ME-662 CONVECTIVE HEAT AND MASS TRANSFER

A. W. Date
Mechanical Engineering Department
Indian Institute of Technology, Bombay
Mumbai - 400076
India

LECTURE-35 BOUNDARY LAYER FLOW MODEL
Definitions

Governing Equations

Conserved property eqns for all types of mass transfer

Boundary Conditions

- Mass Conservation Principle
- Energy Conservation Principle

\[ N_w = g \times B \] for small and large mass transfer rates
Conv MT takes place due to concentration gradients of the transferred species.

Since Reynolds flow model mimics the real flow, Interface mass transfer flux $N_w$ (kg/m$^2$-s) from

$$N_w = g B$$

$N_w$ is positive when mass transfer takes place from the neighbouring phase into the considered phase across the interface & vice versa.
Assuming Steady-state mass transfer

\[
\frac{\partial}{\partial x} (\rho_m u \psi) + \frac{\partial}{\partial y} (\rho_m v \psi) = \frac{\partial}{\partial y} \left[ \Gamma \psi \frac{\partial \psi}{\partial y} \right] + S_\psi
\]

<table>
<thead>
<tr>
<th>\psi</th>
<th>\Gamma_\psi</th>
<th>S_\psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>u</td>
<td>\mu_{m,\text{eff}}</td>
<td>- dp / dx</td>
</tr>
<tr>
<td>\omega_k</td>
<td>\rho_m D_{\text{eff}}</td>
<td>R_k</td>
</tr>
<tr>
<td>\eta_\alpha</td>
<td>\rho_m D_{\text{eff}}</td>
<td>0</td>
</tr>
<tr>
<td>h_m</td>
<td>k_{m,\text{eff}} / cp_m</td>
<td>- \partial (\sum m''_{y,k} h_k) / \partial y</td>
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Bulk Mass
Momentum
Species transfer
Element transfer
Energy

where \( m''_{y,k} = - \rho_m D_{\text{eff}} \partial \omega_k / \partial y \). Sources \( \text{Dp}/\text{Dt}, \dot{Q}_{\text{rad}}, \dot{Q}_{\text{others}} \) and \( \mu_{\text{eff}} (\partial u / \partial y)^2 \) are ignored in the energy equation. All equations are coupled requiring numerical solutions. Simplifications of \( \omega_k \) and \( h_m \) equations are possible under certain assumptions so that they are rendered to conserved property equations.
Conserved Property Eqn - L35(\frac{3}{14})

\[
\frac{\partial (\rho_m u \psi)}{\partial x} + \frac{\partial (\rho_m v \psi)}{\partial y} = \frac{\partial}{\partial y} \begin{bmatrix} \Gamma \psi \\ \partial \psi \end{bmatrix}
\]

\[N_w = g \times B \quad \text{with} \quad B = \frac{\psi_\infty - \psi_w}{\psi_w - \psi_T}\]

1. In Inert MT without HT, \( \psi = \omega_v \) and \( \Gamma = \rho_m D \)
2. In Inert MT with HT, \( \psi = \omega_v \) and \( h_m \) and
   \( \Gamma_{mh} = \rho_m D = \rho_m \alpha_m \) with \( \text{Le} = 1 \)
3. In MT with SCR, \( \psi = \) appropriate \( \Phi \) and \( h_m \) and
   \( \Gamma_{mh} = \rho_m D = \rho_m \alpha_m \) with \( \text{Le} = 1 \) and equal \( c_{p,k} = c_{pm} \)
4. In MT with ACR, \( \psi = \) appropriate \( \Phi \) and \( \Gamma_m = \rho_m D \)

In each case, we need Boundary Conditions at \( y = 0 \) in w-state.
For Inert Mass Transfer, consider mass conservation between T- and w-states. Then

\[ N_w \omega_{k,T} = N_w \omega_{k,w} - \rho_m D \frac{\partial \omega_k}{\partial y} \bigg|_w \]

\[ N_w = \frac{\rho_m D \left( \frac{\partial \omega_k}{\partial y} \right)_w}{\omega_{k,w} - \omega_{k,T}} \]

For Conserved property \( \Phi \)

\[ N_w \Phi_T = N_w \Phi_w - \rho_m D \frac{\partial \Phi}{\partial y} \bigg|_w \]

\[ N_w = \frac{\rho_m D \left( \frac{\partial \Phi}{\partial y} \right)_w}{\Phi_w - \Phi_T} \]

where \( \Phi = \omega_{fu} - \omega_{O_2}/r_{st} = \omega_{fu} + \omega_{pr}/(1 + r_{st}) \) for an SCR
or \( \Phi = \sum a_{\alpha} \eta_{\alpha} \) and \( a_{\alpha} \) are suitable chosen coefficients for an ACR
Consider control volume between $T$- and $w$- states. Then

$$N_w \ h_{m,T} = N_w \ h_{m,w} + \left( \sum_k - \rho_m D \frac{\partial \omega_k}{\partial y} \right)_{w, k} - q_w \ \text{where}$$

$$q_w = k_m \frac{\partial T}{\partial y} w = c_{pm} \Gamma_h \frac{\partial T}{\partial y} w \ \text{hence}$$

$$N_w = \left( \sum_k \Gamma_m \left( \frac{\partial \omega_k}{\partial y} \right) w, k \right) + c_{pm} \Gamma_h \left( \frac{\partial T}{\partial y} \right)_w$$

This is the general energy conservation principle. The final form of the Numerator will depend on mass transfer application.
For Inert MT with HT, \( \text{Le} = 1 \) gives \( \Gamma_h = \Gamma_m \). Hence

\[
c_{pm} \Gamma_h (\frac{\partial T}{\partial y})_w = \Gamma_h (\sum_k \omega_k c_{p,k}) (\frac{\partial T}{\partial y})_w = \Gamma_h (\sum_k \omega_k \frac{\partial h_k}{\partial y})_w
\]

Hence,

\[
N_w = \frac{\Gamma_m (\sum_k (\frac{\partial \omega_k}{\partial y})_w h_k) + \Gamma_h (\sum_k \omega_k \frac{\partial h_k}{\partial y})_w}{h_{m,w} - h_{m,T}}
\]

\[
= \frac{\Gamma_{mh} (\sum_k \{\frac{\partial (\omega_k h_k)}{\partial y}\}_w)}{h_{m,w} - h_{m,T}}
\]

\[
N_w = \frac{\Gamma_{mh} (\frac{\partial h_m}{\partial y})_w}{h_{m,w} - h_{m,T}}
\]
For MT with HT and SCR, taking $\text{Le} = 1$, $c_{p,k} = c_{pm}$ and $\Delta T = (T - T_{\text{ref}})$, we have

$$h_{fu} = c_{pm} \Delta T + \omega_{fu} \Delta h_c \quad \text{and} \quad h_{O_2} = h_{pr} = c_{pm} \Delta T$$

Hence,

$$\left( \sum_k \Gamma_m \left( \frac{\partial \omega_k}{\partial y} \right)_w h_k \right) = \Gamma_m \left( \frac{\partial \omega_{fu}}{\partial y} \right)_w \Delta h_c$$

because

$$c_{pm} \Delta T \Gamma_h \sum_k \left( \frac{\partial \omega_k}{\partial y} \right)_w = 0$$

and

$$c_{pm} \Gamma_h \left( \frac{\partial T}{\partial y} \right)_w = \Gamma_h \left( \frac{\partial h_m}{\partial y} \right)_w - \Gamma_h \Delta h_c \left( \frac{\partial \omega_{fu}}{\partial y} \right)_w$$

Hence, substitution with $\Gamma_m = \Gamma_h = \Gamma_{mh}$ gives

$$N_w = \frac{\Gamma_{mh} \left( \frac{\partial h_m}{\partial y} \right)_w}{h_{m,w} - h_{m,T}}$$
Finally, for single component convective mass transfer

\[
\left( \sum_k \Gamma_m \left( \frac{\partial \omega_k}{\partial y} \right)_w h_k \right) = 0 \quad \text{because} \quad \omega_k = 1
\]

and \( c_{pm} \Gamma_h \left( \frac{\partial T}{\partial y} \right)_w = \Gamma_h \left( \frac{\partial h_m}{\partial y} \right)_w \)

Hence

\[
N_w = \frac{\Gamma_h (\partial h_m/\partial y)_w}{h_{m,w} - h_{m,T}}
\]

If specific heats in all states are equal

\[
N_w = \frac{\Gamma_h (\partial T/\partial y)_w}{T_w - T_T}
\]
Thus in all cases of mass transfer, mass and energy conservation principles give identical formula for \( N_w \)

Combining with Reynolds flow model which claims to mimic the real boundary layer flow model, we have

\[
N_w = \frac{\Gamma \left( \frac{\partial \psi}{\partial y} \right)_w}{\psi_w - \psi_T} = g \times \left[ \frac{\psi_\infty - \psi_w}{\psi_w - \psi_T} \right] = (\rho m V)_w
\]

Hence,

\[
N_w \propto (\psi_\infty - \psi_w) \propto \Gamma \left( \frac{\partial \psi}{\partial y} \right)_w \propto V_w
\]

This shows that even when \( \Gamma \) is uniform, the \( \psi \)-eqn is non-linear because velocity field ( \( u \) and \( v \) ) is a function of \( V_w \) and \( (\psi_\infty - \psi_w) \). This is akin to Natural Convection in which \( u \) and \( v \) are functions of \( (T_\infty - T_w) \).
In Natural convection, the momentum and energy eqns are coupled through buoyancy source in the momentum eqn. In contrast, in Mass transfer, momentum, energy and species eqns are coupled through boundary conditions.

The coupling between momentum and $\Psi$-eqns can be ignored when $N_w \propto V_w \to 0$. Thus

$$g^* \equiv \left( \frac{N_w}{B_\Psi} \right)_{N_w \to 0} = -\Gamma_\Psi \left( \frac{\partial \Psi}{\partial y} \right)_w \frac{\psi_w - \psi_\infty}{\psi_w - \psi_\infty}$$

where $g^*$ now depends only on the $\Psi$-profiles. This definition is analogous to that used to define heat transfer coefficient.

When $N_w$ is large, coupling is strong and $N_w = g \times B_\Psi$. Hence, $g$ must be a function of $B_\Psi$ and $g^* \equiv g_{B_\Psi \to 0}$.
By analysing Experimental data on mass transfer with and without combustion, Spalding\(^1\) showed that within experimental scatter,

\[
\frac{g}{g^*} = \frac{N_w/B}{(N_w/B)_{N_w\to 0}} = F(B) \text{ only}
\]

This eqn shows that \((g/g^*)\) is not influenced by Re, Pr or Sc numbers. The validity of this assertion will be examined later.

Thus, all that is required is the value of \(g^*\) (evaluated from \(h_{cof,v_w=0}/c_{pm}\)) and \(F(B)\) to obtain \(g\).

\(^1\)Spalding D B Convective Mass Transfer, Edward Arnold Ltd, London (1963)
Using computer solutions of the BL eqns as well as experimental data, Spalding showed that

\[
\frac{g}{g^*} = F(B) \approx \frac{\ln (1 + B)}{B}
\]

This relationship was also predicted by the Stefan and Couette flow models.

Consider 2 surfaces \( y_0 \) and \( y_i \) in the considered phase. Let local Reynolds flux \( g^{**} \) cross the \( y_o \) surface carrying properties at \( y_o \). Similarly, let Reynolds flux \( g^{**} + N_w \) cross the \( y_o \) surface carrying properties at \( y_i \).
The physical idea behind introduction of $g^{**}$ is that real flow processes like heat conduction, mass diffusion, turbulence etc do behave like the Reynolds flow but on a much smaller scale $\Delta y = (y_o - y_i) \to 0$.

Thus, writing mass conservation over $y_0$- and T-states

$$N_w \Psi_T + g^{**} \Psi_{yo} = (g^{**} + N_w) \Psi_{yi} \to \frac{N_w}{g^{**}} = \frac{\Psi_{yo} - \Psi_{yi}}{\Psi_{yi} - \Psi_T} = \frac{d \Psi_y}{\Psi_{yi} - \Psi_T}$$

Considering a large number of $\Delta y$ between $\infty$- and $w$-states

$$N_w \sum_{w}^{\infty} \frac{1}{g^{**}} = \int_{0}^{\infty} \frac{d \Psi_y}{\Psi_{yi} - \Psi_T} = \ln \left[ 1 + \frac{\Psi_{\infty} - \Psi_{w}}{\Psi_{w} - \Psi_T} \right] = \ln(1 + B_{\Psi})$$
Thus, as $B_\Psi \to 0$, $\sum_{w} (g^{**})^{-1} = B_\Psi/N_w$

Therefore, comparison with the observation of slide 11 shows that as $B_\Psi \to 0$, $\sum_{w} (g^{**})^{-1} = (g^*)^{-1}$. Hence,

$$N_w = g \times B_\Psi = g^* \ln (1 + B_\Psi)$$ and $$\frac{g}{g^*} = F(B) = \frac{\ln (1 + B_\Psi)}{B_\Psi}$$

This formula can be used for large mass transfer rates obtained in liquid-fuel burning and transpiration cooling. Small mass transfer rates are encountered in Combustion of solid fuel or evaporative cooling/air-conditioning.

The validity of the formula will be checked in the next lecture.