4 Method of Characteristics for PDEs

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4.1 Introduction

The solution methods for PDEs rely on converting or transforming them to ordinary differential equations (ODEs) (or algebraic equations), which are subsequently solved. Here, we will discuss the method of characteristics for solving PDEs.

4.2 First Order PDEs

We will attempt solving a quasi-linear PDE (linear in derivatives) of the form

\[ a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \]  

(4.1)
where \( u(x, y) \) is the field variable, and \( x, y \) are the coordinates. Geometrically, the solution \( u(x, y) \) of (4.1) represents a family of surfaces in the \((x, y, u)\) space, as shown in Fig 4.1. Such surfaces may be represented in a general implicit form as \( \Phi(x, y, u, c_1, c_2) = 0 \), where \( c_1 \) and \( c_2 \) are two (arbitrary) constants of integration. These constants may be determined from the boundary (or initial) conditions to uniquely determine a surface from the family.

The method of characteristics is based on the conversion (or the convertibility) of the left hand side of the PDE (4.1) to a total derivative on a family of curves in the \((x, y)\) plane, which are known as the characteristic curves or characteristics. This will lead to an ODE, which may be solved to obtain the general solution of the PDE.
Consider the ODE
\[
\frac{du(x(\tau), y(\tau))}{d\tau} = c(x, y, u)
\]
\[\Rightarrow u_x \frac{dx}{d\tau} + u_y \frac{dy}{d\tau} = c(x, y, u) \quad (4.2)
\]
Comparing (4.2) with (4.1), we can determine the parametrization \((x(\tau), y(\tau))\) of the characteristic curve, as shown in Fig. 4.2, by solving the differential equations
\[
\frac{dx}{d\tau} = a(x, y, u) \quad \text{and} \quad \frac{dy}{d\tau} = b(x, y, u). \quad (4.3)
\]
Thus, together one can write
\[
\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} = d\tau. \quad (4.4)
\]
If we represent the two independent solutions of the ODEs in (4.4) as \(v_1(x, y, u) = c_1\) and \(v_2(x, y, u) = c_2\), then the general solution of the PDE (4.1) can be written as \(\Phi(v_1, v_2) = 0\), where \(\Phi(\cdot, \cdot)\) is an arbitrary function. The form of the function \(\Phi\) is decided by the boundary (or initial) conditions.

![Fig. 4.2: Visualization of the solution procedure of a PDE using the method of characteristics](image)
Suppose the condition to be satisfied by the solution is specified as \( u = F(x, y) \) on the curve \( G(x, y) = 0 \), as shown in Fig. 4.2. We may parametrize the curve as \((x, y) = (x(s), y(s))\), and represent \( u = F[x(s), y(s)] = u(s) \). Then, the unique solution can be obtained by expressing \( v_1[x(s), y(s), u(s)] = c_1, v_2[x(s), y(s), u(s)] = c_2 \), eliminating \( s \) between them to obtain \( \Phi(c_1, c_2) = 0 \), and finally writing \( \Phi(v_1(x, y, u), v_2(x, y, u)) = 0 \).

Consider the PDE \( u_{,t} + cu_{,x} = -\alpha u \) with the initial condition \( u(x, 0) = f(x) \).

The ODEs obtained from (4.4) are solved to obtain

\[
\begin{align*}
\frac{dt}{1} &= \frac{dx}{c} = \frac{du}{-\alpha u} = d\tau \\
\frac{dx}{dt} &= c \quad \Rightarrow \quad v_1 = x - ct = c_1 \\
\frac{du}{dt} &= -\alpha u \quad \Rightarrow \quad v_2 = u = c_2 e^{-\alpha t}
\end{align*}
\]

Thus, we have a family of characteristic lines \( x - ct - c_1 = 0 \) of constant slope \( c \), as shown in Fig. 4.3. The integral \( u = c_2 e^{-\alpha t} \) on the characteristics

\[ u(x, 0) = f(x) \]

\[ u = e^{-\alpha t}f(x - ct) \]

Fig. 4.3: Visualization of the solution

the PDE \( u_{,t} - cu_{,x} = -\alpha u \)
indicates that, along the characteristics, the amplitude of the wave will diminish exponentially at a rate $e^{-\alpha t}$. We can parametrize the initial condition as $x(s) = s$, $t = 0$, and $u(s) = f(s)$. Substituting this parametrization in the solutions $v_1$ and $v_2$, we have, respectively, $s - c \cdot 0 = c_1$ and $f(s) = c_2$. Eliminating $s$ between the two, we obtain $\Phi(c_1, c_2) = f(c_1) - c_2 = 0$. Hence, the final solution is $f(x - ct) - u e^{\alpha t} = 0$, i.e., $u = e^{-\alpha t} f(x - ct)$.

The inviscid Burgers equation is given by PDE $u_{,t} + uu_{,x} = 0$ with the initial condition $u(x, 0) = e^{-x^2}$. The ODEs are obtained from (4.4), which can be solved to obtain

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$$

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad v_1 = u = c_1$$

$$\frac{dx}{dt} = u = c_1 \quad \Rightarrow \quad v_2 = x - ut = c_2$$

The family of characteristics is this case are given by $x - ut - c_2 = 0$. Here,
the slope depends on the amplitude of the wave. Thus, points in the wave with larger amplitude will travel faster, as observed in Fig. 4.4. However, the amplitude of the wave is transported undiminished along the characteristics. Due to these features, the characteristics intersect, as observed in Fig. 4.4 leading to formation of a shock (slope goes to infinity). Beyond this time (of characteristics crossing), the solution becomes ill-defined due to multivalued amplitude. The initial condition may be represented in the parametric form as \( x(s) = s, \ t = 0, \) and \( u(s) = e^{-s^2}. \) Substituting this in the solutions \( v_1 \) and \( v_2, \) we have, respectively, \( e^{-s^2} = c_1 \) and \( s - u \cdot 0 = c_2. \) Eliminating \( s \) between the two, we obtain \( \Phi(c_1, c_2) = e^{-c_2^2} - c_1 = 0. \) Hence, the final solution is \( e^{-(x-ut)^2} - u = 0. \) This solution of \( u \) is in the implicit form.

### 4.3 General Solution of the Wave Equation

In this section, we consider the wave equation \( w_{tt} - c^2 w_{xx} = 0. \) For using the method of characteristics, we convert the second order PDE to two first order PDEs using the definitions \( p = u_t \) and \( q = u_x \) to obtain

\[
    p_x - q_t = 0 \quad \text{and} \quad p_t + c^2 q_x = 0. \tag{4.5}
\]

The first of these equations is the compatibility condition. In order to obtain the characteristics on which both the PDEs are simultaneously converted to ODEs, we introduce the compatibility condition as a constraint into the
4.3 General Solution of the Wave Equation

second PDE through a Lagrange multiplier to write

\[ p_{,t} + \lambda p_{,x} - c^2 q_{,x} - \lambda q_{,t} = 0. \]  

(4.6)

In order to convert this PDE to an ODE of the form

\[ \frac{dp(t(\tau), x(\tau))}{d\tau} + \frac{dq(t(\tau), x(\tau))}{d\tau} = 0, \]

we must have, simultaneously,

\[ \frac{dt}{1} = \frac{dx}{\lambda} = d\tau \quad \text{and} \quad \frac{dt}{-\lambda} = \frac{dx}{-c^2} = d\tau \]

\[ \Rightarrow \frac{dx}{dt} = \frac{\lambda}{1} = \frac{c^2}{\lambda}, \quad \Rightarrow \lambda = \pm c. \]  

(4.7)

Thus, for two values of \( \lambda \), we can convert (4.6) to an ODE. These correspond to two characteristics with slopes \( \pm c \), as shown in Fig. 4.5. On these two characteristics, we may rewrite (4.6) as

\[ \Phi_{,t} - c\Phi_{,x} = 0 \quad \text{and} \quad \Psi_{,t} + c\Psi_{,x} = 0, \]

(4.8)

where \( \Phi = p + cq \) and \( \Psi = p - cq \). As discussed in the previous section, the solutions of the above equations with initial conditions \( \Phi(x, 0) = f(x) \) and

Fig. 4.5: Characteristics of the wave equation
Ψ(x, 0) = g(x) are given by, respectively, Φ(x, t) = f(x - ct) and Ψ(x, t) = g(x + ct). Then, one can calculate \( p = (\Phi + \Psi)/2 \) (or \( q = (\Phi - \Psi)/2c \)). It may be checked from (4.5) that \( p \) itself satisfies the wave equation \( p_{tt} - c^2 p_{xx} = 0 \). Thus, the general solution of the wave equation may be written as

\[
  w(x, t) = f(x - ct) + g(x + ct),
\]

which represents the superposition of two waves traveling in opposite directions.