ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-16 FULLY-DEVELOPED LAMINAR FLOWS-2
1. Friction Factor - Triangular Cross Section
2. Friction Factor - Arbitrary Cross-sections
Governing Eqn

\[
\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} = 1
\]  

(1)

\[ u^* = \frac{u}{(4a^2 \frac{1}{\mu} \frac{dp}{dz})} \]

\[ x^* = \frac{x}{2a} \quad y^* = \frac{y}{2a} \]

with BCs

\[ u^* = 0 \text{ at } x^* = 0 \text{ and } 1 \]

\[ u^* = 0 \text{ at } y^* = \pm f(x^*) = \pm m x^* \]

Solution is obtained by Variational Method due to Kantarovich. Thus let,

\[ u^* = (f^2 - y^{*2}) F(x^*) \]

The objective is to find \( F(x^*) \).

where \( m = \tan \Phi \). We restrict attention to \( 2\Phi < 90^0 \), so that \( m < 1 \).
Variational Method - L16($\frac{2}{21}$)

The variational

$$\delta l = \int_{0}^{1} \int_{-f}^{f} \left( \frac{\partial^2 u^*}{\partial x^*^2} + \frac{\partial^2 u^*}{\partial y^*^2} - 1 \right) \delta u^* \, dx^* \, dy^* = 0 \quad (2)$$

Note that $df / dx^* = m = \tan \Phi = \text{const.}$ Hence, in the present case, $d^2 f / dx^*^2 = 0$. Then, letting $dF / dx^* = F'$ etc,

$$\frac{\partial^2 u^*}{\partial x^*^2} = (f^2 - y^*^2) F'' + 4 \, m \, f \, F' + 2 \, m^2 \, F$$

$$\frac{\partial^2 u^*}{\partial y^*^2} = -2 \, F$$

Substitution for $u^*$ and the derivatives and, further carrying out the integrations gives (see next slide)
Solution-1 - L16($\frac{3}{21}$)

\[ \delta l = \frac{4}{3} f^3 \delta \int_0^1 \left[ \frac{4}{5} f^2 F'' + \left\{ 4 m f F' + 2(m^2 - 1) \right\} F - 1 \right] Fdx^* = 0 \]

This implies that terms in the square bracket equal zero. Or,

\[ \frac{4}{5} f^2 F'' + \left\{ 4 m f F' + 2(m^2 - 1) \right\} F - 1 = 0 \]

Define \( F^* = F - 0.5 (m^2 - 1)^{-1} \). Then, since, \( f = m x^* \),

\[ x^2 F^{*''} + 5 x^* F^{*'} + \frac{5}{2} \left( \frac{m^2 - 1}{m^2} \right) F^* = 0 \]
Solution-2 - L16($\frac{4}{21}$)

The last eqn can be transformed to read as

$$\frac{1}{x^*^3} \frac{d}{dx^*} \left[ x^*^5 \frac{dF^*}{dx^*} \right] + \frac{5}{2} \left( \frac{m^2 - 1}{m^2} \right) F^* = 0$$

The solution is: $F^* = F - 0.5 \left( m^2 - 1 \right)^{-1} = A \, x^{*R_1} + B \, x^{*R_2}$. Or,

$$F = 0.5 \left( m^2 - 1 \right)^{-1} + A \, x^{*R_1} + B \, x^{*R_2}.$$ Or,

$$u^* = (m^2 \, x^{*2} - y^{*2}) \left\{ 0.5 \left( m^2 - 1 \right)^{-1} + A \, x^{*R_1} + B \, x^{*R_2} \right\}$$

where $R_1 = 0.5 \left[ -4 + \left\{ 16 - 10 \left( \frac{m^2-1}{m^2} \right) \right\}^{0.5} \right]$.

and $R_2 = 0.5 \left[ -4 - \left\{ 16 - 10 \left( \frac{m^2-1}{m^2} \right) \right\}^{0.5} \right]$.

Constants A and B are to be determined from the boundary condition $u^* = 0$ at $x^* = 0$ and 1.
Solution-3 - L16($\frac{5}{21}$)

Condition at $x^* = 1$ gives, $A + B = -0.5 \ast (m^2 - 1)^{-1}$.

Now, for $m < 1$, $R_2 < 0$. Therefore, condition at $x^* = 0$ gives, $B = 0$. Hence, the final solutions is:

$$u^* = -0.5 \ast (m^2 - 1)^{-1} \ast (m^2 \ast x_{*}^2 - y_{*}^2) \ast (x_{*}^R_1 - 1).$$

Integration gives

$$\bar{u}^* = \frac{\int_0^1 \int_{-f}^f u^* \, dx^* \, dy^*}{\int_0^1 \int_{-f}^f dx^* \, dy^*} = \frac{1}{6} \left( \frac{m^2}{m^2 - 1} \right) \left( \frac{R_1}{R_1 + 1} \right)$$

Further, it can be shown that $D_h/(2a) = 2 \ast m \ast (m + \sqrt{m^2 + 1})^{-1}$. Hence,

$$f_{fd} \, Re = \frac{1}{2} \frac{D_h}{\bar{u}^* \left( \frac{D_h}{2a} \right)^2} = \frac{12 \ast (m^2 - 1)}{(m + \sqrt{m^2 + 1})^2} \left( \frac{4}{R_1} + 1 \right)$$
Parametric Solutions - L16($\frac{6}{21}$)

<table>
<thead>
<tr>
<th>$2\Phi$</th>
<th>m</th>
<th>$R_1$</th>
<th>$D_h/2a$</th>
<th>$f_{fd} \cdot Re$</th>
</tr>
</thead>
<tbody>
<tr>
<td>85</td>
<td>0.9163</td>
<td>0.11598</td>
<td>0.80639</td>
<td>13.219</td>
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<tr>
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<td>0.39708</td>
<td>0.7568</td>
<td>13.288</td>
</tr>
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<td>1.00</td>
<td>0.6667</td>
<td>13.333</td>
</tr>
<tr>
<td>50</td>
<td>0.4663</td>
<td>1.60517</td>
<td>0.59414</td>
<td>13.308</td>
</tr>
<tr>
<td>40</td>
<td>0.3640</td>
<td>2.5135</td>
<td>0.50971</td>
<td>13.2267</td>
</tr>
<tr>
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<td>4.0266</td>
<td>0.4112</td>
<td>13.073</td>
</tr>
<tr>
<td>20</td>
<td>0.1763</td>
<td>7.0503</td>
<td>0.29591</td>
<td>12.8309</td>
</tr>
<tr>
<td>10</td>
<td>0.08749</td>
<td>16.114</td>
<td>0.16034</td>
<td>12.4808</td>
</tr>
<tr>
<td>5</td>
<td>0.04366</td>
<td>34.234</td>
<td>0.08359</td>
<td>12.258</td>
</tr>
</tbody>
</table>

$2\Phi = 60$ degrees corresponds to an **Equilateral Triangle**.
Methods of this type are not general. For different ducts such as elliptical or triangular with rounded corners, different strategies must be invoked. Therefore, we seek a **general method** applicable to all types of complex ducts. (see next slide)
Arbitrary Cross-Sections - L16(7/21)

Governing Eqn
\[
\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{d\rho}{dx} = \text{Const}
\]

Define
\[
\frac{\mu}{\frac{1}{\mu} \frac{d\rho}{dx}} = u^* - \left( \frac{z^2 + y^2}{4} \right)
\]

Hence, Laplace Eqn
\[
\frac{\partial^2 u^*}{\partial z^2} + \frac{\partial^2 u^*}{\partial y^2} = 0
\]

No-slip condition implies
\[
u_b^* = \left( \frac{z_b^2 + y_b^2}{4} \right)
\]
Soln of Laplace Eqn - L16\( \left( \frac{8}{21} \right) \)

Soln is given by

\[
\mathbf{u}^* (z, y) = \sum_{i=1}^{N} c_i \, g_i (z, y)
\]

where \( c_i \) are coefficients to be determined and the functions \( g_i \) are prescribed by exploiting the following property of the Laplace equation:

For any positive integer \( n \), the real and imaginary parts of the complex variable \((z + i \, y)^n\) are each exact solutions \((g_n(z, y))\) of the Laplace’s equation.

Thus, by successively assigning \( n = 0, 1, 2, \ldots 8 \) (say), the first seventeen solutions are given by (see next slide)
Functions $g_n(z, y) - L16\left(\frac{9}{21}\right)$

\[
\begin{align*}
g_1 &= 1 \quad (n = 0) \\
g_2 &= z \quad (n = 1) \\
g_3 &= y \quad (n = 1) \\
g_4 &= z^2 - y^2 \quad (n = 2) \\
g_5 &= 2z y \quad (n = 2 \text{ etc}) \\
g_6 &= z^3 - 3z y^2 \\
g_7 &= 3y z^2 - y^3 \\
g_8 &= z^4 + y^4 - 6z^2 y^2 \\
g_9 &= 4z^3 y - 4z y^3 \\
g_{10} &= z^5 - 10z^3 y^2 + 5z y^4 \\
g_{11} &= y^5 - 10y^3 z^2 + 5y z^4 \\
g_{12} &= z^6 - 15z^4 y^2 + 15z^2 y^4 - y^6 \\
g_{13} &= 6z^5 y + 6z y^5 - 20z^3 y^3 \\
g_{14} &= z^7 - 21z^5 y^2 + 35z^3 y^4 - 7y^6 z \\
g_{15} &= -y^7 + 21y^5 z^2 - 35y^3 z^4 + 7z^6 y \\
g_{16} &= z^8 + y^8 - 28z^6 y^2 - 28y^6 z^2 + 70z^4 y^4 \\
g_{17} &= 8z^7 y - 56z^5 y^2 + 56z^3 y^5 - 8x y^7
\end{align*}
\]
We choose 16 boundary points (say)
The coefficients $c_{i=1,2,...,16}$ are determined from 16 boundary conditions. Thus,

$$u^*(z_b, y_b) = (\frac{z_b^2 + y_b^2}{4}) = \sum_{i=1}^{N} c_i g_i (z_b, y_b)$$

The coefficients are determined by LU-decomposition followed by forward elimination and backward substitution Procedure\(^1\)

\(^1\)Another method is to use Gramm-Schmidt Ortho-normalisation method
Example - Elliptical Duct L16($\frac{11}{21}$)

See next slide for the computed data for $b = 1$, $a = 2$
### Coordinates and functions \( L_{16}(\frac{12}{21}) \)

<table>
<thead>
<tr>
<th>( z_b )</th>
<th>( y_b )</th>
<th>( RHS_i )</th>
<th>( c_i )</th>
<th>( z_b )</th>
<th>( y_b )</th>
<th>( RHS_i )</th>
<th>( c_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.40</td>
<td>-1.5</td>
<td>-0.6614</td>
<td>0.6719</td>
<td>0.0</td>
</tr>
<tr>
<td>1.5</td>
<td>0.6614</td>
<td>0.6719</td>
<td>0.00</td>
<td>-1.0</td>
<td>-0.8660</td>
<td>0.4375</td>
<td>0.0</td>
</tr>
<tr>
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<td>0.8660</td>
<td>0.4375</td>
<td>0.00</td>
<td>-0.5</td>
<td>-0.9682</td>
<td>0.2969</td>
<td>0.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9682</td>
<td>0.2969</td>
<td>0.15</td>
<td>0.0</td>
<td>-1.0000</td>
<td>0.2500</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>1.0000</td>
<td>0.2500</td>
<td>0.0</td>
<td>0.5</td>
<td>-0.9682</td>
<td>0.2969</td>
<td>0.0</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.9682</td>
<td>0.2969</td>
<td>0.0</td>
<td>1.0</td>
<td>-0.8660</td>
<td>0.4375</td>
<td>0.0</td>
</tr>
<tr>
<td>-1.0</td>
<td>0.8660</td>
<td>0.4375</td>
<td>0.0</td>
<td>1.5</td>
<td>-0.6614</td>
<td>0.6719</td>
<td>0.0</td>
</tr>
<tr>
<td>-1.5</td>
<td>0.6614</td>
<td>0.6719</td>
<td>0.0</td>
<td>[ -2.0 ]</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Note that only \( c_1 \) and \( c_4 \) are non-zero. This is found to be true for all values of \( a \) and \( b \).

Data for \( b = 1, a = 2 \).

\[ RHS_i = \frac{z_b^2 + y_b^2}{4} \]
Hence, the solution for $b=1$ and $a = 2$ is

\[
\frac{u}{\mu} \frac{dp}{dx} = 0.4 + 0.15 \left( z^2 - y^2 \right) - \left( \frac{z^2 + y^2}{4} \right)
\]

\[
\frac{\bar{u}}{\mu} \frac{dp}{dx} = \frac{\int_0^a \int_0^{y_b} u \, dz \, dy}{\int_0^a \int_0^{y_b} dz \, dy} = 0.2
\]

\[
y_b = b \left[ 1 - \frac{z_b^2}{a^2} \right]^{0.5} \quad \text{(Eqn of Ellipse)}
\]

\[
\frac{u}{\bar{u}} = 2 - 0.5 z^2 - 2 y^2
\]
Generalisation gives:

\[
\frac{u}{-\frac{1}{\mu} \frac{dp}{dx}} = c_1 + (c_4 - 0.25) z^2 - (c_4 + 0.25) y^2
\]

\[
\bar{u} \quad -\frac{1}{\mu} \frac{dp}{dx} = c_1 + 0.25 (c_4 - 0.25) a^2 - 0.25 (c_4 + 0.25) b^2
\]

\[
f_{fd} \quad Re = \frac{D_h^2}{(2 \times \bar{u} \quad -\frac{1}{\mu} \frac{dp}{dx})}
\]

\[
D_h = 4 \left( \frac{A}{P} \right) \quad \text{and} \quad A = a b \pi
\]

\[
P = \pi (a + b) \left[ 1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{64} + \frac{\lambda^6}{256} + \frac{25 \lambda^8}{16384} \right] \quad \lambda = \frac{a - b}{a + b}
\]
### Results - Ellipse - L16($\frac{15}{21}$)

<table>
<thead>
<tr>
<th>b</th>
<th>a</th>
<th>$C_1$</th>
<th>$C_4$</th>
<th>$A / b^2$</th>
<th>P/b</th>
<th>$D_h/b$</th>
<th>$f_{fd} \ Re$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.25</td>
<td>0.00</td>
<td>$\pi$</td>
<td>2 $\pi$</td>
<td>2.0</td>
<td>16.00</td>
</tr>
<tr>
<td>1</td>
<td>1.25</td>
<td>0.3049</td>
<td>0.0549</td>
<td>3.927</td>
<td>7.0904</td>
<td>2.254</td>
<td>16.098</td>
</tr>
<tr>
<td>1</td>
<td>1.67</td>
<td>0.3676</td>
<td>0.1176</td>
<td>5.236</td>
<td>8.059</td>
<td>2.461</td>
<td>16.479</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.4</td>
<td>0.15</td>
<td>6.283</td>
<td>9.688</td>
<td>2.594</td>
<td>16.823</td>
</tr>
<tr>
<td>1</td>
<td>2.5</td>
<td>0.431</td>
<td>0.181</td>
<td>7.854</td>
<td>11.506</td>
<td>2.730</td>
<td>17.294</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0.4808</td>
<td>0.2308</td>
<td>15.708</td>
<td>21.008</td>
<td>2.991</td>
<td>18.605</td>
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<td>0.245</td>
<td>31.416</td>
<td>40.623</td>
<td>3.0934</td>
<td>19.329</td>
</tr>
</tbody>
</table>

$a = 1$ and $b = 1$ corresponds to circular tube
Triangle with Rounded Corners - L16(\frac{16}{21})

17 points are chosen. b = rounding radius, a = unrounded side
### Coefficients $c_i$ - L16($\frac{17}{21}$)

<table>
<thead>
<tr>
<th>$RHS_i$</th>
<th>$c_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0006</td>
<td>0.756E-02</td>
</tr>
<tr>
<td>0.0006</td>
<td>0.198E+00</td>
</tr>
<tr>
<td>0.0006</td>
<td>-0.371E-01</td>
</tr>
<tr>
<td>0.0427</td>
<td>0.680E+00</td>
</tr>
<tr>
<td>0.1592</td>
<td>-0.763E+00</td>
</tr>
<tr>
<td>0.1795</td>
<td>-0.838E+01</td>
</tr>
<tr>
<td>0.1860</td>
<td>0.384E+01</td>
</tr>
<tr>
<td>0.1908</td>
<td>0.317E+02</td>
</tr>
<tr>
<td>0.1894</td>
<td>-0.804E+01</td>
</tr>
<tr>
<td>0.1467</td>
<td>-0.580E+02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$RHS_i$</th>
<th>$c_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1467</td>
<td>-0.580E+02</td>
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<tr>
<td>0.1894</td>
<td>0.890E+01</td>
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<td>0.1908</td>
<td>0.585E+02</td>
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<tr>
<td>0.1860</td>
<td>-0.528E+01</td>
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<tr>
<td>0.1795</td>
<td>-0.321E+02</td>
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<td>0.1592</td>
<td>0.139E+01</td>
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<tr>
<td>0.0427</td>
<td>0.761E+01</td>
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<tr>
<td>0.0006</td>
<td>0.224E-01</td>
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$b = 0.05$, $a = 1$. $z_b$ and $y_b$ are not listed.
# Triangle with Rounded Corners - L16($\frac{18}{21}$)

<table>
<thead>
<tr>
<th>b</th>
<th>a</th>
<th>$A / a^2$</th>
<th>$P/a$</th>
<th>$D_h/a$</th>
<th>$f_{fd} \text{ Re}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
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<td>0.62277</td>
<td>15.66</td>
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<td>15.74</td>
</tr>
</tbody>
</table>

$b=0$ corresponds to equilateral triangle with sharp corners
16 points are chosen. \( b = \text{radius} \), \( \theta = \text{Apex angle} \)
Coefficients and \( f_{fd} \text{Re} - L16\left(\frac{20}{21}\right) \)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( D_h/b )</th>
<th>( c_3 )</th>
<th>( c_4 )</th>
<th>( c_7 )</th>
<th>( c_{11} )</th>
<th>( c_{15} )</th>
<th>( f_{fd}\text{Re} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
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<td>.426</td>
<td>.250</td>
<td>-.0816</td>
<td>-.0083</td>
<td>-.001</td>
<td>15.765</td>
</tr>
<tr>
<td>60</td>
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<td>.231</td>
<td>.250</td>
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<td>-.0104</td>
<td>-.0031</td>
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</tr>
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<td>.250</td>
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<td>-.005</td>
<td>15.643</td>
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<tr>
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<td>-.0132</td>
<td>-.008</td>
<td>15.598</td>
</tr>
<tr>
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<td>.0076</td>
<td>.250</td>
<td>-.083</td>
<td>-.0133</td>
<td>-.0889</td>
<td>15.56</td>
</tr>
</tbody>
</table>

\( c_1, c_2, c_5, c_6, c_8, c_9, c_{10}, c_{12}, c_{13}, c_{14}, c_{16} \to 0. \)

\( \theta = 90 \) corresponds to a duct of semi-circular cross section.
The method developed for Ducts of Arbitrary Cross-sections is most general. It can be applied to any **Singly Connected Duct Cross Section**. In lecture 18, we shall apply this method to FD heat transfer.

Important References:
