ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-27 TURBULENCE MODELS-2
1. Low $Re_t$ Two-Eqn model
2. High $Re_t$ Stress-Eqn model
3. Low $Re_t$ Stress-Eqn model
4. Algebraic Stress-Eqn model
5. Scalar Transport model
   1. Eddy Diffusivity model
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6. Modeling Combustion and Turbulence Interaction
Low $Re_t$ e-$\epsilon$ model L27($\frac{1}{19}$)

For low $Re_t = \nu_t/\nu$

\[
\rho \frac{De}{\partial t} = \frac{\partial}{\partial x_i} \left\{ \left( \mu + \frac{\mu_t}{\sigma_e} \right) \frac{\partial e}{\partial x_i} \right\} + \mu_t \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \frac{\partial u_i}{\partial x_j} - \rho \epsilon^* \\
\rho \frac{D\epsilon^*}{\partial t} = \frac{\partial}{\partial x_i} \left\{ \left( \mu + \frac{\mu_t}{\sigma_\epsilon} \right) \frac{\partial \epsilon^*}{\partial x_i} \right\} \\
- \frac{\epsilon^*}{e} \left\{ C_1 \mu_t \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \frac{\partial u_i}{\partial x_j} - C_2 \rho \epsilon^* \right\} \\
+ 2 \nu \mu_t \left( \frac{\partial^2 u_i}{\partial x_k \partial x_l} \right)^2 \\
\mu_t = C_D^* \left( \frac{\rho e^2}{\epsilon^*} \right), \quad C_D^* = C_D \exp \left\{ \frac{-3.4}{(1 + Re_t/50)^2} \right\} \\
C_1^* = C_1, \quad C_2^* = C_2 \left[ 1 - 0.3 \exp \left\{ - Re_t^2 \right\} \right] \\
\epsilon^* = \epsilon - 2 \nu \left( \frac{\partial e^{0.5}}{\partial x_i} \right)^2
\]
Model constants\(^1\) are sensitised to low \(Re_t\) region near the wall. They tend to high \(Re_t\) values beyond sub-layers.

The correction to \(C_D\) is chosen to give values of \(\nu_t\) in agreement with the Van-Driest mixing length formula.

The correction to \(C_2\) is selected from exptl. data on the decay of isotropic turbulence at low \(Re_t\) (at large times, \(e \propto t^{-n}\) where \(n \simeq 2.5\) to 2.8).

The correction to \(\epsilon\) is introduced to account for the non-isotropic contribution to the dissipation.

Wall-functions are no longer necessary and \(e\) and \(\epsilon\) Eqns can be solved with \(e_{\text{wall}} = \epsilon_{\text{wall}}^* = 0\). However, to capture the low \(Re_t\) effects, very fine mesh (\(> 60\) grid nodes) become necessary in the \(y^+ < 100\) region.

Stress Eqn Model- L27(\frac{3}{19})

Six transport equations for the one-point correlation $u'_i u'_j$ are derived from equation for $B_{ij}$ by setting separation $\xi_k = 0$ (lecture 23)

$$\frac{D u'_i u'_j}{Dt} = - \left[ \frac{u'_j u'_k}{u'_i u'_j} \frac{\partial u_i}{\partial x_k} + \frac{u'_i u'_k}{u'_i u'_j} \frac{\partial u_j}{\partial x_k} \right]$$

$$\{ P_{ij} \}$$

$$- \frac{\partial}{\partial x_k} \left[ \frac{u'_i u'_j u'_k}{u'_i u'_j} + \frac{p'}{\rho} \left\{ u'_i \delta_{jk} + u'_j \delta_{ik} \right\} \right]$$

$$\{ D_{ij} \}$$

$$+ \frac{p'}{\rho} \left\{ \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right\} - 2 \nu \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}$$

$$\{ PS_{ij} \}$$

$$\{ \epsilon_{ij} \}$$
Modeling $\overline{u_i'}u_j'$ Eqn - L27(\(\frac{4}{19}\))

1. Invoking the idea of local isotropy at high $Re_t$ the destruction rate is equally distributed among all its components. Hence $\epsilon_{ij} = (2/3) \epsilon \delta_{ij}$ where $\epsilon$ is obtained from its eqn.

2. Pressure-Strain Correlation $PS_{ij}$ acts in two ways: Firstly, it sustains the division of TKE ($e$) into its three components $\overline{u_i'^2}$ and secondly, it destructs the absolute magnitude of the shear stresses. Hence, without further elaboration

$$- PS_{ij} = C_{p1} \frac{\epsilon}{e} (\overline{u_i'}v_j' - \frac{2}{3} e \delta_{ij}) + C_{p2} (P_{ij} - \frac{P_{ii}}{3})$$

$$+ C_{p3} (P_{ij}' - \frac{2}{3} P \delta_{ij}) + C_{p4} e \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + PS_w$$

$$PS_w = \frac{e^{3/2}}{\epsilon L_B} \left[ C_{p1}' \frac{\epsilon}{e} (\overline{u_i'}v_j' - \frac{2}{3} e \delta_{ij}) + C_{p2}' (P_{ij} - P_{ij}') \right]$$

$$P_{ij}' = -\overline{u_i'u_k'} \frac{\partial U_k}{\partial x_j} - \overline{u_i'u_j'} \frac{\partial U_k}{\partial x_i}$$

(see next slide)
This algebraic expression for $PS_{ij}$ is derived from its exact Eqn\(^2\). The term containing $C_{p1}$ is called return-to-isotropy. The $PS_w$ term is called the wall-reflection term which accounts for the effects of pressure reflections from the wall. The recommended constants are: $C_{p1} = 1.5$, $C'_{p1} = 0.12$, $C_{p2} = 0.764$, $C'_{p2} = 0.01$, $C_{p3} = 0.109$, $C_{p4} = 0.182$, $L_B = $ wall distance. Finally the Triple Velocity correlation $\overline{u'_i u'_j u'_k}$ in the Diffusion term $D_{ij}$ is modeled from its exact Eqn and $(p'/\rho) \left\{ \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right\} \simeq 0$.

$$\overline{-u'_i u'_j u'_k} = C_s \frac{e}{\epsilon} \left\{ \frac{\partial \overline{u'_j u'_k}}{\partial x_l} + \frac{\partial \overline{u'_k u'_i}}{\partial x_l} + \frac{\partial \overline{u'_i u'_j}}{\partial x_l} \right\}$$

where $C_s \simeq 0.08$ to $0.11$ (from num expts) 

Implementation of Stress-Eqn model requires solution of 6 differential eqns for $u'_i u'_j$, 2 Eqns for $e$ and $\epsilon$ coupled with the 3 RANS Eqns. This is a formidable problem.

The modeled forms presented above show that spatial gradients of $u'_i u'_j$ occur only in the diffusion and convection - these terms make the Eqns differential ones.

Alg. Stress Models are developed using the idea that

$$ \frac{u'_i u'_j}{e} \sim \frac{D u'_i u'_j}{D t} - \text{Diff} (u'_i u'_j) = \frac{D e}{D t} - \text{Diff} (e) = -(2/3) (1 - C_{p1}) \delta_{ij} + \left( \frac{P}{\epsilon} \right) F $$

$$ F = (1 - C_{p2}) \frac{P_{ij}}{P} - C_{p3} \frac{P'_{ij}}{P} - C_{p4} \frac{e}{P} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) $$

$$ + \frac{2}{3} (C_{p2} + C_{p3}) \delta_{ij} \quad (\text{computational expense reduced}) $$
\[
\overline{u'_i u'_j} = -(2/3) e \delta_{ij} + e \times F
\]
\[
F = \frac{\nu_t}{e} S_{ij} + C_1 \frac{\nu_t}{e} (S_{ik} S_{jk} - \frac{1}{3} S_{kl} S_{kl} \delta_{ij})
+ C_2 \frac{\nu_t}{e} (\Omega_{ik} S_{jk} + \Omega_{jk} S_{ik}) + C_3 \frac{\nu_t}{e} (\Omega_{ik} \Omega_{jk} - \frac{1}{3} \Omega_{kl} \Omega_{kl} \delta_{ij})
+ C_4 \frac{\nu_t}{e} \left( \frac{\epsilon^*}{2} \right)^2 (S_{kl} \Omega_{lj} + S_{kj} \Omega_{li}) S_{kl}
+ C_5 \frac{\nu_t}{e} \left( \frac{\epsilon^*}{2} \right)^2 (\Omega_{il} \Omega_{lm} S_{mj} + S_{il} \Omega_{lm} \Omega_{mj} - \frac{2}{3} S_{lm} \Omega_{mn} \Omega_{nl} \delta_{ij})
+ \frac{\nu_t}{e} \left( \frac{\epsilon^*}{2} \right)^2 (C_6 S_{ij} S_{kl} S_{kl} + C_7 S_{ij} \Omega_{kl} \Omega_{kl}) \quad (\text{see next slide})
\]
Low $Re_t$ ASM Contd - L27(\frac{8}{19})

\[ \Omega_{ij} = (\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) \quad S_{ij} = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \]

\[ \mu_t = f_\mu C^*_D e^2/\epsilon^* \quad \rightarrow \quad f_\mu = 1 - \exp \left[ -\left( \frac{Re_t}{90} \right)^{0.5} - \left( \frac{Re_t}{90} \right)^2 \right] \]

\[ C^*_D = 0.3 \times (1 + 0.35 \left\{ \max (S, \Omega) \right\}^{1.5})^{-1} \]

\[ \times \left( 1 - \exp \left\{ -\frac{0.36}{\exp (-0.75 \max (S, \Omega))} \right\} \right) \]

\[ \overline{S} = \left( \frac{e}{\epsilon^*} \right) \sqrt{0.5 \ S_{ij} \ S_{ij}} \quad \overline{\Omega} = \left( \frac{e}{\epsilon^*} \right) \sqrt{0.5 \ \Omega_{ij} \ \Omega_{ij}} \]

Constants are: $C_1 = -0.1$, $C_2 = 0.1$, $C_3 = 0.26$, $C_4 = -10 \ (C^*_D)^2$, $C_5 = 0$, $C_6 = -5 \ (C^*_D)^2$ and $C_7 = 5 \ (C^*_D)^2$. The model is tested for very complex strain fields - swirling flows, curved channels and jet-impingement on a wall (Craft T. J., Launder B. L. and Suga K, IJHFF, 17(12), p 108, 1996)
Scalar Transport - L27($\frac{9}{19}$)

From Lecture 21,

\[
\rho_m \ c_{pm} \left[ \frac{\partial \hat{T}}{\partial t} + \frac{\partial \hat{u}_j \hat{T}}{\partial x_j} \right] = - \frac{\partial \hat{q}_j}{\partial x_j} + \mu \ \hat{\Phi}_v \quad \text{(Instantaneous)}
\]

\[
\rho_m \ c_{pm} \left[ \frac{\partial T}{\partial t} + \frac{\partial u_j \ T}{\partial x_j} \right] = - \frac{\partial}{\partial x_j} \left( - k_m \ \frac{\partial T}{\partial x_j} + \rho_m \ c_{pm} \ \overline{u_j' \ T'} \right) + \mu_{eff} \ \Phi_v + \rho_m \ \epsilon \quad \text{(Time averaged)}
\]

\(\rho_m \ c_{pm} \ \overline{u_j' \ T'}\) must be obtained from

1. Eddy Diffusivity model, or
2. Transport Eqn for \(\overline{u_j' \ T'}\)
Analogous to $\mu_t$, we define Turbulent thermal conductivity $k_t$ so that

$$-u'_i T' = \left( \frac{k_t}{\rho c_p} \right) \frac{\partial T}{\partial x_i} = \alpha_t \frac{\partial T}{\partial x_i} = \frac{\nu_t}{Pr_T} \frac{\partial T}{\partial x_i}$$

where $Pr_T =$ Turbulent Prandtl number $\approx 0.9$ when $Re_t$ is high.

Hence, energy Eqn will read as

$$\frac{D T}{D t} = \frac{\partial}{\partial x_k} \left\{ \left( \frac{\nu}{Pr} + \frac{\nu_t}{\sigma_t} \right) \frac{\partial T}{\partial x_k} \right\} + \frac{Q_{gen}}{\rho c_p}$$

where $Q_{gen} = \mu_{eff} \Phi_v + \rho_m \epsilon$. Usually, $\rho_m \epsilon << \mu_{eff} \Phi_v$. 

The model is very convenient because $\nu_t$ is obtained from mixing length, or one- or two-eqn models and $Pr_T$ is an absolute constant.

The disadvantage is that $\alpha_t = 0$ where $\nu_t = 0$. In several flows, significant temperature gradients and hence heat transfer exist in regions where $\nu_t = 0$.

Like $\nu_t$, $\alpha_t$ is also isotropic. But, measurement of decay of non-axi-symmetric temperature profiles in a fully developed turbulent flow in a pipe suggests that the ratio of tangential to radial diffusivities ($\alpha_{t, \theta}/\alpha_{t, r}$) $\gg 1$ near the wall.

Therefore, in general, $u_i' T'$ must be obtained directly from its differential transport equation.
Eqn for $\overline{u_i' T'}$ is derived by multiplying Eqn for $\hat{T}$ by $u_i'$ and Eqn for $\hat{u}_i$ by $T'$ - addition and time-averaging gives:

$$
\frac{\partial \overline{u_i' T'}}{\partial t} + u_k \frac{\partial \overline{u_i' T'}}{\partial x_k} = - \left[ \overline{u_i' u_k'} \frac{\partial T}{\partial x_k} + \overline{u_k' T'} \frac{\partial u_i}{\partial x_k} \right] \{ P_T \}
$$

$$
- \frac{\partial}{\partial x_k} \left[ \overline{u_i' u_k'} T' + \frac{\rho'}{\rho} \overline{T'} \delta_{ik} - \alpha \frac{\partial \overline{u_i' T'}}{\partial x_k} \right] \{ D_T \}
$$

$$
+ \frac{\rho'}{\rho} \left\{ \frac{\partial T'}{\partial x_i} \right\} - (\nu + \alpha) \frac{\partial \overline{u_i'} \partial T'}{\partial x_k \partial x_k} \{ RD_T \} \{ Dis_T \}
$$
Like $PS_{ij}$, Redistribution term $RD_T$ is modeled as

$$RD_T = -C_{T1} \frac{\epsilon}{e} \overline{u'_i T'} + C_{T2} \overline{u'_k T'} \frac{\partial u_i}{\partial x_k}$$

$$= -0.5 \frac{\epsilon}{e} \overline{u'_n T'} \frac{e^{3/2}}{\epsilon L_B} \quad \text{(for Pr > 1)}$$

$$= - \left\{ C_{T1} + 0.5 \left( \frac{Pr + 1}{Pr} \right) \right\} \frac{\epsilon}{e} \overline{u'_i T'} \quad \text{(for Pr << 1)}$$

At high $Re_t$ or (Peclet), the task of Destruction is performed by $RD_T$. Hence, $Dis_T = 0$.

In the diffusion term, effect of $p'$ is either neglected or taken to be $0.2 \times \overline{u'_i u'_k T'}$ where

$$-\overline{u'_i u'_k T'} = C_T \frac{e}{\epsilon} \left[ \overline{u'_j u'_k} \frac{\partial \overline{u'_j T'}}{\partial x_j} + \overline{u'_i u'_k} \frac{\partial \overline{u'_j T'}}{\partial x_j} \right]$$
Solving $\overline{u_i' T'}$ Eqn - L27($\frac{14}{19}$)

1. The model constants are: $C_{T1} = 3.6$, $C_{T2} = 0.266$ and $C_T = 0.11$.
2. Required correlations are taken as

$$- \overline{u_i' T'} = \frac{\nu_t}{Pr_T} \frac{\partial T}{\partial x_i} \quad \text{and} \quad - \overline{u_i' u_j'} = \nu_t S_{ij}$$

3. $\nu_t$ is determined from $e$ and $\epsilon$ Eqns
4. For complete range of Prandtl numbers, $Pr_T$ is modeled as

$$Pr_T = 0.85 + 0.0309 \left\{ \frac{Pr + 1}{Pr} \right\}$$
Algebraic Flux Model - L27(15/19)

1. Eqn for scalar fluctuations is derived as

\[
\frac{D T'^2 / 2}{Dt} = - \frac{\partial}{\partial x_i} \left[ \frac{u'_i T'^2}{2} - \alpha \frac{\partial}{\partial x_i} \left\{ \overline{T'^2} \right\} \right] - u'_i \overline{T'} \frac{\partial T}{\partial x_i} - \alpha \left( \frac{\partial T'}{\partial x_i} \right)^2
\]

where \( \alpha \left( \frac{\partial T'}{\partial x_i} \right)^2 = \epsilon_T \propto \frac{e}{\epsilon} \overline{T'^2} \)

2. The AFM is derived from

\[
\frac{D u'_i T'}{Dt} - \text{Diff} (u'_i T') = \frac{\left[ (P - \epsilon)_e + (P - \epsilon)\overline{T'^2} \right]}{2} u'_i \overline{T'} e \sqrt{\overline{T'^2}}
\]

\[
\overline{T'^2} = \frac{C'_T}{\epsilon} \frac{e}{u'_i T'} \frac{\partial T}{\partial x_k} \text{ prod } = \text{ diss assumed}
\]

where \( C'_T \approx 1.6 \) for \( Pr \geq 1. \)
Mean $T$ profiles for pipe flow agreed with DNS.

Location of peak $\bar{T}^2$ shows that production shifts towards larger $y^+$ as $Pr$ decreases.

$\bar{u'}T'$ budget is similar to $e$-budget.

$\bar{v'}T'$ budget resembles $\bar{u'}v'$ budget justifying Eddy Diff model for this case.
In Combustion, it is necessary to solve differential eqns for all participating species $k$.

\[
\frac{\partial (\rho m \omega_k)}{\partial t} + \frac{\partial (\rho m u_j \omega_k)}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \rho m D_{\text{eff}} \frac{\partial \omega_k}{\partial x_j} \right] + R_k
\]

where $R_k = \text{rate of species generation/consumption}$. $D_{\text{eff}} = \frac{\nu}{Sc} + \frac{\nu_t}{SC_t}$ and $SC_t \approx 0.9$.

The simplest postulate is called the Simple Chemical Reaction (SCR) that is written as

1 kg of Fuel + $R_{st}$ kg of Oxidant = $(1 + R_{st})$ kg of Product

There are only three species Fuel, Oxidant air and Products and $R_{st} = \left( \frac{A}{F} \right)_{\text{stoich}}$

$R_{ox} = R_{st} \times R_{fu}$ and $R_{pr} = -(1 + R_{st}) \times R_{fu}$. In laminar flow

$R_{fu} = -A \exp \left( \frac{-E}{R_u T} \right) \omega_{fu}^m \omega_{ox}^n$ (A and E are fuel-specific)
In turbulent combustion, however, it is observed that outer edges of flames are very intermittent and jagged. Experimentally it is observed that even if time-averaged $\overline{\omega}_{fu}$ and $\overline{\omega}_{ox}$ are high, $R_{fu}$ rates are not as high as would be expected from the Arrhenius formula.

This is because, the fuel and oxidant at a given point are present at different times. Clearly, therefore, time scales of chemical reaction and turbulence are important. These are characterised by $S_L/u_{rms}$ where $S_L$ is the laminar flame speed of the fuel.

These ideas are captured in

$$R_{fu} = - C_{ebu} \rho_m \sqrt{(\omega'_{fu})^2 \frac{\epsilon}{e}} \simeq - C_{ebu} \rho_m \overline{\omega}_{fu} \frac{\epsilon}{e}$$

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In practical computing, the applicability of the EBU has been enhanced by the following variant:

\[ R_{fu} = -\rho_m \frac{\epsilon}{\bar{e}} \min \left\{ A \frac{\bar{\omega}_{fu}}{R_{st}}, \frac{A}{R_{st}} \bar{\omega}_{ox}, \frac{A'}{1 + R_{st}} \bar{\omega}_{prod} \right\} \]

where \( A = 4 \) and \( A' \approx 2 \).

In the next lecture, we shall discuss two important aspects of turbulent flows: (a) Laminar-to-Turbulent Transition and (b) Effect of Wall Roughness.