Mechanics of Laminated Composite Structures

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Lecture 26
Analysis of a Laminated Composite
Lecture Overview

• Introduction

• Resultant Forces and Moments

• Piece-wise integration for calculating resultant forces and moments.
Introduction

• In previous lecture, mathematical relations have been developed, which define:
  – Variation of strains over the thickness of a laminate
  – Variation of stresses over the thickness of a laminate

• Given mid-plane strains and curvature, these relations may be used to calculate stresses in a plate.

• In a large number of real applications, we may not necessarily know mid-plane strains and curvatures for a composite, and sans their knowledge, predicting stresses in composite laminates is not possible just by using equations developed earlier.

• However, in several cases we do know the value of externally applied loads and moments on plates. Thus, there is a need to develop relations which connect mid-plane strains, mid-plane curvatures, stresses, and external forces and moments.
Consider a small part of composite plate, which is acted upon by forces and moments on its edges as shown in Fig. 26.1 due to different stresses. Here, $N_x$, $N_y$ and $N_{xy}$ are **resultant forces** per unit length acting on the edges of the composite plate in directions as shown in Fig. 26.1. Similarly, $M_x$, $M_y$ and $M_{xy}$ are **resultant moments** per unit length acting on the edges of the composite plate in directions as shown in Fig. 26.1.

**Fig. 26.1**: Forces and Moments at Mid-Plane of a Plate
Resultant Forces and Moments

- Using principles of equilibrium, we can relate stresses to force resultants by integrating appropriate stress components through the plate thickness. Thus, we get.

\[
N_x = \int_{-t/2}^{t/2} \sigma_{xx} dz \\
N_y = \int_{-t/2}^{t/2} \sigma_{yy} dz \\
N_{xy} = \int_{-t/2}^{t/2} \tau_{xy} dz
\]

(Eq. 26.1)

- In above equation, \( t \), represents the thickness of composite plate.
Resultant Forces and Moments

- Similarly, using principles of equilibrium, we can relate stresses to moment resultants by integrating appropriate products of stress components and distance from mid-plane, through the plate thickness. Thus, we get.

\[
M_x = \int_{-t/2}^{t/2} \sigma_{xx} z dz \\
M_y = \int_{-t/2}^{t/2} \sigma_{yy} z dz \\
M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z dz
\]

(Eq. 26.2)

- In these equations, unit of force resultants \((N_x, N_y\) and \(N_{xy}\)) is N/m, while that for moment resultant \((M_x, M_y\) and \(M_{xy}\)) is N. Also, Fig. 26.1 depicts conventions used for positive resultant forces and resultant moments.
Resultant Forces and Moments

![Possible Variation of Stresses, Moduli, and Stresses Over a Hypothetical 4-Layer Laminate](image)

<table>
<thead>
<tr>
<th>LAMINATE STACK UP</th>
<th>LINEAR VARIATION OF STRAIN OVER PLATE THICKNESS</th>
</tr>
</thead>
</table>
| 1  
2  
3  
4  | ![Graph 1](image)                                 |
| X     | ![Graph 2](image)                                 |
| Y     | ![Graph 3](image)                                 |
| ![Graph 4](image) | ![Graph 5](image)                                 |
| ![Graph 6](image) | ![Graph 7](image)                                 |
| ![Graph 8](image) | ![Graph 9](image)                                 |
| ![Graph 10](image) | ![Graph 11](image)                                |
| ![Graph 12](image) | ![Graph 13](image)                                |
| ![Graph 14](image) | ![Graph 15](image)                                |

- **MODULUS “JUMPS” BETWEEN TWO LAYERS, AND IS CONSTANT OVER A LAYER**
- **STRESS IS PIECE-WISE LINEAR OVER EACH LAYER THICKNESS (NOT ACROSS LAYERS)**
Resultant Forces and Moments

- Consider Fig. 26.2. Looking at it, and also realizing from previous analysis, we infer that variation of stress is:
  - Discontinuous over the thickness of the whole laminate.
  - Linearly continuous over the thickness of a single layer. Such a variation of stresses over laminate thickness.

- Thus, the integrands for resultant forces and moments, as defined in Eqs. 26.1 and 26.2 are not continuous functions of z.

- Rather, they are piece-wise continuous over the thickness of composite plate. Hence, their integration over entire thickness requires one to:
  - Piece-wise integrate the function over each lamina’s thickness.
  - Add up lamina specific integrals for all the layers.

  This is accomplished subsequently.
Resultant Forces and Moments

- Consider Fig. 26.3, which shows cross-section of the stack up of an $n$-layered orthotropic laminate. Here, the $z$ coordinate of top and bottom surfaces of $k^{th}$ laminate is $z_k$, and $z_{k-1}$. Further, as per the convention in this schematic, top most layer, with a $z$-coordinate of $t/2$ is considered the 1$^{st}$ layer, while the bottom most layer is considered the $n^{th}$ layer.
Resultant Forces and Moments

- For such a schematic, the relations of resultant forces and moments, using Eqs. 26.1 and 26.2 can be written as:

\[
\begin{align*}
\begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix} &= \int_{-t/2}^{t/2} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} \, dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{pmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \tau_{xy}^k \end{pmatrix} \, dz \\
\text{(Eq. 26.3)}
\end{align*}
\]

and,

\[
\begin{align*}
\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} &= \int_{-t/2}^{t/2} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} \, dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{pmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \tau_{xy}^k \end{pmatrix} \, dz \\
\text{(Eq. 26.4)}
\end{align*}
\]
Resultant Forces and Moments

- Equations 26.4 and 26.5 are *summations* of integrals. If there are $n$ layers in the composite, then there will be $n$ summations.

- In this way, contribution of each layer is summed up while calculating resultant forces and moments.

- After integration and summation, coordinate $z$ no longer appears in expressions for resultant forces and moments.

- These force resultants, $N_x$, $N_y$, and $N_{xy}$, and moment resultants, $M_x$, $M_y$, and $M_{xy}$, get applied on a composite plate’s mid-plane, thereby generating stresses and strains in the plate.

- It should be noted here that even though these force and moment resultant do not vary with respect to the $z$-direction, they are indeed functions of $x$ and $y$ coordinates.