Objectives

In this class:
- Derivation of conservation of momentum equation is completed.

Conservation of Momentum

Derivation-8
- Now consider the influx of momentum due to mass entering the control volume. Let velocity be $u, v, w$ in the $x, y$ and $z$ directions.
- Momentum entering the control volume in the ‘y’ direction due to mass entering the ‘$y = 0$’, ‘$z = 0$’ and ‘$x = 0$’ faces is $(\rho v dxdz)v$, $(\rho wdxdv)v$ and $(\rho udxdw)v$.
- Momentum leaving due to mass leaving the control volume at $y = dy$, $z = dz$ and $x = dx$ is obtained from the Taylor series expansion with only the leading term retained.

Conservation of Momentum

Derivation-9
- All the momentum terms in ‘y’ direction due to mass entering or leaving the control volume are given on the figure below; term on $x = 0$ face omitted for clarity.

Conservation of Momentum

Derivation-10
- Net influx of momentum in ‘y’ direction due to mass influx

$$
\rho v dxdz + \rho wdxdy + \rho udxdy

- \left( \rho v \frac{\partial}{\partial y} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial y} + \rho \frac{\partial u}{\partial x} \right) dxdydz

= - \left( \frac{\partial}{\partial y} \rho \frac{\partial v}{\partial y} + \frac{\partial}{\partial z} \rho \frac{\partial w}{\partial y} + \frac{\partial}{\partial x} \rho \frac{\partial u}{\partial y} \right) dxdydz
$$

(11.1)
- In addition to surface forces due to the stresses, assume body forces are present.

Conservation of Momentum

Derivation-11
- Assume body forces are present. Body force vector (per unit mass) is denoted by:
Net influx of momentum into control volume is due to:
- mass entering (equation 11.1)
- force on the control volume faces (equation 10.8)
- Body force (equation 11.2)

Net accumulation rate is \( \frac{\partial}{\partial t} \rho \mathbf{v} dxdydz \)

### Conservation of Momentum

#### Derivation-12

The overall momentum balance equation therefore becomes

\[
- \left( \frac{\partial}{\partial y} \rho v^2 + \frac{\partial}{\partial z} \rho wv + \frac{\partial}{\partial x} \rho uv \right) dxdydz + \left[ \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} + \frac{\partial}{\partial x} \tau_{xy} \right] dxdydz
- \frac{\partial}{\partial t} \rho \mathbf{v} dxdydz + \rho \mathbf{X} dxdydz = 0
\]

#### Derivation-13

Newton examined results of a large number of experiments and proposed the following relationship for shear stress: \( \tau = \mu \frac{du}{dy} \) for 1D.

This shear stress can be generalized using the nomenclature adopted earlier to get:

\[
\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \text{ for } i \neq j
\]

A relationship between velocities and stress is established using the above equation.

#### Derivation-14

The following relationship, called the Stokes constitutive relationship, will be used here without deriving it.

\[
\tau_{ij} = \left( -P - \frac{2}{3} \mu \nabla \cdot \mathbf{u} \right) \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

\[
\delta_{ij} = 0 \text{ for } i \neq j
\]

\[
\delta_{ij} = 1 \text{ for } i = j
\]

\( \mathbf{u} = u_{\hat{x}} \mathbf{i} + u_{\hat{y}} \mathbf{j} + u_{\hat{z}} \mathbf{k} \)

#### Derivation-15

Now, consider the stress terms in the momentum equation and substitute the Stokes relationship to get:

\[
\frac{\partial}{\partial y} \tau_{xy} + \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial z} \tau_{xy} = \text{ From momentum equation}
\]
After substituting Stokes relationship (11.6)

\[- \frac{\partial P}{\partial y} + \frac{\partial}{\partial y} \left( \frac{2}{3} \mu \nabla \vec{u} \right) + \frac{\partial}{\partial y} \left( 2 \mu \frac{\partial \nu}{\partial y} \right) + \frac{\partial}{\partial x} \left( \mu \left( \frac{\partial u}{\partial y} + \frac{\partial \nu}{\partial x} \right) \right) + \frac{\partial}{\partial z} \left( \mu \frac{\partial \nu}{\partial z} + \frac{\partial \omega}{\partial y} \right) \]

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**Conservation of Momentum**

**Derivation-16**

- In addition if \( \mu \) is assumed constant the equation becomes:

\[
\frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial z} \tau_{yz} = - \frac{\partial P}{\partial y} - \frac{2}{3} \mu \frac{\partial}{\partial y} \left( \nabla \vec{u} \right) \\
+ 2 \mu \frac{\partial}{\partial y} \left( \frac{\partial \nu}{\partial y} \right) + \mu \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} + \frac{\partial \nu}{\partial x} \right) + \mu \frac{\partial}{\partial z} \left( \frac{\partial \nu}{\partial z} + \frac{\partial \omega}{\partial y} \right) \]  

(11.7)

- For an incompressible fluid it has been shown earlier that (refer equn (10.7a))

\[
\nabla \vec{u} = 0 \]  

(10.7 a)

**Conservation of Momentum**

**Derivation-17**

- Since velocity is a continuous function, cross differentiation is permissible:

\[
\frac{\partial}{\partial y} \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \frac{\partial u}{\partial y} \]  

(11.8)

- Use equn (10.7a) and equn (11.8) in equn (11.6):

\[
\frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial z} \tau_{yz} = - \frac{\partial P}{\partial y} - \frac{2}{3} \mu \frac{\partial}{\partial y} \left( \nabla \vec{u} \right) \\
\quad = 0 \\
+ \mu \left( \frac{\partial^{2} \nu}{\partial x^{2}} + \frac{\partial^{2} \nu}{\partial y^{2}} + \frac{\partial^{2} \nu}{\partial z^{2}} \right) + \mu \left( \frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial y} + \frac{\partial \omega}{\partial z} \right) \]

(11.9)

**Conservation of Momentum**

**Derivation-18**

- Substituting Equ 11.9 in equ 11.3:

\[
\frac{\partial}{\partial x} (\rho \nu) + \frac{\partial}{\partial x} \rho u \nu + \frac{\partial}{\partial y} \rho v \nu + \frac{\partial}{\partial z} \rho w \nu = \\
\rho \nu \frac{\partial P}{\partial y} + \mu \left( \frac{\partial^{2} \nu}{\partial x^{2}} + \frac{\partial^{2} \nu}{\partial y^{2}} + \frac{\partial^{2} \nu}{\partial z^{2}} \right) \]  

(11.10)

- Above equation is called the conservative form of the momentum equation since it is the ‘original’ form obtained from the conservation equations and no simplifications are as yet applied.

**Conservation of momentum**

**Derivation-19**

- Expand LHS of equn (11.10) to get:
\[
\frac{\partial}{\partial t} (\rho v) + \frac{\partial}{\partial x} \rho u v + \frac{\partial}{\partial y} \rho v^2 + \frac{\partial}{\partial z} \rho w v
\]
\[
= \rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] + v \left[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right] = 0
\]

- Second term is zero from continuity (eqn 10.6)

**Conservation of momentum**

**Derivation - 20**

- The 'y' component of the momentum equation therefore becomes (Note that \( \nu = \frac{\mu}{\rho} \)):

\[
\left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] =
\[
X_y + \nu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] - \frac{1}{\rho} \frac{\partial p}{\partial y}
\] (11.11)

**Conservation of Momentum**

**Derivation - 21**

- The above Y-momentum equation is written in a compact form in the following fashion:

\[
\frac{dv}{dt} = X_y + \nu \nabla^2 v - \frac{1}{\rho} \frac{\partial p}{\partial y}
\]

\[
\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}
\]

\[
\frac{dv}{dt} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}
\]

- X and Z momentum can be similarly derived

**Conservation of momentum**

**Derivation - 22**

- The final set of momentum equations are:

\[
\frac{dv}{dt} = X_y + \nu \nabla^2 v - \frac{1}{\rho} \frac{\partial p}{\partial y}
\] (11.12)

\[
\frac{du}{dt} = X_x + \nu \nabla^2 u - \frac{1}{\rho} \frac{\partial p}{\partial x}
\] (11.13)

\[
\frac{dw}{dt} = X_z + \nu \nabla^2 w - \frac{1}{\rho} \frac{\partial p}{\partial z}
\] (11.14)

- The above equations are derived for laminar, incompressible, constant viscosity, Newtonian fluids

**Recap**

**In this class:**

- Derivation of conservation of momentum equation is completed.