State Space Approach in Modelling

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Answer of the Last Assignment

Following Mason’s law, there are two forward paths in the SFG:

\[ T_1 = G_1 \cdot G_2 \cdot G_3 \text{ and} \]
\[ T_2 = G_4 \]

There are four loops:
\[ L_1 = -G_1 \cdot H_1 \]
\[ L_2 = -G_3 \cdot H_2 \]
\[ L_3 = -G_1 \cdot G_2 \cdot G_3 \cdot H_3 \]
\[ L_4 = -G_4 \cdot H_3 \]

\[ \Delta = 1 - (L_1 + L_2 + L_3 + L_4) + L_1 \cdot L_2 \]

\[ \Delta_1 = 1 \]
\[ \Delta_2 = 1 \]

Hence, the transfer function could be expressed as \((T_1 + T_2)/ \Delta\)
The Lecture Contains

- State Space Modeling
- EOM of a SDOF system in State Space Form
- Response of a State Space System
- Examples to Solve
State-Space Modelling

The *state* of a model of a dynamic system is a set of independent physical quantities, the specification of which (in the absence of excitation) completely determines the future positions of the system.

*Dynamics* describes how the state evolves. The *dynamics* of a model is an update rule for the system state that describes how the state evolves, as a function on the current state and any external inputs.

\[
\dot{X} = \begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \vdots \\
    \dot{x}_n
\end{bmatrix} = AX(t) + BU(t)
\]
When we talk about electro-mechanical systems modeled by differential equations, such as masses and springs, electric circuits or satellites (rigid bodies) rotating in space, we can attach some additional intuition: the variables in the state should be adequate to specify the **energy** of the system.

For example, take a ball free-falling to earth: we can specify the position of the ball by specifying the height \( h \) above the ground, but we also need to include the velocity of the ball \( \frac{dh}{dt} \) to specify the total energy \( E = \frac{1}{2}m\left(\frac{dh}{dt}\right)^2 + mgh \). Therefore, the state of the ball is \( (h, \frac{dh}{dt}) \).
State Space Modelling of a Single Degree of Freedom System

- Consider a SDOF system (with mass $M$, stiffness $K$ and Damping constant $C$) such that the equation of motion corresponding to force excitation is given by:

$$M \ddot{x} + C \dot{x} + K x = F(t)$$

- The following pair of states or their linear combinations could be considered for the modelling:

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}, \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ \cdot\cdot\cdot \end{bmatrix}$$
The EOM in State Space Form

• Consider for example, the position and velocity as the state coordinates.

• The state vector could be written as:

\[
X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}
\]

• Based on these states, the EOM could be rewritten as:

\[
\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (f/m)
\]

\[
\dot{X} = AX + BU
\]

\[
A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad U = f/m
\]
Output of a State-space System

- Many a times states of a system are not directly measurable and hence are not of direct interest. For example, if you consider, state-space representation of a finite element model pertaining to a Spacecraft. The number of states could be as high as three to four thousand! However, one cannot have so many sensors to measure all the states. In such cases, we fix a feasible number of outputs that are observable/measurable.

- Suppose for a system of $n$-states there are $r$ outputs that are measurable. Then the output vector $Y(t)$ of size $r$ could be represented as a linear combination of input to the system and the states as follows:

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_r(t) \end{bmatrix} = C X(t) + D U(t)$$

Where $C$ & $D$ are constants for an LTIV system. For majority of dynamic systems it is observed that $D = 0$, meaning outputs are not directly affected by the system inputs.
Time Domain Solution for a Vector State Equation

- \( X(t) = e^{At} X_0 + \int_0^t e^{A(t-\tau)} B U(\tau) \, d\tau \)
- \( e^{At} = I + At + (At)^2/2! + (At)^3/3! \)
- \( X(s) = (sl-A)^{-1} B U(s) \)
- Find out the eigen values and eigen vectors of \( sl-A \), Obtain the transformation matrix and convert the state matrix into diagonal form
- Solve using a Discrete Time -Model
Special References for this Lecture

- *Feedback Control of Dynamic Systems* – Franklin, Powell and Naeini, Pearson

  Education Asia

- *Control Systems Engineering* – Norman S Nise, John Wiley & Sons

- *Modern Control Engineering* – K. Ogata, Prentice Hall

- Control System Design – B Friedland, Dover
Find out the EOM for the following mechanical system in state space form: