LECTURE-14 LAMINAR INTERNAL FLOWS

1. Relevance
2. Important Definitions
3. Prediction of Developing Flow
In Heat Exchangers, it is important to have knowledge of pressure drop (or friction factor \( f \)) and heat transfer coefficient (or Nusselt Number \( \text{Nu} \)) on the Tube Side to facilitate their design.

Modern Heat exchangers employ ducts of both Circular and Non-Circular cross-section. Sometimes Curved Ducts are preferred or are necessitated to conserve space.

Duct passages with Internal Insertions such as Twisted tape or Coils are also popular. Optimally Internally Structured Surfaces such as rib-roughnesses, grooves and indentations are used for augmentation of \( \text{Nu} \).

Solution of Transport Equations of mass, momentum and energy provide means for obtaining \( f \) and \( \text{Nu} \). In simple ducts, analytical solutions are possible. In more complex ones, CFD solutions become necessary.
Non-circular Cross-Sections - L14\(\frac{2}{17}\)

ANNULUS

ANNULAR SECTOR DUCT

PLATE–FIN HEAT EXCHANGER

INTERNAL FLOW

HOT

FINS

COLD

SQUARE AND TRIANGULAR
CROSS–SECTION

Nuclear Rod Cluster
Curved Ducts - L14($\frac{3}{17}$)

TUBE WITH A TWISTED TAPE INSERT

SPIRAL PLATE HEAT EXCHANGER
Non-circular ducts with Internal Ribs, Pin-Fins and Wall Perforations

Internally and Externally Spiral Groove Tube
1. It is of interest to determine $L_v = F(Re)$
2. Analytical treatment difficult except in simple cases (example follows)
3. Fully-developed flow is identified with $\frac{\partial u}{\partial x} = 0$ and $\frac{dp}{dx} = \text{const.}$
Consider laminar flow between infinite parallel plates separated by distance 2b.

In the entrance region, using BL approximations, the governing eqns and Boundary conditions are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}
\]

\[
u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{d p}{d x}(x) + \nu \frac{\partial^2 u}{\partial y^2} \tag{2}
\]

\[u(0, y) = \bar{u}, \quad v(0, y) = 0 \quad \text{(inlet) \quad \bar{u} = \frac{1}{b} \int_0^b u \, dy} \]

\[\frac{\partial u}{\partial y}(x, b) = 0, \quad v(x, b) = 0 \quad \text{(symmetry)} \]

\[u(x, 0) = 0, \quad v(x, 0) = 0 \quad \text{(plate wall)} \tag{3}\]
Dimensionless Eqns - L14(\frac{7}{17})

\begin{align*}
\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} &= 0 \quad (4) \\
Re \left[ \frac{\partial (u^* u^*)}{\partial x^*} + \frac{\partial (u^* v^*)}{\partial y^*} \right] &= -Re \frac{d p^*}{d x^*} + \frac{\partial^2 u^*}{\partial y^*^2} \quad (5) \\
u^* &= \frac{u}{\bar{u}} \quad v^* = \frac{v}{\bar{u}} \quad p^* = \frac{p}{\rho \bar{u}^2} \quad (6) \\
x^* &= \frac{x}{D_h} \quad y^* = \frac{y}{D_h} \quad (7) \\
Re &= \frac{\bar{u} D_h}{\nu} \quad D_h = 4b \quad (8)
\end{align*}

Eqn 5 shows that pressure drop in the duct-entrance-length is caused by viscous friction as well as momentum change caused by changes in velocity profiles.
Analytical solutions are not possible because of the coupling involved in non-linear convection terms. Therefore, following Langhaar, let

$$Re \left[ \frac{\partial (u^* u^*)}{\partial x^*} + \frac{\partial (u^* v^*)}{\partial y^*} \right] = \beta^2(x^*) u^* \quad (9)$$

Hence, the momentum eqn can be written as

$$\frac{\partial^2 u^*}{\partial y^{*2}} - \beta^2 u^* = Re \frac{d p^*}{d x^*} \quad (10)$$

where \( d p^*/d x^* = f_l \) the local Fanning Friction factor.

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Further Manipulations - I L14(9/17)

1. To make further progress, Define

\[ u' = u^* + \frac{Re \ d p^*}{\beta^2 \ d x^*} \]

2. Then, the momentum eqn will read as

\[ \frac{\partial^2 u'}{\partial y^*^2} - \beta^2 u' = 0 \] (11)

with \( u^* = 0 \) at \( y^* = 0 \) and \( \partial u'/\partial y^* = 0 \) at \( y^* = 1/4 \)

3. This is the familiar *Fin-Equation* with a solution

\[ u' = C_1 \ exp (\beta \ y^*) + C_2 \ exp (-\beta \ y^*) \] (12)

\[ C_1 = \frac{(Re/\beta^2) (d p^* / d x^*)}{1 + \exp (\beta/2)} \quad C_2 = C_1 \ exp (\beta/2) \] (13)
To evaluate $d p^*/d x^*$, we use definition of $\bar{u}$. This gives

$$\int_0^{1/4} u^* dy^* = \int_0^{1/4} (u' - \frac{Re}{\beta^2} \frac{d p^*}{d x^*}) dy^* = \frac{1}{4}$$

Substitution for $u'$ gives

$$Re \frac{d p^*}{d x^*} = f_i Re = \beta [4 C_1 \{\exp (\beta/2) - 1\} - 1] \quad (14)$$
Centerline Velocity $u_c$ - L14(11/17)

Consider equation 10 again. Then at $y^* = 1/4$ (or at centerline)

\[
\left( \frac{\partial^2 u^*}{\partial y^{*2}} \right)^{1/4} - \beta^2 u_c^* = Re \frac{dp^*}{dx^*}
\]

(15)

where, it can be shown that

\[
\left( \frac{\partial^2 u^*}{\partial y^{*2}} \right)^{1/4} = 2 C_1 \beta^2 \exp(\beta/4)
\]

and hence,

\[
u_c^* = -C_1 \left[ \exp(\beta/4) - 1 \right]^2
\]

(16)
Final Solution $\beta \sim x L14(\frac{12}{17})$

1. Integrating equation 5 and noting that $u_{y*}=0 = v_{y*}=1/4 = 0$ gives

$$\text{Re} \frac{d}{d x^*} \int_0^{1/4} (u^* u^*) \, dy^* = -\left( \frac{\text{Re} \, d \rho^*}{4 \, d x^*} + \frac{\partial u^*}{\partial y^*} \bigg|_{y^*=0} \right)$$  \hspace{1cm} (17)

2. Substitution gives

$$\text{Re} \frac{d F_1(\beta)}{d x^*} = F_2 (\beta) \quad \text{or} \quad x^* = \text{Re} \int_{F_1(x^*=0)}^{F_1(x^*=x^*)} \frac{1}{F_2} \, d F_1$$  \hspace{1cm} (18)

where $F_1 = C_1^2 \left[ l_1 + l_2 - l_3 \right]$

$l_1 = (\exp(\beta/2) + 1)^2/4.0 \quad l_2 = (\exp \beta + \beta \exp(\beta/2) - 1)/(2 \beta)$

$l_3 = 2(\exp(\beta) - 1)/\beta$

$F_2 = -\beta \cdot C_1 \left[ \beta \left\{ 1 + \exp(\beta/2) \right\} + 1 - \exp(\beta/2) \right]$
Evaluation of the Integral - L14(13/17)

1. Objective: To evaluate

\[ x^* = \text{Re} \int_{F_1(x^*=0)}^{F_1(x^*=x^*)} \frac{1}{F_2} \, dF_1 \]

2. We assign different numerical values to \( \beta \) and generate functions \( F_1(\beta) \) and \( F_2(\beta) \)

3. Then, integration is performed by Trapezoidal rule

4. Here, \( 0 < \beta < 60 \) were chosen in steps of 1 and found to be sufficient. Note that as \( \beta \to \infty \), \( x^* \to 0 \) and as \( \beta \to 0 \), \( x^* \to \infty \)

5. For each \( \beta \), solutions \( u^*_c \) and \( f_i \text{ Re} \) are also evaluated

6. Solutions for select values of \( \beta \) are shown on the next slide
## Tabulated Solution - L14($\frac{14}{17}$)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$C_1$</th>
<th>$(x/Dh) / Re$</th>
<th>$u_c^*$</th>
<th>$f_i \times Re$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60.0</td>
<td>-1.002e-13</td>
<td>4.60e-6</td>
<td>1.071</td>
<td>1928</td>
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<td>-1.509e-11</td>
<td>6.82e-6</td>
<td>1.0869</td>
<td>1358</td>
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<td>1.178e-5</td>
<td>1.111</td>
<td>888</td>
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<tr>
<td>30.0</td>
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<td>2.50e-5</td>
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<td>519</td>
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<td>20.0</td>
<td>-5.670e-5</td>
<td>7.74e-5</td>
<td>1.233</td>
<td>250</td>
</tr>
<tr>
<td>10.0</td>
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<tr>
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<td>-35.24</td>
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<td>1.4991</td>
<td>24.33</td>
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<td>1.4996</td>
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<td>1.49998</td>
<td>24.006</td>
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<td>0.0</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>1.50</td>
<td>24.00</td>
</tr>
</tbody>
</table>
1. Development length is 
   \((L_v/D_h)/Re \approx 0.01\)

2. Fully Developed Friction Factor is \((f \, Re)_{fd} = 24.0\)

3. Fully Developed Centerline velocity is \(u_c/\bar{u} = 1.5\)

4. These are well-known results from UG Texts

5. More results on \(L_v\) on the next slide

Sometimes Apparent Friction Factor is evaluated as

\[
f_{app} = -\frac{1}{2} \left( \frac{p_x - p_{x=0}}{x} \right) \frac{D_h}{\rho \, \bar{u}^2} = \frac{1}{x} \int_0^x f_l \, dx \quad (19)
\]
### Flow Development Lengths - $L_{14(\frac{16}{17})}$

<table>
<thead>
<tr>
<th>Duct Cross-section</th>
<th>Geometry parameter</th>
<th>Value of parameter</th>
<th>$L_v/D_h/Re_{D_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular</td>
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<td>0.05</td>
<td>0.05</td>
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<tr>
<td>Annulus</td>
<td>Radius ratio</td>
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<td>$r_i/r_o$</td>
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<td>0.01</td>
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<tr>
<td></td>
<td>$(b/a)$</td>
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<td>0.0227</td>
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Some References - L14(17/17)


