Lecture 3.6
Unstructured Mesh Generation
Unstructured Mesh Generation Schemes

- The important steps involved in the unstructured mesh generation are:

  - Division of the geometry into elements of desired (e.g. triangular) shape and placement of nodes on the boundary and interior of the elements.

  - Numbering of nodes and elements and definition of nodal connectivity for each element. A certain order is maintained while specifying the connectivity of elements.

  - Improvement of element aspect ratios by mesh smoothing and controlling technique.

  - Development of algorithms to achieve the above steps.
Two methods of unstructured mesh generation are popular.

- **Advancing-front method**
  In this method the triangles/cells are generated sequentially from an ever shrinking set of dynamic curves, starting at the boundaries and advancing into the body interior.

- **Delaunay Triangulation**
  In this method the domain is decomposed into a set of convex (vornoi) polygons which in turn yield an unique set of near equilateral triangles.

The delaunay triangles have the property that each of their vertices is surrounded by a polygon (called vornoi polygon) whose sides are formed by perpendicular bisectors of the sides of the triangles. The vornoi polygons are non intersecting and cover the whole domain consisting of the node points.
• Among the two methods the method delaunay triangulation gives a better mesh and is therefore discussed in detail in the following sections. Although the discussion is centered around 2D space the same ideas are applicable to 3D bodies as well.

• The most widely used method of generating Delaunay triangles is by Circumcircle test or Watson algorithm.

• The following steps are used in this Delaunay triangulation method.
  (1) Initial boundary triangulation.

    (2) Boundary Delaunay triangulation.

    (3) Formation of an envelope of equilateral triangles around chosen boundaries.

    (4) Grid refinement by generation of interior nodes.

    (5) Grid smoothing
Initial Boundary Triangulation

- During initial boundary triangulation, the computational domain is filled with a set of coarse triangles using the nodes generated on the boundaries of the domain.

- The generation of boundary triangles with the given boundary discretization lists should satisfy the following four basic requirements:

  (a) Confinement condition: Boundary triangles should not be formed outside the domain.

  (b) Proximity condition: The three nodes forming the boundary triangle must be as close as possible.
(c) Non-overlapping condition: The boundary triangles should not overlap with one another.

(d) Full coverage condition: The boundary triangles should cover the entire domain.

• The initial boundary triangles generated for a turbine stator geometry using this method is given in Fig. 3.6.1.

Fig. 3.6.1: Initial boundary triangulation of a turbine stator domain
Boundary Delaunay Triangulation

• A set of triangles is said to be of the Delaunay type, if the circumcircle of any triangles does not contain any nodes other than the three nodes forming the triangle.

• As the boundary triangles formed in Fig. 3.6.1 are, in general, not the Delaunay, they should be converted into the same number of Delaunay triangles.

• From the list of triangles a pair of boundary triangles having a common edge is considered for checking their Delaunay property.

• For example, consider two triangles $T_1(1-2-3)$ and $T_2 (3-4-1)$ having common edges (1-3) and (3-1), Fig. 3.6.2(a).
Here, since the node 4 is outside the circumcircle of $T_1, T_1$ and $T_2$ are Delaunay triangles. In Fig. 3.6.2(b), the node 4 is inside the circumcircle of the triangle $T_1$, and therefore they are not Delaunay.

A diagonal swapping algorithm is then applied for making the pair of triangles Delaunay. In the diagonal swapping algorithm, the union polygon of the triangles $T_1$ and $T_2$ is obtained by removing the common edge (1-3).

![Diagram](image)

Fig. 3.6.2: Illustration of diagonal swapping algorithm (a) Delaunay triangles (b) Non-Delaunay triangles (c) Triangles after the diagonal swapping of triangles in (b).
• The union polygon is a quadrilateral (1-2-3-4); it is divided into two new triangles $D_1(1\text{-}2\text{-}4)$ and $D_2(2\text{-}3\text{-}4)$ by joining the diagonal (2-4), rather than the earlier diagonal (1-3).

• The new triangles $D_1$ and $D_2$, shown in Fig. 3.6.2 (c) are Delaunay as node 3 is outside the circumcircle of $D_1$.

• Fig. 3.6.3 shows the Delaunay triangle set obtained after implementing the above diagonal swapping algorithm on the boundary triangulation in Fig. 3.6.1.

Fig. 3.6.3: Boundary Delaunay triangulation of a turbine stator domain
Grid Refinement

• The triangles formed after boundary triangulation, albeit Delaunay, are irregular and large in size (Fig. 3.6.3).

• Therefore these triangles should be further refined by placing more number of interior nodes, so that an acceptable final mesh is obtained.

• A good refined mesh should have triangles as nearly equilateral as plausible with desirable node spacing.

• Furthermore it will be easy and convenient to implement certain boundary conditions if equilateral triangles are formed all around the boundaries.

• For instance, the non-penetrating condition through solid walls (i.e. $\partial p/\partial n = 0$) for the compressible flow through a turbine blade passage requires specification of pressure on the boundary edge of the airfoil surface.
- If an equilateral triangle is formed with a boundary edge, the value of pressure acting on the edge is simply obtained by equating it to the value of pressure at the cell center.

- In the similar fashion, implementation of any Neumann conditions and estimation of the surface gradients is easier with the use of equilateral triangles.

- However, construction of equilateral triangles throughout the domain is not practical for graded meshes (meshes of size varying from one location to other) and for complicated shapes having multiple boundaries.

- As these two factors are inherent to the turbomachinery flows, the process of refinement in the present work is performed in two stages. First, by providing an envelope of equilateral triangles along the body contour, and second, by adopting a graded mesh in the remaining portion of the domain with the help of an automatic interior node placement scheme.
Formation of an Envelop of Equilateral Triangles

- A point that is required for the formation of an equilateral triangle with a boundary edge is obtained by rotating the same edge through $60^0$ in the anti clock-wise direction about its first node.

- Such points, named as “peer nodes”, are inserted by means of the Watson algorithm for all the boundary edges that should be covered by the envelope of equilateral triangles. Figure 3.6.4 shows the partly refined Delaunay triangulation obtained after the formation of an envelope with the equilateral triangles.

Node Spacing Function and Grid Size Control

- In the refinement process, extra nodes are sought to be placed at convenient locations in the interior of the computational domain and hence more number of new triangles. The size of the triangles formed in the refinement process is controlled by a distance parameter called spacing function with the help of a spacing test.
• The spacing function of a boundary node is specified as the minimum of the distances between the node and it’s two neighbour nodes on the boundary.

• Spacing functions of all the interior nodes are calculated as a harmonic average of the spacing functions of its neighbour nodes.

• The procedure for calculating the value of spacing function is illustrated in a later section. The grid size control is obtained by means of a spacing test as described in the following.

• Consider a location arbitrarily chosen within a triangle. If the distances between the chosen location and the vertices of the triangle are greater than or equal to the values of respective spacing functions of the vertices, a node can be placed at that location. This is called spacing test.
Locations for Interior Node Placement

• In an automatic node placement scheme, interior nodes are inserted into existing triangles.

• The quality of final triangulation is very much dependent on the location considered for the node placement. Previous researchers have considered a single criterion for the location of node placement.

• Frey has suggested that an interior node may be placed inside a triangle at a location between its circumcentre and in-centre. The position of the node referred to as Frey's point is based on the normalised shape ratio of the triangle.

• Another location named as peer point, is obtained by rotating the shortest edge of the triangle through 600 in the anti clock-wise direction about its first node.
• The advantage of incorporating the peer point is that at least one equilateral triangle is obtained at a time when a new node is inserted.

• In order to test the effect of location for node placement on the final triangulation, refinement of the boundary triangulation, (Fig. 3.6.3) has been carried out by node placement at any of the aforesaid locations namely, the circumcentre, Frey's point and the peer point.

• It can be seen from Table 3.6.1 that with node placement at peer point results in 20.47% of the triangles with aspect ratio (the ratio of the length of the longest edge and shortest edge of a triangle) more than 0.9.

• With node placement at circumcentre and Frey’s point, 14.01% and 17.37% of the triangles have aspect ratio more than 0.9.
• However the total number of triangles obtained with node placement at peer point is 1783, which is less compared to circumcentre or Frey's point.

• Note that the spacing test ensures that the actual distance between neighbouring nodes is more than or equal to the spacing function.

• The actual distances between neighbouring nodes obtained for all the three locations are more than the spacing function (close to two times the spacing function in certain areas).

• This is due to the fact that many locations have not qualified the spacing test and are rejected with the single criterion for node placement.
• A good triangulation should create triangles with the distances between neighbouring nodes as close to the Spacing functions at the respective nodes as possible.

• The criteria of the node insertion scheme is to use all the above said locations together for placing a node.

• If any one location fails to satisfy the spacing test, another location will be considered.

Fig. 3.6.4: Partially refined boundary triangulation with an envelope of equilateral triangles around the blade surface.
Table 3.6.1: Comparison of cell information for different node placement schemes

<table>
<thead>
<tr>
<th>Node displacement criterion</th>
<th>% cells with Aspect ratio, A</th>
<th>Total cells</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A&gt;0.90</td>
<td>A&gt;0.80</td>
</tr>
<tr>
<td>Single location</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Circumcenter</td>
<td>14.01</td>
<td>50.13</td>
</tr>
<tr>
<td>Frey’ point</td>
<td>17.37</td>
<td>53.95</td>
</tr>
<tr>
<td>Peer point</td>
<td>20.47</td>
<td>56.02</td>
</tr>
<tr>
<td>Three location</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present method</td>
<td>22.1</td>
<td>58.40</td>
</tr>
</tbody>
</table>

- Thus the probability of a node placement inside a triangle with the proposed scheme increases when compared with any one single criterion.

- The sequence with which the three locations are to be tested for spacing is evolved after extensive experimentation with different combinations, as in the following.
• Spacing test for the peer point shall be done first and a node is placed at the peer point, if it satisfies the spacing test requirements. If not the second and the third choices for node placement locations are the circumcentre and the Frey’s point.

• The advantage of the present three-location node placement strategy in respect of the quality of triangulation is clear in Table 3.6.1.

• The percentage of the triangles above aspect ratio of 0.9, 0.8, 0.7 and 0.6 are larger for the present method as compared with single location criteria.

• A systematic way for inserting nodes into a Delaunay triangulation using the Watson’s method is explained in the following section.
Node Insertion Algorithm

- The Watson algorithm is adopted to facilitate direct and rapid formation of Delaunay triangles.

- Consider that a chosen node P is to be inserted into the existing Delaunay Triangles. In Watson algorithm, the triangles whose circumcircles containing the node P are first identified. For example in Fig. 3.6.5 (a) triangles (1-2-4), (2-3-4), (4-5-1), (5-6-1) and (6-7-1) contain P in their circumcircles.

- A union polygon called insertion polygon of these triangles, (1-2-3-4-5-6-7) is obtained as in Fig. 3.6.5 (b) by removing the common edges viz. (2-4), (4-1), (5-1) and (6-1). A set of new triangles are then formed by connecting edges of this union polygon with the point P.
• The new triangles (Fig. 3.6.5 (c)) formed in this fashion are Delaunay. Note that the difference between the number of new triangles and the triangles used for making the insertion polygon should be equal to two.

• Otherwise, the node under consideration can not be inserted, as it may be lead to overlapping of cells.

• Finally the triangle list is updated by adding the new triangles to the end of the list and by removing those triangles used for constructing the insertion polygon from the list.

Fig. 3.6.5: Watson’s node insertion algorithm
Spacing Function of an Inserted Interior Node

• The spacing function of an inserted interior node is required for refinement in the subsequent stages.

• This is calculated from the known spacing functions at the vertices of the triangle in which the node is placed.

• An arithmetic average of the spacing functions at the vertices for the spacing function at the inserted node is expected to produce sudden change in the sizes of the cells in graded meshes.

• However a harmonic averaging as given in the following is expected to give a more uniform variation of the spacing function within the domain. The spacing function at an interior node P on a triangle 1-2-3 is
\[ S_p = \frac{S_1 + S_2 + S_3}{L_{p1} + L_{p2} + L_{p3}} \]  

where \( S_1, S_2 \) and \( S_3 \) are the spacing functions at points 1, 2 and 3 respectively. \( L_{p1}, L_{p2} \) and \( L_{p3} \) are the distances between the node \( P \) and the vertices of the triangle 1, 2 and 3 respectively.

- The process of insertion of interior nodes for refinement is repeated for all the triangles recursively until further refinement is not possible.
Mesh Smoothing

• The quality of triangles produced by Delaunay triangulation is improved by a technique called Laplacian smoothing.

• The interior nodes, excluding the peer nodes used for getting the envelope of equilateral triangles, are repositioned by retaining their connectivity in the following manner.

• First, the polygons which will be enclosing all such interior nodes are identified. These interior nodes are then repositioned to the centroids of their enclosing polygons.

• Since all the interior nodes are repositioned, several iterations are needed to get converged positions for the interior nodes.
• Although smoothing generates good quality triangles, it results in two undesirable effects: (i) some of the cells may lose their Delaunay property and (ii) the spacing between nodes may change.

• The bigger triangles resulted by smoothing may have scope for further refinement.

• The refinement algorithm is then applied again to incorporate more number of nodes inside the cells that became unduly larger than the spacing function by smoothing.

• This process of smoothing and refining is continued till further addition of nodes is not possible.
• To reinstate the Delaunay property on those cells which have lost the same during smoothing, the diagonal swapping algorithm described in boundary Delaunay triangulation may be reapplied.

• Figures 3.6.6 and 3.6.7 show the triangulation before and after smoothing of the turbine stator geometry. The number of interior nodes that are added during the smoothing stage is 31.

• Consequently the number of cells have been increased from 2461 before smoothing to 2523 after smoothing. A significant improvement in the quality of triangulation is thus evident.
Fig. 3.6.6: The triangular grid before mesh smoothing
Fig. 3.6.7: The triangular grid after mesh smoothing
Unstructured mesh generation schemes with advancing front and Delaunay triangulation methods are presented. Technique for mesh smoothing is discussed with an example.