Module 1 : Dynamics of Rigid Bodies in Plane Motion; Dynamic Force Analysis of Machines

Lecture 2 : Dynamics of Rigid Bodies in Plane Motion; Dynamic Force Analysis of Machines.

Objectives

In this lecture you will learn the following

- Inverse dynamics -- Determination of actuating forces

- Forward dynamics – determination of accelerations given the actuating forces

Introduction

In this lecture, we take up the following two problems.

- Determine forces required to generate given accelerations of a mechanism.

- Determine the acceleration and motion resulting from given forces on a mechanism.

The equations of motion were derived in the previous lecture. Here we use those equations to solve the above two types of problems.

Force Determination

In many situations, it is necessary to determine the forces to be applied on a mechanism to keep it in equilibrium or to accelerate it. Both are part of the same general problem. However, we treat them separately here. First we look at how the force is specified.

Specification of Unknown Force

For an F degrees of freedom mechanism, we need to specify F number of scalar force elements as the unknowns to be determined. To give an example, in the mechanism shown in the previous lecture, as the degree of freedom is 2, we need to specify two force elements. In that case, we can say that the unknowns to be determined are the torque t 2 and the magnitude of the force F 4 . It is assumed that the direction of F 4 is known.

Equilibrium

In many situations we are interested in determining the forces that will keep a given mechanism stationary at a position. When the mechanism is at a position with no velocity, and the forces on the mechanism do not cause any acceleration, the mechanism is said to be in equilibrium in that position. The problem of finding forces causing equilibrium can be stated formally as follows.

> *Given a mechanism and its fixed kinematic and inertia parameters (i.e., link lengths, CG locations, masses etc) and the forces already on the mechanism, determine the additional forces to be applied on the mechanism to prevent it from accelerating.*

The solution can be obtained using the equations of motion of the mechanism. As the accelerations and velocities are zero, all terms of the equations involving acceleration and velocity terms disappear. The unknowns in the equation are the reaction forces and the unknown applied forces.

For the mechanism shown in Fig. 1.2.1, considering the magnitudes of F_4 and $\tau 2$ as the unknowns along with the reaction forces, we can write the nine equations of motion as

$$= R_1 - R_3 u_y$$
$$= -m_2 g + R_2 - R_3 u_x$$
$$= \tau_2 + R_4 - R_1(y_A - y_{C_2}) + R_2(x_A - x_{C_2})$$
$$+ R_3 u_y (y_B - y_{C_2}) + R_3 u_x (x_B - x_{C_2})$$
$$= R_3 u_y + R_5$$
$$= -m_3 g - R_3 u_x + R_6$$
$$= -R_4 - R_3 u_y (y_B - y_{C_3}) - R_3 u_x (x_B - x_{C_3})$$
$$- R_5(y_D - y_{C_3}) + R_6(x_D - x_{C_3})$$
$$= -R_5 + R_7 v_x$$
$$= -m_4 g - R_6 + R_7 v_y$$
$$= R_5(y_D - y_{C_4}) - R_6(x_D - x_{C_4}) - R_7 v_x (y_E - y_{C_4})$$
$$+ R_7 v_y (x_E - x_{C_4}) - F_{4_x}(y_G - y_{C_4}) + F_{4_y}(x_G - x_{C_4})$$

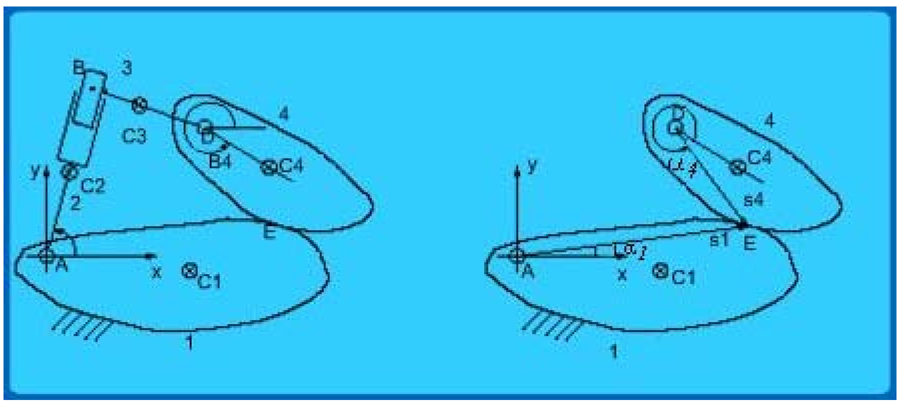


Figure 1.2.1 Typical planar mechanism

Since all the kinematic information about position of various links, their C.Gs is given, the unknowns are only reaction and applied forces namely $R_1, R_2, R_3, R_4, R_5, R_6, R_7, F_{4x}$ or F_{4y} (since direction of F_4 is assumed to be given) and τ_2. It can be seen that the number of unknowns is the same as the number of equations. Note that the unknowns appear in linear form in the equations. Hence the unknowns can be obtained by solving the system of nine linear algebraic equations, using any standard available techniques. Thus the problem of determining the forces required to maintain static equilibrium for a given position of the mechanism is fairly straight-forward.

Solution of the static equilibrium problem (also known as “static force analysis”) is also useful when viewed from another context. Normally, from a specified input-output motion requirement, a mechanism (such as a four-bar mechanism) is synthesized i.e., its link lengths are found. However, purely from this kinematics, link cross-sections and material cannot be decided since these have to be chosen such that the link can withstand the forces being transmitted. Since link cross-sections and material are as yet unknown, link masses and mass moments of inertia required in the general dynamic equations of motion are as yet unknown. Thus, from synthesis, one cannot jump to general dynamic analysis. Static force analysis helps to get some estimate of the forces on the links and the joints, when the accelerations are negligibly small. Using these estimates, we can decide tentative dimensions for link cross-sections and choose an appropriate material. Using these numbers, we can perform dynamic analysis to verify, if under dynamic conditions, the mechanism performs as desired.

Nonzero Accelerations

When the acceleration demanded is nonzero, the problem of determining forces can be stated as follows.

Given a mechanism and its fixed kinematic and inertia parameters, the forces already on the mechanism, and the velocity and acceleration of the mechanism, determine the additional forces to be applied on the mechanism to generate the required acceleration.

Here, the terms "velocity of the mechanism" and "acceleration of the mechanism" means, velocities and accelerations respectively of all links of the mechanism. However since the mechanism has only F degrees of freedom, there are only F independent velocities and accelerations. Thus one approach is to give a set of F independent velocities and accelerations, from which the velocities and acceleration of all links can be determined using standard kinematic velocity and acceleration analysis.

The problem of solving for the forces which generate the given accelerations is solved by substituting the accelerations (and velocities) in the equations of motion and then solving for the unknown reaction forces and additional unknown forces. If reaction dependent friction forces are absent, the unknowns occur in the linear form and hence solution is easy.

Acceleration Determination

Determination of acceleration when the forces are given is necessary to determine the motion of mechanisms subjected to forces. This problem can be stated as follows.

> *Given a mechanism and its fixed kinematic and inertia parameters, all the applied forces on the mechanism, and its position and velocity, determine the acceleration of the mechanism.*

Once the accelerations at this instant of time (t = 0) are found, we can integrate forward in time and determine the possible position and velocity of the mechanism at the next instant of time $(t = \Delta t)$. Again, knowing the position and velocity and the forces, the acceleration can be determined at Δt. Repeating these steps, we can find the complete motion history of the mechanism, from the forces (as a function of time) and initial position/velocity. This problem can also be solved using the equations of motion of the mechanism. However, if we count the unknowns and the number of equations of motion, we find that the unknowns are more than the number of equations. If there are n links, there are n - 1 moving links and hence there are 3(n - 1) equations. The unknowns in the equations of motion are the accelerations which are 3(n - 1) in number and the reaction forces which are equal in number to the number of constraints in the mechanism.

In the particular mechanism example we have been discussing, the unknowns are -- $R_1, R_2, R_3, R_4, R_5, R_6, R_7$, $\ddot{x}_{c2}, \ddot{y}_{c2}, \ddot{\theta}_{c2}, \ddot{x}_{c3}, \ddot{y}_{c3}, \ddot{\theta}_{c3}, \ddot{x}_{c4}, \ddot{y}_{c4}, \ddot{\theta}_{c4}$ and there are nine equations of motion. Thus we need to find more equations to be able to solve the system. We do know that this is a two degree of freedom mechanism and therefore there are only two independent accelerations and seven constraint equations that tie together all others. Therefore we can use the kinematic constraints to generate the necessary additional equations. The number of kinematic constraints and the number of reaction forces are the same – each constraint prevents some motion and hence sets up some reaction force or moment.

Consider the two degrees of freedom example mechanism we have been discussing. To simplify the equations, let us make several simplifications in the mechanism. The global reference frame attached to the fixed link has its origin at A (see Fig.1.2.1). The line on link 2, on which the point B of link 3 is constrained to move, passes through A . The center of mass C_2 is on this line, at a distance r_2 from A . The line BD on link 3 is perpendicular to the line AC_2 on link 2. The center of mass C_3 is on line BD , at a distance r_3 from B . The center of mass C_4 is at a distance r_4 from D . Let the fixed link length BD be called l_3.

Consider as position variables, the angles θ_2 and θ_4 and the distance l_2 from A to B . Here θ 4 is defined as the absolute angle made by the vector DC_4. In addition to the above position variables, to formulate the contact constraint of the cam-follower joint, we introduce the variables (s_1, α_1) which locate the point of contact E on link 1, and (s_4, α_4) which locate the point of contact E on link 4. Note that s_1 is dependent on α_1 and s_4 is dependent on α_4 as the cam profile is given . We can now write the following constraint equations relating the above variables to the Cartesian coordinates of the links (note that some of the former are identical to some of the latter).

$$
\begin{aligned}
x_{C_2} &= r_2 \cos\theta_2 \\
y_{C_2} &= r_2 \sin\theta_2 \\
x_{C_3} &= l_2 \cos\theta_2 + r_3 \cos\theta_3 \\
y_{C_3} &= l_2 \sin\theta_2 + r_3 \sin\theta_3 \\
\theta_3 &= \theta_2 - \frac{\pi}{2} \\
x_{C_4} &= l_2 \cos\theta_2 + l_3 \cos\theta_3 + r_4 \cos\theta_4 \\
y_{C_4} &= l_2 \sin\theta_2 + l_3 \sin\theta_3 + r_4 \sin\theta_4
\end{aligned} \qquad (2)
$$

We can write three constraints related to the cam-follower joint as follows. The first two constraints (i.e., eq (3)) state that the global location of the point of contact E on the two links be identical.

$$
\begin{aligned}
s_1 \cos\alpha_1 &= l_2 \cos\theta_2 + l_3 \cos\theta_3 + s_4 \cos(\theta_4 + \alpha_4) \\
s_1 \sin\alpha_1 &= l_2 \sin\theta_2 + l_3 \sin\theta_3 + s_4 \sin(\theta_4 + \alpha_4)
\end{aligned} \qquad (3)
$$

The last constraint states that the two curves are tangential to each other. For this, we use the normal $\hat{n}_1$ to the profile of link 1 at E and the tangent $\hat{n}_4$ to the profile of link 4 at E . The tangency condition is

$$
n_1^T \hat{t}_4 = 0 \qquad (4)
$$

The expressions for

$$
\hat{n}_1 = \begin{Bmatrix} -s_1^t \sin\alpha_1 - s_1 \cos\alpha_1 \\ s_1^t \cos\alpha_1 - s_1 \sin\alpha_1 \end{Bmatrix}
$$

$$
\hat{t}_4 = \begin{Bmatrix} s_4^t \cos(\theta_4 + \alpha_4) - s_4 \sin(\theta_4 + \alpha_4) \\ s_4^t \sin(\theta_4 + \alpha_4) + s_4 \cos(\theta_4 + \alpha_4) \end{Bmatrix}
$$

$\hat{n}_1$ and $\hat{t}_4$ are (5)

Where $s_1' = ds_1/d\alpha_1$ and $s_4' = ds_4/d\alpha_4$

Now equation (4) can be written as

$$
\begin{aligned}
&\left(s_1' \sin\alpha_1 + s_1 \cos\alpha_1\right)\left(s_4' \cos(\theta_4 + \alpha_4) - s_4 \sin(\theta_4 + \alpha_4)\right) = \\
&\left(s_1' \cos\alpha_1 - s_1 \sin\alpha_1\right)\left(s_4' \sin(\theta_4 + \alpha_4) + s_4 \cos(\theta_4 + \alpha_4)\right)
\end{aligned} \qquad (6)
$$

Recapitulating our discussion thus far, we have nine position variables x_{C2}, y_{C2} etc and we have introduced additional variables l_2, α_1, α_4. We also have seven reactions as unknowns. On the other hand, we have nine equations of motion and the above ten constraint equations (seven contained in eq (2); two contained in eq (3) and one in eq (6)).

The constraint equations in position coordinates are differentiated twice with respect to time to get constraint equations in velocities and accelerations. These are actually the equations used in any standard kinematic position/velocity/acceleration analysis. It is to be observed that the original equations of motion are differential equations. The constraint equations in position coordinates are algebraic equations. So also the velocity and acceleration equations. Thus, for forward dynamics problems, we need to solve a set of differential – algebraic equations. Recall that inverse dynamic analysis (i.e., given all the kinematic variables the problem of finding the actuating forces) involved only algebraic equations. Thus forward dynamics problem of simulating the mechanism motion is far more involved than the inverse dynamic problem.

We will briefly illustrate the forward dynamics problem on a simple example problem.

Example: 2-R planar manipulator

Consider the two link planar manipulator moving in horizontal plane as shown in Fig. 1.2.2. The free body diagrams are shown in Fig. 1.2.3. The equations of motion for the two moving bodies are given by:

$$m_1 \ddot{x}_{G1} = R_{Ox} + R_{Ax} \quad \text{(E1)}$$

$$m_1 \ddot{y}_{G1} = R_{Oy} + R_{Ay} \quad \text{(E2)}$$

$$I_{G1} \ddot{\theta}_1 = T_1 + \left(R_{Ox} - R_{Ax}\right)\frac{l_1}{2}\sin\theta_1 + \left(R_{Ay} - R_{Oy}\right)\frac{l_1}{2}\cos\theta_1 \quad \text{(E3)}$$

$$m_2 \ddot{x}_{G2} = -R_{Ax} \quad \text{(E4)}$$

$$m_2 \ddot{y}_{G2} = -R_{Ay} \quad \text{(E5)}$$

$$I_{G2} \ddot{\theta}_2 = T_2 + \left(-R_{Ax}\right)\frac{l_2}{2}\sin\theta_2 + \left(R_{Ay}\right)\frac{l_2}{2}\cos\theta_2 \quad \text{(E6)}$$

The unknowns for a forward dynamics simulation problem are

$$\ddot{x}_{G1}, \ddot{y}_{G1}, \ddot{\theta}_1, \ddot{x}_{G2}, \ddot{y}_{G2}, \ddot{\theta}_2, R_{Ox}, R_{Oy}, R_{Ax}, R_{Ay}$$

It is observed that there are 10 unknowns in six equations of motion. It is a two degree of freedom mechanism. Thus even though we have taken six coordinates for the two moving links, only two of these are independent. Let us choose them to be θ_1 *and* θ_2. Other coordinates can be expressed in terms of these two independent coordinates through constraint equations. There are four constraint equations and correspondingly four reaction force unknowns. The kinematic constraint equations are given as:

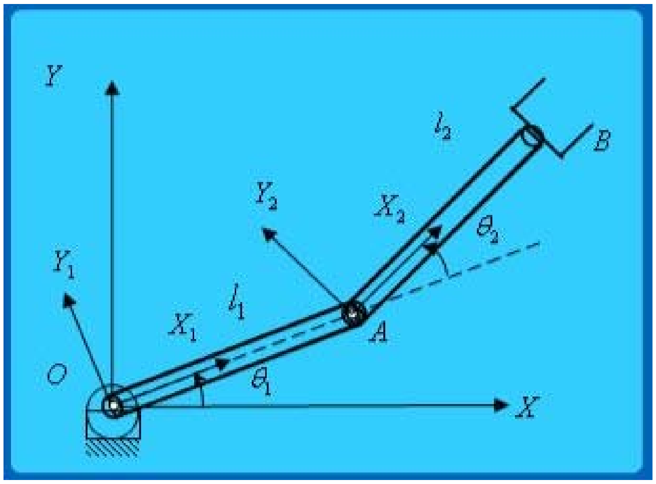
$$x_{G1} = \left(\frac{l_1}{2}\right)\cos\theta_1 \quad \text{(E7)}$$

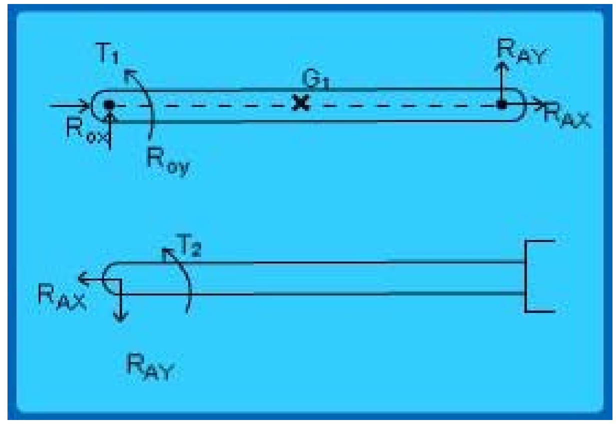
$$y_{G1} = \left(\frac{l_1}{2}\right)\sin\theta_1 \quad \text{(E8)}$$

$$x_{G2} = \left(l_1\right)\cos\theta_1 + \left(\frac{l_2}{2}\right)\cos\theta_2 \quad \text{(E9)}$$

(E10)

$$y_{G2} = (l_1)\sin\theta_1 + \left(\frac{l_2}{2}\right)\sin\theta_2$$





Let us explicitly work out, how these constraint equations can be used in the solution process. One strategy is to keep them as algebraic equations and solve a set of Differential (equations of motion) and Algebraic (constraint) equations. Other approach is to differentiate the constraint equations once for velocity and a second time for acceleration equations. We illustrate this second approach here. Differentiating equation (E7-E10) once, we get the velocity equations as:

$$\dot{x}_{G1} = -\frac{l_1}{2}(\sin\theta_1)\dot{\theta}_1 \quad \text{(E11)}$$

$$\dot{y}_{G1} = \frac{l_1}{2}(\cos\theta_1)\dot{\theta}_1 \quad \text{(E12)}$$

$$\dot{x}_{G2} = -l_1(\sin\theta_1)\dot{\theta}_1 - \frac{l_2}{2}(\sin\theta_2)\dot{\theta}_2 \quad \text{(E13)}$$

$$\dot{y}_{G2} = l_1(\cos\theta_1)\dot{\theta}_1 + \frac{l_2}{2}(\cos\theta_2)\dot{\theta}_2 \quad \text{(E14)}$$

Further differentiation yields the acceleration equations as follows:

$$\ddot{x}_{G1} = -\frac{l_1}{2}(\cos\theta_1)\dot{\theta}_1^2 - \frac{l_1}{2}(\sin\theta_1)\ddot{\theta}_1 \quad \text{(E15)}$$

$$\ddot{y}_{G1} = -\frac{l_1}{2}(\sin\theta_1)\dot{\theta}_1^2 + \frac{l_1}{2}(\cos\theta_1)\ddot{\theta}_1 \quad \text{(E16)}$$

$$\ddot{x}_{G2} = -l_1(\cos\theta_1)\dot{\theta}_1^2 - l_1(\sin\theta_1)\ddot{\theta}_1 - \frac{l_2}{2}(\cos\theta_2)\dot{\theta}_2^2 - \frac{l_2}{2}(\sin\theta_2)\ddot{\theta}_2 \quad \text{(E17)}$$

$$\ddot{y}_{G2} = -l_1(\sin\theta_1)\dot{\theta}_1^2 + l_1(\cos\theta_1)\ddot{\theta}_1 - \frac{l_2}{2}(\sin\theta_2)\dot{\theta}_2^2 + \frac{l_2}{2}(\cos\theta_2)\ddot{\theta}_2 \quad \text{(E18)}$$

Substituting for R_{Ax} *and* R_{Ay} from equations (E4) and (E5) into (E6) and using equations (E17) and (E18) in equation (E6) we get the following:

$$\left[\frac{m_2 l_1 l_2}{2}\cos\left(\theta_2-\theta_1\right)\right]\ddot{\theta}_1+\left[I_{G2}+\frac{m_2 l_2^2}{4}\right]\ddot{\theta}_2=T_2-\left[\frac{m_2 l_1 l_2}{2}\sin\left(\theta_2-\theta_1\right)\right]\dot{\theta}_1^2 \qquad \text{(E19)}$$

Similarly, substituting for R_{Ax} *and* R_{Ay} from equations (E4) and (E5) into (E3) and R_{Ox} and R_{Oy} from equations (E1) and (E2) into (E3) and using equations (E15)-(E18) in

$$\left[I_{G1}+\frac{m_1 l_1^2}{4}+m_2 l_1^2\right]\ddot{\theta}_1+\left[\frac{m_2 l_1 l_2}{2}\cos\left(\theta_2-\theta_1\right)\right]\ddot{\theta}_2=T_1+\left[\frac{m_2 l_1 l_2}{2}\sin\left(\theta_2-\theta_1\right)\right]\dot{\theta}_2^2 \qquad \text{(E20)}$$

equation (E3) we get the following:

These are the two independent equations of motion in the two independent degrees of freedom namely θ_1 *and* θ_2. All the substitutions etc. that have been carried out, may also be done automatically in a formal computer program. Now the solution process proceeds as follows. To begin with we are given the parameters of link lengths, masses etc; we are also given the position and velocity on both θ_1 *and* θ_2; we are also given the torques $T_1(t)$ *and* $T_2(t)$. Using equations (E19 – E20), we can find the accelerations $\ddot{\theta}_1$ *and* $\ddot{\theta}_2$ at this instant of time. Over a sufficiently small time interval Δt, changes in velocity $\dot{\theta}_1$ *and* $\dot{\theta}_2$ and consequently positions θ_1 and θ_2 can be estimated. Using these new position and velocity variables at time $\left(t+\Delta t\right)$, and appropriate values of torques, we can find the accelerations again using equations (E19-E20). This process is repeated for the entire time duration of simulation. At each instant of time, the constraint equations (E7-E10); (E11-E14) and (E15-E18) can be used to estimate other position, velocity and acceleration variables. Using these in equations (E1-E6), we can find the reaction forces at each instant of time. That completes the solution process.

Recap

In this lecture you have learnt the following

- Statement of forward and inverse dynamic problems
- Inverse dynamics problems (in the absence of complications such as friction) are posed as linear algebraic equations and hence readily solved
- Forward dynamics problems where the forces are specified and resulting accelerations for a given position, velocity of the mechanism are to be determined get posed as differential equations which need to be solved for finding the accelerations. Integrating these acceleration will yield the position and velocity information at the next instant of time.

Congratulations, you have finished Lecture 2. To view the next lecture select it from the left hand side menu of the page