Explicit Method for Solving Parabolic PDE

One of the simplest second order Parabolic Differential Equation in one-dimension is the Heat Conduction Equation, written as:

\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } 0 \leq x \leq L, t \geq 0
\]  

(1.1)

which arises in many real problems.

The appropriate, but most simple conditions are:

Initial condition: \( u(x,0) = u_0(x) \)

and two Boundary Conditions namely: \( u(0,t) = u_1(t) \) and \( u(L,t) = u_2(t) \).

Note that the analytical solution of equation (1.1) is usually a trigonometric series, which may create problem in convergence.

The first Finite Difference method is the Explicit Method.

For this, let us discuss first step, which is common to all methods i.e. discretization.

The domain of the solution is \( 0 \leq x \leq L, t \geq 0 \), as shown in fig(1).

\[
\begin{align*}
& x = 0 \quad \Delta x \quad \rightarrow \quad x = L \\
& t \quad \Delta t
\end{align*}
\]

fig(1)

It is to be discretized by drawing vertical and horizontal lines at equal distance say \( \Delta x \) and \( \Delta t \) respectively.

\( \Delta x = \Delta t \) \quad \text{Let } i, j \text{ be defined dummy variables along } x \text{ & } t \text{ axis so that;}

\[
x_i = i \Delta x, \ t_j = j \Delta t \quad \& \quad u(x_i, t_j) = u_{i,j} = (i \Delta x, j \Delta t).
\]

Let the domain from \( x = 0 \) to \( x = L \) be subdivided into \( N \) sub-parts so that \( x = 0 \) corresponds to \( i = 0 \) and \( x = L \) corresponds to \( i = N \) with \( t = 0 \) corresponds to \( j = 0 \).

(1.2)
The initial condition then can be written as: $$u(x,0) = u_0(x) \Rightarrow u_{i,0} = u_0(i\Delta x)$$

The boundary condition will be converted to:

$$u(0,t) = u_1(t) \Rightarrow u_{0,j} = u_1(j\Delta t)$$  \hspace{1cm} (1.3)

$$u(L,t) = u_N(t) \Rightarrow u_{N,j} = u_N(j\Delta t)$$

**Step 2:** Replacing the derivatives by corresponding Finite Difference representation in equation (1.1) which reduces to:

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = c^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$  \hspace{1cm} (1.4)

The truncation error is: $$o(\Delta t) + o(\Delta x)^2$$

Equation (1.4) thus can be rewritten as:

$$u_{i,j+1} = c^2ru_{i+1,j} + (1 - 2c^2r)u_{i,j} + c^2ru_{i-1,j} \quad \text{with, } r = \Delta t/(\Delta x)^2$$  \hspace{1cm} (1.5)

Equation (1.5) is called the Explicit Scheme.

The computational molecule for scheme (1.5) can be shown as in fig (2)

![Computational Molecule](image)

Equation (1.5) is now solved at first time level for $$j = 0$$; for all values of $$i = 1, 2, \ldots, (N-1)$$. Similarly solution at second, third time level is obtained correspondingly for $$j = 1, 2 \ldots$$ It is very important to note that this scheme is not unconditionally stable.

The value of $$r$$ has to be < 1/2 i.e. $$\Delta t < (1/2)(\Delta x)^2$$ which makes $$\Delta t$$ to be sufficiently small. Thus it requires large no. of computations at intermediate time level; even for a small time, as $$\Delta x$$ is itself very small (Since the Finite Difference Method for approximating the derivatives is based
on Taylor’s expansion hence both $\Delta t$ and $\Delta x$ are small. This is one of the great drawbacks of this method.

Though the truncation error tends to 0 as $\Delta t \to 0$ and $\Delta x \to 0$ but the detail discussion about it will be discussed in module 3, lecture 1. The main advantage of this scheme is that it is computationally simple as the computations proceed pointwise, thus even manually manageable.

**Example 1:** Solve the Heat Conduction Equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $0 \leq x \leq 1, t \geq 0$

subject to B.C: $u = 0$ at $x = 0$ and $\frac{\partial u}{\partial x} = 0$ at $x = 1$, for all $t$

and, I.C: $u(x,0) = \sin \frac{3\pi x}{2}$

Using the Explicit Method, choosing $\Delta x = 0.1$ and $\Delta t = 0.0025$ so that $r = 1/4$, obtain the solution for one time level and compare with the exact solution.

The exact solution is $u(x,t) = e^{-\frac{9\pi^2 t}{4}} \sin \frac{3\pi x}{2}$

**Solution**

At a general point $(i,j)$ the given pde is -

$$\left( \frac{\partial u}{\partial t} \right)_{i,j} = \left( \frac{\partial^2 u}{\partial x^2} \right)_{i,j}$$

The Explicit Finite-Difference representation of this equation is:

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2}$$

or

$$u_{i,j+1} = ru_{i-1,j} + (1 - 2r)u_{i,j} + ru_{i+1,j}$$

(1)

where, $r = \frac{\Delta t}{(\Delta x)^2}$

(2)
Initial condition is: \[ u(x,0) = \sin \frac{3\pi x}{2} \]

Boundary conditions are: \( u_{x,j} = 0 \) and \( \left( \frac{\partial u}{\partial x} \right)_{N,j} = 0 \); \( N = 10 \)

Replacing L.H.S of the above boundary condition by Backward Difference,

\[ \frac{u_{N-1,j} - u_{N,j}}{\Delta x} = 0 \Rightarrow u_{N-1,j} = u_{N,j} \Rightarrow u_{10,j} = u_{9,j} \]  

Substituting \( r = 1/4 \) and \( j = 0 \) in equation (1)

\[ u_{i,1} = \frac{1}{4} \left( u_{i-1,0} + 2u_{i,0} + u_{i+1,0} \right) \]  

Substituting \( i = 1,2,\ldots,9 \) in equation (4), we get values at the first time level. These values are used for the solution at the second time level for \( j = 1 \). These values are shown below:

<table>
<thead>
<tr>
<th>( i = 0 )</th>
<th>( i = 1 )</th>
<th>( i = 2 )</th>
<th>( i = 3 )</th>
<th>( i = 4 )</th>
<th>( i = 5 )</th>
<th>( i = 6 )</th>
<th>( i = 7 )</th>
<th>( i = 8 )</th>
<th>( i = 9 )</th>
<th>( i = 10 )</th>
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<tbody>
<tr>
<td>( X = 0 )</td>
<td>( X = 0.1 )</td>
<td>( X = 0.2 )</td>
<td>( X = 0.3 )</td>
<td>( X = 0.4 )</td>
<td>( X = 0.5 )</td>
<td>( X = 0.6 )</td>
<td>( X = 0.7 )</td>
<td>( X = 0.8 )</td>
<td>( X = 0.9 )</td>
<td>( X = 1.0 )</td>
</tr>
<tr>
<td>( j = 0 )</td>
<td>0</td>
<td>.0082</td>
<td>.0164</td>
<td>.0247</td>
<td>.0329</td>
<td>.0411</td>
<td>.0493</td>
<td>.0575</td>
<td>.0657</td>
<td>.0740</td>
</tr>
<tr>
<td>( j = 1 )</td>
<td>0</td>
<td>.0082</td>
<td>.0164</td>
<td>.0247</td>
<td>.0329</td>
<td>.0411</td>
<td>.0493</td>
<td>.0574</td>
<td>.0656</td>
<td>.0739</td>
</tr>
</tbody>
</table>

The Exact solution is:

\[ u(x,t) = e^{-9\pi^2 t} \sin \frac{3\pi x}{2} \]  

(here \( t = 0.0025 \))

Comparison between Explicit and Exact solution:

<table>
<thead>
<tr>
<th>( X )</th>
<th>0.</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit</td>
<td>0</td>
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<td>.0164</td>
<td>.0247</td>
<td>.0329</td>
<td>.0411</td>
<td>.0493</td>
<td>.0575</td>
<td>.0656</td>
<td>.0739</td>
<td>.0739</td>
</tr>
<tr>
<td>Exact</td>
<td>0</td>
<td>.0078</td>
<td>.0156</td>
<td>.0233</td>
<td>.0311</td>
<td>.0389</td>
<td>.0467</td>
<td>.0543</td>
<td>.0622</td>
<td>.0699</td>
<td>.0777</td>
</tr>
</tbody>
</table>
Example 2: Consider the PDE: \[ \frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2}; 0 < x < 1, t > 0 \] (1)

B.Cs  
(i) \( u=0 \) at \( x=0, t>0 \)
(ii) \( \frac{\partial u}{\partial x} = -\frac{1}{2} u; x = 1, t > 0 \)

I.C is \( u = x(1-x) \) when \( t=0 \) & \( 0 \leq x \leq 1 \)

Solve this equation by an explicit method, employing central-difference for the boundary conditions. Take \( \Delta x = h = 0.1 \) & \( r = 0.25 \) and 0.7 and compare the result.

Solution:

The explicit approximation is

\[ \frac{u_{i,j+1} - u_{i,j}}{k} = \frac{ih(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2} \]

\[ \Rightarrow u_{i,j+1} = irhu_{i-1,j} + (1 - 2irh)u_{i,j} + irhu_{i+1,j}; i = 1,2,\ldots,N-1 \] (2) with \( r = k/h^2 \)

Now applying central difference formula to Second B.C., we get

\[ \frac{u_{i+1,j} - u_{i,j}}{2h} = -\frac{1}{2} u_{i,j} \Rightarrow u_{i+1,j} - u_{i-1,j} = -hu_{i,j} \]

At \( x=1 \) i.e. \( i=10 \)
\[ u_{1,j} - u_{9,j} = -hu_{10,j} \Rightarrow u_{1,j} = u_{9,j} - hu_{10,j} = u_{9,j} - .1u_{10,j} \]  \hspace{1cm} (3)

(i) \( r=0.25 \)

Putting \( i=10 \), \( h=0.1 \) in equation (2):

\[ u_{10,j+1} = \frac{1}{4} u_{9,j} + \frac{1}{2} u_{10,j} + \frac{1}{4} u_{11,j} \]  \hspace{1cm} (4)

Eliminating \( u_{11,j} \) from eqn. (3) & eqn. (4), we get

\[ u_{10,j+1} = \frac{1}{2} u_{9,j} + \frac{19}{40} u_{10,j} \]  \hspace{1cm} (5)

The other B.C. is \( u=0 \) at \( x=0 \) & \( t>0 \)

\[ \Rightarrow u_{0,1} = u_{0,2} = u_{0,3} = \ldots \ldots \ldots u_{0,n} = 0 \]

And I.C is: \( u(x,0)=x(1-x); \ 0 \leq x \leq 1 \)

\[ u_{0,0} = u(0,0) = 0, \quad u_{1,0} = u(.1,0) = 0.09, \quad u_{2,0} = u(.2,0) = 0.16 \]
\[ u_{3,0} = u(.3,0) = 0.21, \quad u_{4,0} = u(.4,0) = 0.24, \quad u_{5,0} = u(.5,0) = 0.25, \quad u_{6,0} = 0.24, \quad u_{7,0} = 0.21, \quad u_{8,0} = 0.16, \quad u_{9,0} = 0.09, \quad u_{10,0} = 0 \]

Now putting \( r=0.25 \) & \( h=0.1 \) in (2)

\[ \Rightarrow u_{i,j+1} = 0.025u_{i-1,j} + (1 - 0.05)i)u_{i,j} + 0.025iu_{i+1,j} \]  \hspace{1cm} (6)

1st time level: Putting \( i=1,2,3\ldots9 \) \( j=0 \) in eqn.(6)

\[ u_{1,1} = 0.025u_{0,0} + (1 - 0.05)u_{1,0} + 0.025u_{2,0} = 0.0895 \]
\[ u_{2,1} = \quad 0.05u_{1,0} + 0.9u_{2,0} + 0.05u_{3,0} = 0.1590 \]
\[ u_{3,1} = \quad 0.075u_{2,0} + 0.85u_{3,0} + 0.075u_{4,0} = 0.2085 \]
\[ u_{4,1} = \quad 0.1u_{3,0} + 0.8u_{4,0} + 0.1u_{5,0} = 0.2380 \]
\[ u_{5,1} = 0.2475, \quad u_{6,1} = 0.2370, \quad u_{7,1} = 0.2065 \]
\[ u_{8,1} = 0.1560, \quad u_{9,1} = 0.0855 \]

Putting \( j=0 \) in equation (5)

\[ u_{10,j} = \frac{1}{2} u_{9,0} + \frac{19}{40} u_{10,0} = \frac{1}{2} u_{9,0} = 0.0450 \]

Now, Second time level: Putting \( i=1,2,3\ldots9 \) \( j=1 \) in equation (6)
\[ u_{1,2} = 0.025u_{0,1} + (1 - 0.05)u_{1,1} + 0.025u_{2,1} = 0.0890 \]
\[ u_{2,2} = 0.05u_{1,1} + 0.9u_{2,1} + 0.05u_{3,1} = 0.1580 \]
\[ u_{3,2} = 0.075u_{2,1} + 0.85u_{3,1} + 0.075u_{4,1} = 0.2070 \]
\[ u_{4,2} = 0.1u_{3,1} + 0.8u_{4,1} + 0.1u_{5,1} = 0.2360 \]
\[ u_{5,2} = 0.2450, \quad u_{6,2} = 0.2340, \quad u_{7,2} = 0.2030 \]
\[ u_{8,2} = 0.1520, \quad u_{9,2} = 0.0922 \]

Putting \( j = 1 \) in equation (5)

\[ u_{10,1} = \frac{1}{2}u_{9,1} + \frac{19}{40}u_{10,1} = 0.0641 \]

(ii) \( r = 0.7 \)

Put \( i = 10, r = 0.7, h = 0.1 \) in equation (2)

\[ u_{10,j+1} = 0.7u_{9,j} - 0.4u_{10,j} + 0.7u_{11,j} \quad (7) \]

Eliminate \( u_{11,j} \) from equation (3) and equation (7); we get

\[ u_{10,j+1} = 1.4u_{9,j} - .47u_{10,j} \quad (8) \]

Put \( r = 0.7 \) & \( h = 0.1 \) in equation (2)

\[ u_{i,j+1} = 0.07iu_{i-1,j} + (1 - 0.14i)u_{i,j} + 0.07iu_{i+1,j} \quad (9) \]

1st time level

Putting \( i = 1, 2, 3 \ldots \ldots 9 \), and \( j = 0 \) in (9)

\[ u_{1,1} = 0.07u_{0,0} + (1 - 0.14)u_{1,0} + 0.07u_{2,0} = 0.0886 \]
\[ u_{2,1} = 0.14u_{1,0} + .72u_{2,0} + .14u_{3,0} = 0.1572 \]
\[ u_{3,1} = 0.21u_{2,0} + .58u_{3,0} + .21u_{4,0} = 0.2058 \]
\[ u_{4,1} = 0.28u_{3,0} + .44u_{4,0} + .28u_{5,0} = 0.2344 \]
\[ u_{5,1} = 0.2430, \quad u_{6,1} = 0.2316, \quad u_{7,1} = 0.2002 \]
\[ u_{8,1} = 0.1488, \quad u_{9,1} = 0.0774 \]

Putting \( j = 0 \) in (8)

\[ u_{10,1} = 1.4u_{9,0} - .47u_{10,0} = .126 \]

Second time level
Putting \( i=1,2,3\ldots 9 \), and \( j=1 \) in (9)

\[
\begin{align*}
    u_{1,2} &= 0.07u_{0,1} + (1 - 0.14)u_{1,1} + 0.07u_{2,1} = 0.0872 \\
    u_{3,2} &= 0.14u_{2,1} + 0.72u_{2,1} + 0.14u_{3,1} = 0.1544 \\
    u_{4,2} &= 0.21u_{2,1} + 0.58u_{3,1} + 0.21u_{4,1} = 0.2016 \\
    u_{5,2} &= 0.28u_{3,1} + 0.44u_{4,1} + 0.28u_{5,1} = 0.2288 \\
    u_{3,2} &= 0.2360, \quad u_{6,2} = 0.2232, \quad u_{7,2} = 0.1904 \\
    u_{8,2} &= 0.1376, \quad u_{9,2} = 0.1530
\end{align*}
\]

Putting \( j=1 \) in (8)

\[
    u_{10,2} = 1.4u_{9,1} - 47u_{10,1} = 0.0491
\]

The results are written in Tabular Form:

<table>
<thead>
<tr>
<th>I</th>
<th>For first time level</th>
<th>For second time level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>r=.25</td>
<td>r=.7</td>
</tr>
<tr>
<td>1</td>
<td>.0895 .0886</td>
<td>.0009 .089 .0872 .0018</td>
</tr>
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<td>.00180 .158 .1544 .0036</td>
</tr>
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<td>.0027 .207 .2016 .0054</td>
</tr>
<tr>
<td>4</td>
<td>.238 .2344</td>
<td>.0036 .236 .2288 .0072</td>
</tr>
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<td>5</td>
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<td>.0045 .245 .236 .0090</td>
</tr>
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<td>.0054 .234 .2232 .0108</td>
</tr>
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<td>.0063 .203 .1904 .0396</td>
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<td>.0072 .152 .1376 .0144</td>
</tr>
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</tr>
<tr>
<td>10</td>
<td>.045 .036</td>
<td>.009 .064 .095 -.0308</td>
</tr>
</tbody>
</table>

Comparison for different values of ‘r’ in Explicit Method:
First Time Level

Second Time Level