morph PVDF beam is solved using the formulated 2-D isoparametric element.
- First the static actuation is performed, that is the effect of mass is neglected.
- The beam is modeled using 300 2-D plane stress finite element
- In the first problem, the voltage is increased from 50V to 200V. This decreases the static deflection considerably
Some case studies- Sensing of cracks in a composite

- Delamination in composites is one of the crucial mode of damage
- Delamination essentially behaves as a stable crack and can be characterized by Strain Energy Release Rate (SERR)
- The main objective here is to understand the distributed sensing behavior of the piezoelectric sensor patches embedded in composites with growing delamination and subjected to static and dynamic loading

- Methods of computation of Strain Energy Release Rate (G)
  
  * Direct Method (Watwood, 1969)
  * Crack Closure Integral (Irwin)
  * Modified Crack Closure Integral (Rybicki, 1977)
  * J-integral (Contour integration) (Rice, 1968)
  * Equivalent Domain Integral (EDI) (Area integration)
Fracture mechanics of composite laminates

• For isotropic structures,

\[ G = \frac{K^2}{E'} \]

where

\[ E' = E \quad \text{(plane stress)} \]
\[ E' = \frac{E}{1 - \nu^2} \quad \text{(plane strain)} \]

• For composite structures,

\[
\begin{bmatrix}
G_1 \\
G_2
\end{bmatrix} = \left( \frac{1 + \rho}{2E_1E_2} \right)^{\frac{1}{2}} \begin{bmatrix}
l^{\frac{1}{4}}k_1^2 \\
l^{\frac{1}{4}}k_2^2
\end{bmatrix} \quad \lambda = \frac{E_1}{E_2} \quad \rho = \frac{(E_1E_2)^{\frac{1}{2}}}{2G_{12}} - (\nu_1\nu_2)^{\frac{1}{2}}
\]

• Generalized form of the Equivalent domain integral:

\[
J = \int_{S(G_0=0)} \left[ \sigma_{ij} \frac{\partial u_i}{\partial x_j} \frac{\partial \eta}{\partial x_1} - (U + T) \frac{\partial \eta}{\partial x_1} + \rho \left( \ddot{u}_i \frac{\partial u_i}{\partial x_1} - \dot{u}_i \frac{\partial \dot{u}_i}{\partial x_1} \right) \right] dS
\]
DCB model with embedded self-sensing PZT layers

• All the fibers are unidirectional (0 degrees fiber orientation)
• Delamination occurs along the mid-depth of the section
• Graphite-Epoxy and PZT-Piezoelectric sensors are used
• FE modeling: 540 nodes and 440 elements
  • Static: Mode-I and Mode-II
  • Dynamic: Mode-II

Objective
• To study the generated voltage on the sensor patch to the approaching crack tip stress fields
• The sensitivity in this case, can be defined as the ratio of EDI (J) and Equivalent Voltage (Va) from a particular sensor patch. The generated sensitivity data can be used as a calibrating parameter in SHM to detect the growing damage just by measuring the voltage from the sensor group
Sensing under mode-I static loading

The distribution of $\sigma_{xx}$ ahead and behind the crack tip
Sensing under mode-I static loading (contd..)

Front Sensor response

Bottom Sensor response
Sensing under mode-II static loading

Sensitivity for sensor patch ahead of crack tip

Voltage distribution across sensor surface for varied Delamination length
Sensing under mode-II static loading

Sensitivity for sensor patch in the upper sublaminate

Voltage distribution across sensor surface in the upper sublaminate
Sensing under mode-II static loading

Sensitivity for sensor patch in the lower sublaminate

Voltage distribution across sensor surface in the lower sublaminate
Summary

• In this lecture we covered the following:

1. We introduced the piezoelectric material constitutive model and discussed its effect on FE modeling.

2. We developed two different FE model, one based on beam modeling, where the explicit coupling between the electrical and mechanical dof is ignored. In the second model, a 2-D 4-noded isoparametric plane stress model with electrical dof in addition to mechanical dof is developed.

3. Some examples involving both these models is shown.

4. In addition, one case study on the sensing of cracks using piezoelectric material is presented.