11 Boundary conditions

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11.1 Boundary conditions for the normal components of the fields

When an electromagnetic field faces an abrupt change in the permittivity and permeability, certain conditions on electric and magnetic fields on the interface are to be respected for the continuity. These conditions of continuity are known as the boundary conditions for the electromagnetic field. Consider the pillbox in the following figure where two different media are characterised by their permittivities and permeabilities, viz \( \epsilon_1, \mu_1 \) and \( \epsilon_2, \mu_2 \). The interface is shown with a curved surface. The height of the pillbox is \( h \) and the two flat surfaces of the pillbox in two different media are shown by \( \delta A_1 \) and \( \delta A_2 \). We start with the Maxwell’s equation, \( \nabla \cdot \mathbf{D} = \rho \), and integrate it over the
Fig. 11.1: Pillbox on the interface of two media

pillbox.

\[ \int \nabla \cdot \mathbf{D} \, dV = \int \rho \, dV \]  \hspace{1cm} (11.1)

\[ \int \mathbf{D} \cdot d\mathbf{S} = \int \rho \, dV \]  \hspace{1cm} (11.2)

where \( \mathbf{S} \) is the total surface area of the pillbox. Now the left hand side of the above equation is

\[ \mathbf{D}^{(2)} \cdot \mathbf{\hat{n}}_2 \delta A_2 + \mathbf{D}^{(1)} \cdot \mathbf{\hat{n}}_1 \delta A_1 + \text{flux through the curved surface of the pillbox}, \]  \hspace{1cm} (11.3)

where the superscripts identify the fields in different media. We now reduce the height of the pillbox eventually making it to zero. In this case the area of the curved surface reduces to zero and hence the flux through it is also zero. Since there is a finite charge inside the box, \( h \to 0 \) will make \( \rho \to \infty \). So in this limit \( \rho h = \bar{\rho} = \text{finite} \) and

\[ \int \rho \, dV = \int \bar{\rho} \, dS, \]
11.2 Boundary conditions for the tangential components of the fields

where \( \bar{\rho} \) is surface charge density on the interface.

In this situation, \( \hat{n}_2 = -\hat{n}_1 = \hat{n} \) and \( \delta A_2 = \delta A_1 = \delta A \). The right hand side of the equation for small \( \delta A \) becomes \( \bar{\rho} \delta A \). Using the above relations in (11.2), we have the condition,

\[
(D^{(2)} - D^{(1)}) \cdot \hat{n} = \bar{\rho} \tag{11.4}
\]

The above condition says that there is an abrupt jump in the normal component of the displacement vector while crossing the medium if there is a non zero surface charge density on the interface. Similarly we proceed with the Maxwell equation \( \nabla \cdot B = 0 \) and obtain the following boundary condition,

\[
(B^{(2)} - B^{(1)}) \cdot \hat{n} = 0, \tag{11.5}
\]

which says the normal component of the magnetic field is always continuous.

11.2 Boundary conditions for the tangential components of the fields

Let us consider the following small closed curve, \( PQRS \), across the interface of two media. The area, \( PQRS \), has the unit vector \( \hat{b} \), which is normal to the surface \( PQRS \).

Now we start with the Maxwell’s equation, (3.2),

\[
\nabla \times E = -\frac{\partial B}{\partial t}.
\]
The scalar product of the above equation with $\hat{b} ds$ is integrated over the surface $PQRS$.

$$
\int_{PQRS} \nabla \times \mathbf{E} \cdot \hat{b} ds = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{b} ds, \quad (11.6)
$$

$$
\oint_{PQRS} \mathbf{E} \cdot d\mathbf{r} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{b} ds. \quad (11.7)
$$

The left hand side of equation (11.9) is

$$
\mathbf{E}^{(2)} \cdot \hat{t}_2 \delta l_2 + \mathbf{E}^{(1)} \cdot \hat{t}_1 \delta l_1 + \text{contributions from } QR \text{ and } SP,
$$

where $\hat{t}_2$ and $\hat{t}_1$ are the unit vectors along $PQ$ and $RS$ respectively. $\delta l_2$ and $\delta l_1$ are the lengths of $PQ$ and $RS$ respectively.

Now in the limit $QR \to 0$ and $SP \to 0$ the right hand side of the equation (11.9) vanishes (as the area of $PQRS$ vanishes) as well as in the left hand side the contributions from $QR$ and $SP$ also vanish and then we have (with $\delta l_1 = \delta l_2$)

$$
\mathbf{E}^{(2)} \cdot \hat{t}_2 + \mathbf{E}^{(1)} \cdot \hat{t}_1 = 0, \quad (11.8)
$$

$$
\hat{b} \times \hat{n}_2 \cdot (\mathbf{E}^{(2)} - \mathbf{E}^{(1)}) = \hat{b} \cdot \hat{n}_2 \times (\mathbf{E}^{(2)} - \mathbf{E}^{(1)}) = 0. \quad (11.9)
$$
Since the \( \hat{\mathbf{b}} \) arbitrary we get the following condition for the tangential component of the electric field (for \( \hat{\mathbf{n}}_2 = \hat{\mathbf{n}} \), normal to the surface),

\[
\hat{\mathbf{n}} \times (\mathbf{E}^{(2)} - \mathbf{E}^{(1)}) = 0. \tag{11.10}
\]

Similarly proceeding with the Maxwell’s equation (3.4), we get the condition on the tangential component of the \( \mathbf{H} \) field as

\[
\hat{\mathbf{n}} \times (\mathbf{H}^{(2)} - \mathbf{H}^{(1)}) = \bar{\mathbf{j}}, \tag{11.11}
\]

\( \bar{\mathbf{j}} \) is the surface current density.

**Problem 1:** Obtain the boundary condition (11.11).