38 Bäcklund transformation and multi-soliton solutions

Contents

38.1 Bäcklund transformation
   38.1.1 Auto Bäcklund transformation ........................................ 2
   38.1.2 Liouville’s equation ...................................................... 4

38.2 Auto-Bäcklund transformation for the KdV equation

38.3 Non-linear superposition and multi-soliton solution

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38.1 Bäcklund transformation

Bäcklund transformations, named after Victor Bäcklund, connect partial differential equations (PDE) and their solutions. That is to say if solution of PDE is known one can exploit the Bäcklund trasformation to generate a solution of another PDE. There is a class of Bäcklund transformations where
the connected PDEs are the same in that situation we call the class as auto-
Bäcklund transformations. The Bäcklund transformations are immensely
used in integrable theories for the generation of new solutions. They may
connect a difficult PDE to a simple one for which solutions are easy and then
one may obtain a solution of the difficult PDE through the Bäcklund trans-
formations. In case of solitons using these transformation one can generate
multi-soliton solutions from the known solutions. Many a times non-trivial
solutions are generated from trivial solutions applying Bäcklund transforma-
tion. In this chapter we learn the transformations with the help of several
examples.

38.1.1 Auto Bäcklund transformation

The standard example of an auto Bäcklund transformation is Cauchy Rie-
mann conditions.

\[ u_x = v_y, \quad u_y = -v_x. \]  \hspace{1cm} (38.1)

These equations connect Laplace equation to itself, so that both \( u \) and \( v \) are
solutions of Laplace equation, i.e.

\[ u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0. \]  \hspace{1cm} (38.2)

The above equation can be obtained easily by differentiating the (38.1).
Knowing a solution of \( u_{xx} + u_{yy} = 0 \) one can quickly obtain a solution of
\( v_{xx} + v_{yy} = 0 \) using the Bäcklund transformations (38.1).
Example 1: Let us start with a simple solution, \( v = y \) which satisfies the Laplace equation. The transformations quickly generate a solution \( u = x + c \).

Example 2: Next let us take \( v = xy \), which again is a solution of Laplace equation. The transformations (38.1) now read

\[
\begin{align*}
u_x &= v_y = x, & u_y &= -v_x = -y.
\end{align*}
\]

(38.3)

The above equations can be integrated at once to give

\[
u = \frac{1}{2} x^2 - y^2 + c.
\]

(38.4)

Example 3: As the third example consider \( u = \ln \sqrt{x^2 + y^2} \) which satisfies Laplace equation. The Cauchy Riemann conditions for this case become

\[
\begin{align*}
v_y &= u_x = \frac{x}{x^2 + y^2} & v_x &= -u_y = -\frac{y}{x^2 + y^2}.
\end{align*}
\]

(38.5)

The above can be integrated to produce \( v = \tan^{-1}(y/x) + c \).

In all the above examples we have obtained a new solution of the Laplace equation for a given solution using Bäcklund transformations (which in this case are identical to the Cauchy Riemann conditions).

Problem 1: Given \( u = e^x \cos y \), find another solution \( v \) of the Laplace equation.
38.1.2 Liouville’s equation

Now consider the following transformations,

\[ u_x + v_x = \sqrt{2} \exp\left[\frac{1}{2}(u - v)\right], \quad (38.6) \]
\[ u_y - v_y = \sqrt{2} \exp\left[\frac{1}{2}(u + v)\right]. \quad (38.7) \]

Now differentiating the first equation (38.6) w.r.t. \( y \) and the second (38.7) w.r.t. \( x \) and adding we have the Liouville’s equation

\[ u_{xy} = \exp(u), \quad (38.8) \]

while the subtraction leads to the wave equation,

\[ v_{xy} = 0. \quad (38.9) \]

So in this case Bäcklund transformations connect to different PDEs. The solutions of wave equation are easily obtained and then one can exploit them to find the solutions of the Liouville’s equation. A solution of \( v_{xy} = 0 \), is

\[ v = \phi(x) + \psi(y), \quad (38.10) \]
where $\phi$ and $\psi$ are two arbitrary functions of $x$ and $y$ respectively. Now substituting $v$ in the transformation (38.6) we get,

$$u_x + \phi_x = \sqrt{2} \exp\left[\frac{1}{2}(u - \phi - \psi)\right]$$
$$\exp[-\frac{1}{2}(u + \phi)](u_x + \phi_x) = \sqrt{2} \exp[-\phi - \frac{1}{2}\psi]$$
$$\exp[-\frac{1}{2}(u + \phi)] = -\frac{1}{\sqrt{2}} \exp(-\frac{\psi}{2}) \left[\int \exp(-\phi)dx + f(y)\right]$$
$$\exp(-\frac{u}{2}) = -\frac{1}{\sqrt{2}} \exp(\frac{\phi - \psi}{2}) \left[\int \exp(-\phi)dx + f(y)\right].$$

(38.11)

To identify $f(y)$ we integrate (38.7) w.r.t. $y$ in a similar fashion and obtain,

$$\exp(-\frac{u}{2}) = -\frac{1}{\sqrt{2}} \exp(\frac{\phi - \psi}{2}) \left[\int \exp(\psi)dy + g(x)\right],$$

(38.12)

where $g(x)$ is some function of $x$. Comparing (38.11) and (38.12) we find

$$\exp(-\frac{u}{2}) = -\frac{1}{\sqrt{2}} \exp(\frac{\phi - \psi}{2}) \left[\int \exp(-\phi)dx + \int \exp(\psi)dy\right]$$

(38.13)

giving

$$\exp(u) = \frac{2 \exp(\psi(y) - \phi(x))}{\left[\int \exp(-\phi)dx + \int \exp(\psi)dy\right]^2}.$$  

(38.14)

### 38.2 Auto-Bäcklund transformation for the KdV equation

Now we would see how the transformation works for KdV equation. Let us substitute $u = z_x$ in the KdV equation (36.8)

$$z_{tx} + 6z_xz_{xx} + z_{xxxx} = \partial_x[z_t + 3z_x^2 + z_{xx}] = 0.$$  

(38.15)
An integration results in

\[ z_t + 3z_x^2 + z_{xxx} = f(t). \]  \hspace{1cm} (38.16)

Introducing a new variable \( w \) by shifting the \( v \) as

\[ w = z - \int f(t')dt', \]  \hspace{1cm} (38.17)

and substituting in (38.16), without loss of generality, we get

\[ w_t + 3w_x^2 + w_{xxx} = 0. \]  \hspace{1cm} (38.18)

Auto-Bäcklund transformation for the above equation is the following

\[ v_x + w_x = \beta - \frac{1}{2}(v - w)^2 \] \hspace{1cm} (38.19)

\[ v_t + w_t = (v - w)(v_{xx} - w_{xx}) - 2(v_x^2 + v_xw_x + w_x^2) \] \hspace{1cm} (38.20)

The constant \( \beta \) is called the Bäcklund parameter. It can now be verified that the variable \( v \) satisfies the same equation as \( w \), i.e. the equation (38.18).

**Problem 2:** Show explicitly that \( v \) indeed satisfies the equation \( v_t + 3v_x^2 + v_{xxx} = 0 \).

We are now in a position to generate a non trivial solution \( w \) of the equation (38.18) form a trivial solution \( v = 0 \). Substituting \( v \) in equations (38.19) and (38.20),

\[ w_x = \beta - \frac{1}{2}w^2 \quad \text{and} \quad w_t = ww_{xx} - 2w_x^2. \] \hspace{1cm} (38.21)
The first equation of (38.21) can be instantly integrated to give,

\[ w = \sqrt{2\beta} \tanh[\sqrt{\frac{\beta}{2}} x + \alpha(t)]. \] (38.22)

There is a second solution which is not regular and develops singularity for the vanishing argument. It is given by

\[ \bar{w} = \sqrt{2\beta} \coth[\sqrt{\frac{\beta}{2}} x + \alpha(t)]. \] (38.23)

Substitution of \( w \) in the second equation of (38.21) determines, \( \alpha(t) = -\sqrt{2\beta} \beta t + \delta \), where \( \delta \) is the integration constant. So the solution \( w \) becomes,

\[ w = \sqrt{2\beta} \tanh[\sqrt{\frac{\beta}{2}}(x - 2\beta t - \delta \sqrt{\frac{2}{\beta}})]. \] (38.24)

Now setting \( 2\beta = c \) and \( \delta \sqrt{\frac{2}{\beta}} = x_0 \) we identify the solution \( u \) as KdV soliton (36.21),

\[ u = w_x = \frac{c}{2} \text{sech}^2[\sqrt{\frac{c}{2}}(x - ct - x_0)]. \] (38.25)

**Problem 3:** Fill intermediate steps of the above calculation.

Whereas the irregular solution reads,

\[ \bar{w} = \sqrt{2\beta} \coth[\sqrt{\frac{\beta}{2}}(x - 2\beta t - \delta \sqrt{\frac{2}{\beta}})], \] (38.26)

so

\[ \bar{u} = \bar{w}_x = -\frac{c}{2} \text{cosech}^2[\sqrt{\frac{c}{2}}(x - ct - x_0)]. \] (38.27)

The irregular solution will become useful when in the next section while constructing the multi-soliton solution by non-linear superposition.
38.3 Non-linear superposition and multi-soliton solution

We have seen in the previous section that the Bäcklund parameter, $\beta$ for KdV equation turns out to be the velocity of the soliton modulo a factor 2. Since the KdV equation is non-linear if we construct two solitons with two different velocities and simply add them then the sum would not satisfy the KdV and hence would not be a solution of KdV equation. To construct multi-soliton solution we first construct two solitons with different Bäcklund parameters (i.e. velocities) and then superpose them non-linearly. Let us first have two different solutions $w_1$ and $w_2$ with Bäcklund parameters $\beta_1$ and $\beta_2$ respectively using (38.19) from a solution $w$,

$$w_{1x}(\beta_1) = -w_x + \beta_1 - \frac{1}{2}(w_1(\beta_1) - w)^2 \quad \text{(38.28)}$$
$$w_{2x}(\beta_2) = -w_x + \beta_2 - \frac{1}{2}(w_2(\beta_2) - w)^2 \quad \text{(38.29)}$$

Now construct a third solution $w_{12}(\beta_1, \beta_2)$ from $w_2(\beta_2)$ by another Bäcklund transformation through the parameter $\beta_1$, i.e.

$$w_{12x}(\beta_1, \beta_2) = -w_{2x}(\beta_2) + \beta_1 - \frac{1}{2}(w_{12}(\beta_1, \beta_2) - w_2(\beta_2))^2 \quad \text{(38.30)}$$

Similarly constructing a fourth solution $w_{21}(\beta_2, \beta_1)$ from $w_1(\beta_1)$ by another Bäcklund transformation through the parameter $\beta_2$,

$$w_{21x}(\beta_2, \beta_1) = -w_{1x}(\beta_1) + \beta_2 - \frac{1}{2}(w_{21}(\beta_2, \beta_1) - w_1(\beta_1))^2. \quad \text{(38.31)}$$
If these two processes commute then we have \( w_{12}(\beta_1, \beta_1) = w_{21}(\beta_2, \beta_1) = W \). We now solve (38.30) and (38.31) with the help of (38.28) and (38.29) consistently. Subtracting (38.29) from (38.28) we get
\[
w_{1x}(\beta_1) - w_{2x}(\beta_2) = \beta_1 - \beta_2 - \frac{1}{2}[w_1(\beta_1) - w_2(\beta_2)] [w_1(\beta_1) + w_2(\beta_2) - 2w]. \tag{38.32}
\]
Again substituting r.h.s of (38.30) in the l.h.s of (38.31), since we have assumed the l.h.s. of both to be the same, we get
\[
w_{1x}(\beta_1) - w_{2x}(\beta_2) = \beta_2 - \beta_1 - \frac{1}{2}[w_1(\beta_1) - w_2(\beta_2)] [w_1(\beta_1) + w_2(\beta_2) - 2W]. \tag{38.33}
\]
Equating the r.h.s. of (38.32) and (38.33) one has,
\[
2(\beta_1 - \beta_2) = (W - w)[w_1(\beta_1) - w_2(\beta_2)]. \tag{38.34}
\]
From the above equation one extracts the multi-soliton solution
\[
W = w + \frac{2(\beta_1 - \beta_2)}{[w_1(\beta_1) - w_2(\beta_2)]}. \tag{38.35}
\]
Since \( W \) is constructed through successive Bäcklund transformation it will satisfy the equation
\[
W_t + 3W_x^2 + W_{xxx} = 0. \tag{38.36}
\]
**Problem 4:** Verify the above statement by direct substitution of (38.38) in the above equation. Two obtain the multi-soliton solution of KdV one has to differentiate \( W \) w.r.t. \( x \). i.e
\[
u(\beta_1, \beta_2, x, t) = W_x. \tag{38.37}
\]
When one takes the $w_1 = w$, regular one soliton solution and $w_2 = \bar{w}$, irregular solution or *vice versa* one constructs a regular two-soliton solution, i.e.,

$$W = \frac{2(\beta_1 - \beta_2)}{[w(\beta_1) - \bar{w}(\beta_2)]},$$  

(38.38)

one the other hand using both $w_1$ and $w_2$ regular one would generate an irregular solution. This process can be used iteratively to produce multi-soliton solution.

**Problem 5:** *Use (38.24) and (38.26) to construct a regular two-soliton solution of the KdV equation.*