TOPOLOGY

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Chapter 1

Topological Spaces

1.1 Basic Concepts

We start with the assumption that we intuitively understand what is meant by a set. For us, set is a collection of well defined objects. We have a set $X$ and let $\mathcal{J}$ be a collection of subsets of $X$ satisfying:

- (T1) $\emptyset \in \mathcal{J}$, $X \in \mathcal{J}$, where $\emptyset$ is the empty set (or say null set).
- (T2) Suppose we have an arbitrary nonempty set $J$ and to each $\alpha \in \mathcal{J}$ we have a subset $A_\alpha$ of $X$ such that $A_\alpha \in \mathcal{J}$, then our $\mathcal{J}$ has the property that $\bigcup_{\alpha \in \mathcal{J}} A_\alpha \in \mathcal{J}$, where $\bigcup_{\alpha \in \mathcal{J}} A_\alpha = \{x \in X : x \in A_\alpha \text{ for at least one } \alpha \in \mathcal{J}\}$.
- (T3) If $A_1, A_2$ are in $\mathcal{J}$ then $A_1 \cap A_2$ is also in $\mathcal{J}$ (that is $A_1, A_2 \in \mathcal{J}$ implies $A_1 \cap A_2 \in \mathcal{J}$).

In such a case, the given collection $\mathcal{J}$ is called a topology on $X$ and the pair $(X, \mathcal{J})$ is called a topological space.

Remark 1.1.1. If $A$ is a member of $\mathcal{J}$ and $x \in A$ then we say that $A$ is a neighbourhood (also known as open neighbourhood) of $x$. That is for each $x \in X$, $\mathcal{J}$ contains the collection $\mathcal{N}_x = \{U \in \mathcal{J} : x \in U\}$ of all open neighbourhoods of $x$. ♦

Suppose we are given a set $X$. Now our aim is to find collections $\mathcal{B}$ and $\mathcal{J}$ of subsets of $X$ satisfying:

(i) $\mathcal{B} \subseteq \mathcal{J}$, (ii) $\mathcal{J}$ satisfies (T1), (T2), (T3), and (iii) $\mathcal{J} = \{U \subseteq X : x \in U \text{ implies there exists } B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}$. 

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In such a case, \( \mathcal{J} \) is said to be a topology on \( X \) generated by the collection \( \mathcal{B} \) and \( \mathcal{B} \) is said to be a basis for the topology \( \mathcal{J} \). Each member of \( \mathcal{J} \) is called an open subset of \( X \) and each member of \( \mathcal{B} \) is called an essential neighbourhood or a basic open set in \( X \). Since \( X \in \mathcal{J} \), by (iii) for each \( x \in X \) there exists \( B \in \mathcal{B} \) such that \( x \in B \). Also note that if \( B_1, B_2 \in \mathcal{B} \) then \( B_1 \cap B_2 \in \mathcal{J} \). Hence for any \( x \in B_1 \cap B_2 \) there exists \( B_3 \in \mathcal{B} \) such that \( x \in B_3 \subseteq B_1 \cap B_2 \). Therefore \( \mathcal{B} \) satisfies the following:

- (B1) For every \( x \in X \) there exists \( B \in \mathcal{B} \) such that \( x \in B \).
- (B2) \( B_1, B_2 \in \mathcal{B} \) and \( x \in B_1 \cap B_2 \) implies there exists \( B_3 \in \mathcal{B} \) such that \( x \in B_3 \subseteq B_1 \cap B_2 \).

Suppose a collection \( \mathcal{B} \) of subsets of a given set \( X \) satisfies the conditions (B1), (B2). Then using (iii) we can define \( \mathcal{J} \) and such a collection \( \mathcal{J} \) satisfies (T1), (T2), and (T3).

Let us prove the following theorem:

**Theorem 1.1.2.** Suppose a collection \( \mathcal{B} \) of subsets of a given set \( X \) satisfies:

(B1) For every \( x \in X \) there exists \( B \in \mathcal{B} \) such that \( x \in B \).

(B2) \( B_1, B_2 \in \mathcal{B} \) and \( x \in B_1 \cap B_2 \) implies there exists \( B_3 \in \mathcal{B} \) such that \( x \in B_3 \subseteq B_1 \cap B_2 \).
Then the collection $\mathcal{J}$ defined as $\mathcal{J} = \{ U \subseteq X : x \in U \text{ implies there exists } B \in \mathcal{B} \text{ such that } x \in B \subseteq U \}$ is a topology on $X$.

**Proof.** To prove (T1): From (B1), $x \in X$ implies there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$. Hence by the definition of $\mathcal{J}$, $X \in \mathcal{J}$. Now we will have to prove that the null set $\phi \in \mathcal{J}$. How to prove? Our statement namely $x \in U$ implies there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$ is a conditional statement. That is, we have statements say $p$ and $q$. Now consider the truth table

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The so-called null set $\phi$ (or empty set) is a subset of $X$. Whether $\phi$ satisfies the stated property? What is the stated property with respect to our set $\phi$? If $x \in \phi$ then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq \phi$, where are we in the truth table? Whether there is $x \in \phi$? The answer is no. So our statement $x \in \phi$ is false. In such a case whether $q$ is true or false it does not matter and $p \Rightarrow q$ is true. So the conclusion is that the null set $\phi$ has the stated property, therefore by the definition of $\mathcal{J}, \phi \in \mathcal{J}$.

To prove (T2): Suppose $\mathcal{J}$ is a nonempty set and for each $\alpha \in \mathcal{J}$, $A_\alpha \in \mathcal{J}$. Now we will have to prove that $\bigcup_{\alpha \in \mathcal{J}} A_\alpha \in \mathcal{J}$.

If $\bigcup_{\alpha \in \mathcal{J}} A_\alpha = \phi$, then $\phi \in \mathcal{J}$ (follows from (T1)). So let us assume that $\bigcup_{\alpha \in \mathcal{J}} A_\alpha \neq \phi$. Let $x \in \bigcup_{\alpha \in \mathcal{J}} A_\alpha$, then there exists $\alpha_0 \in \mathcal{J}$ such that $x \in A_{\alpha_0}$. Now $x \in A_{\alpha_0}$ and $A_{\alpha_0} \in \mathcal{J}$ therefore by the definition of $\mathcal{J}$ there exists $B \in \mathcal{B}$ such
that $x \in B \subseteq A_{\alpha_0}$. But $A_{\alpha_0} \subseteq \bigcup_{\alpha \in J} A_{\alpha}$. Hence $x \in \bigcup_{\alpha \in J} A_{\alpha}$ implies there exists $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcup_{\alpha \in J} A_{\alpha}$ (since $x \in B \subseteq A_{\alpha_0}$).

Therefore by the definition of $\mathcal{J}$, $\bigcup_{\alpha \in J} A_{\alpha} \in \mathcal{J}$. That is, if $J$ is a nonempty set and $A_{\alpha} \in \mathcal{J}$ for all $\alpha \in J$ then $\bigcup_{\alpha \in J} A_{\alpha} \in \mathcal{J}$.

To prove (T3): Let $A_1, A_2 \in \mathcal{J}$. Again if $A_1 \cap A_2 = \emptyset$ then by (T1), $\emptyset \in \mathcal{J}$ and hence $A_1 \cap A_2 \in \mathcal{J}$.

Suppose $A_1 \cap A_2 \neq \emptyset$. Now let $x \in A_1 \cap A_2$ then $x \in A_1$ and $x \in A_2$. Now $x \in A_1, A_1 \in \mathcal{J}$ implies there exists $B_1 \in \mathcal{B}$ so that $x \in B_1 \subseteq A_1$. Also $x \in A_2, A_2 \in \mathcal{J}$ implies there exists $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq A_2$. Now $B_1, B_2 \in \mathcal{B}$ are such that $x \in B_1 \cap B_2$. Hence by (B2) there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. But $B_1 \cap B_2 \subseteq A_1 \cap A_2$. Hence $B_3 \in \mathcal{B}$ is such that $x \in B_3 \subseteq A_1 \cap A_2$.

That is $x \in A_1 \cap A_2$ implies there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq A_1 \cap A_2$ implies $A_1 \cap A_2 \in \mathcal{J}$ (by the definition of $\mathcal{J}$). Now $\mathcal{J}$ satisfies (T1), (T2), (T3) and therefore $\mathcal{J}$ is a topology on $X$.

\[ \square \]

**Remark 1.1.3.** The topology $\mathcal{J}$ defined as in theorem 1.1.2 is called the topology generated by $\mathcal{B}$. If we want to define a topology on a set $X$ then we search for a collection $\mathcal{B}$ of subsets of $X$ satisfying (B1), (B2) and once we know such a collection $\mathcal{B}$ then we know how to get the topology generated by $\mathcal{B}$. Such a collection $\mathcal{B}$ is called a basis for a topology on $X$ and the topology generated by $\mathcal{B}$ is normally denoted by $\mathcal{J}_\mathcal{B}$.

\[ \diamond \]

**Definition 1.1.4.** If $\mathcal{B}$ is a collection of subsets of a given set $X$ satisfying (B1), (B2) then $\mathcal{B}$ is called a **basis** for a topology on $X$. 

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1.2 The Metric Topology

Let $X$ be a nonempty set and $(x, y) \in X \times X$. With each $(x, y) \in X \times X$ we associate a non-negative real number which we denote by $d(x, y)$. We want to identify $d(x, y)$ as the distance between the elements $x, y$ in $X$. So it is natural to expect that

- (M1) $d(x, y) = 0$ if and only if $x = y$;
- (M2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (M3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

It is to be noted that to each element $(x, y)$ in $X \times X$ we associate a unique element $d(x, y)$ in $\mathbb{R}^+ = [0, \infty)$. That is $d(x, y)$ is the image of $(x, y) \in X \times X$. Hence $d$ is a function from $X \times X$ into $\mathbb{R}^+$ i.e. $d : X \times X \to \mathbb{R}^+$.

If $X$ is a nonempty set and $d : X \times X \to \mathbb{R}^+$ is a function satisfying the above conditions (M1), (M2), (M3) then we say that $d$ is a metric on $X$. In such a case, the pair $(X, d)$ is called a metric space.

Let us fix $x \in X$. Now we want to collect all those elements of the space $X$ which are not far away from $x$ and such a set is known as a neighbourhood of $x$. Well, what do you mean by “not far away from $x$”? The term “not far away” is a relative term. So we fix an $r > 0$ (in some sense radius of our neighbourhood) and then take an element, say $y$ from $X$. If the distance between $x$ and $y$ is strictly less than $r$, that is $d(x, y) < r$, then we say that $y$ is in our neighbourhood of $x$. Let us define $B(x, r) = \{y \in X : d(x, y) < r\}$, and call this set as one of our neighbourhoods of $x$. If we change $r$, we get different neighbourhoods of $x$ and $B(x, r)$ is also known as the ball centered at $x$ and radius $r$. When $X = \mathbb{R}^3$ and $d$, the distance function, is the usual Euclidean distance, i.e. for any $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$
where \(d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}\), then \(B(x, r)\) is the usual Euclidean ball centered at \(x\) and radius \(r > 0\).

**Remark 1.2.1.** One can have different metrics on \(\mathbb{R}^3\) (or \(\mathbb{R}^n, n \geq 1\)) and for \(x = (x_1, x_2, x_3) \in \mathbb{R}^3, r > 0\), \(B(x, r)\) may be a cube or a solid sphere or an ellipsoid (excluding the points on the boundary) or a singleton \(\{x\}\) or the whole space \(\mathbb{R}^3\) under suitable metrics. Now consider a subset \(A\) of \(X\). Suppose \(A\) has the property: if \(x \in A\) then there exists at least one neighbourhood of \(x\) say \(B(x, r)\) which is contained in our set \(A\). That is, \(x \in A\) implies there exists \(r > 0\) such that \(B(x, r) \subseteq A\) (such a \(r > 0\) depends on \(x \in A\). i.e. same \(r\) may not work for every \(x \in A\)).

**Note.** Our statement namely \(x \in A\) implies there exists \(r > 0\) such that \(B(x, r) \subseteq A\) is a conditional statement. The so-called empty set (or null set) \(\emptyset\) is a subset of our space \(X\). Whether empty set \(\emptyset\) has the stated property? What is the stated property? Well, following the same argument as given in theorem 1.1.2 we see that \(\emptyset\) has the stated property. Now it is easy (if not obvious) to prove:

- \(X\) has the stated property.
- \(A, B \subseteq X\) such that \(A, B\) have the stated property then \(A \cap B\) has the stated property.
- Consider a nonempty set \(J\). Suppose for each \(\alpha \in J, A_\alpha \subseteq X\) and \(A_\alpha\) has the stated property, then \(\bigcup_{\alpha \in J} A_\alpha\) has the stated property. That is the collection \(J_d\) defined as \(J_d = \{A \subseteq X : x \in A\) implies there exists \(r > 0\) such that \(B(x, r) \subseteq A\}\) is a topology on \(X\), known as the topology induced by the given metric \(d\).

In this sense we say that every metric space \((X, d)\) is a topological space. \(\star\)
Theorem 1.2.2. In a metric space \((X, d)\) for each \(x \in X, r > 0\), \(B(x, r)\) is an open subset of \((X, J_d)\).

Proof. Let \(y \in B(x, r)\). Then \(d(x, y) < r\). Let \(s = r - d(x, y)\). If \(z \in B(y, s)\), then \(d(y, z) < s = r - d(x, y)\). So \(d(x, y) + d(y, z) < r\). By the triangle inequality, \(d(x, z) < r\). That is \(z \in B(x, r)\). Thus \(B(y, s) \subset B(x, r)\). Hence \(B(x, r)\) is open. ■

It is interesting to note that \(\mathcal{B} = \{B(x, r) : x \in X, r > 0\}\) is a basis for a topology on \(X\) and it is clear from the definition of \(J_d\) that the topology \(J_\mathcal{B}\) generated by \(\mathcal{B}\) is same as \(J_d\).

Now let us give some important examples of topological spaces.

Let \(X\) be a set and let \(J_t = \{\emptyset, X\}\), \(J_D = \{A : A\) is subset of \(X\}\), \(J_f = \{A : X \setminus A = A^c\) is a finite subset of \(X\) or \(A^c = X\}\), \(J_c = \{A : X \setminus A\) is a countable subset of \(A\) or \(A^c = X\}\). It is easy to prove that \(J_t, J_D, J_f, J_c\) are topological spaces on \(X\), \(J_t\) is known as the trivial or indiscrete topology on \(X\), \(J_D\) is known as the discrete topology on \(X\), \(J_f\) is known as the cofinite topology on \(X\), \(J_c\) is known as the co-countable topology on \(X\).

Recall that a set \(A\) is a countable set if and only if \(A\) is a finite set or \(A\) is a countably infinite set. Also note that \(A\) is a countably infinite set if and only if there exists a bijective function \(f\) from \(\mathbb{N}\) to \(A\), where \(\mathbb{N}\) is the set of all natural numbers. Also it is known that a nonempty set \(A\) is a countable set if and only if there exists a surjective function say \(f : \mathbb{N} \rightarrow A\).

Now let us prove that \(J_c = \{A : X \setminus A\) is a countable subset of \(A\) or \(A^c = X\}\) is a topology on \(X\).
Proof. Now \( \phi^c = X \) implies \( \phi \in \mathcal{J}_c \), \( X^c = \phi \) and \( \phi \) is a countable set implies \( X \in \mathcal{J}_c \). Hence

\[
\phi, X \in \mathcal{J}_c. \quad (1.1)
\]

Let \( J \) be a nonempty set and for each \( \alpha \in J \), \( A_\alpha \in \mathcal{J}_\alpha \).

Claim: \( \bigcup_{\alpha \in J} A_\alpha \in \mathcal{J}_c \).

Now \( (\bigcup_{\alpha \in J} A_\alpha)^c = \bigcap_{\alpha \in J} A_\alpha^c \). Hence we will have to prove that either \( \bigcap_{\alpha \in J} A_\alpha^c = X \) or \( \bigcap_{\alpha \in J} A_\alpha^c \) is a countable subset of \( X \). If \( \bigcap_{\alpha \in J} A_\alpha^c = X \) then we are through (from (T1)).

Suppose not. This implies for at least one \( \alpha_0 \in J \), \( A_{\alpha_0}^c \neq X \), \( A_{\alpha_0} \in \mathcal{J}_c \) implies \( A_{\alpha_0}^c \) is a countable set. Since \( \bigcap_{\alpha \in J} A_\alpha^c \subseteq A_{\alpha_0}^c \), \( \bigcap_{\alpha \in J} A_\alpha^c \) is a countable set (subset of a countable set is countable). We have proved that either \( \bigcap_{\alpha \in J} A_\alpha^c = X \) or \( \bigcap_{\alpha \in J} A_\alpha^c \) is a countable set. Hence

\[
\bigcup_{\alpha \in J} A_\alpha \in \mathcal{J}_c. \quad (1.2)
\]

Let \( A_1, A_2 \in \mathcal{J}_c \) implies that \( A_1^c \) is a countable set or \( A_1^c = X \) and \( A_2 \in \mathcal{J}_c \) implies \( A_2^c \) is a countable set or \( A_2^c = X \). Now \( (A_1 \cap A_2)^c = A_1^c \cup A_2^c = X \) when \( A_1^c \) or \( A_2^c = X \) or \( A_1^c \cup A_2^c \) is a countable set since in this case both \( A_1^c \) and \( A_2^c \) are countable sets. Hence \( (A_1 \cap A_2)^c = X \) or it is a countable set. This implies that

\[
A_1 \cap A_2 \in \mathcal{J}_c. \quad (1.3)
\]

From Eqs. (1.1), (1.2), and (1.3), \( \mathcal{J}_c \) is a topology on \( X \). Now let us give some examples to illustrate the natural way of obtaining topologies once we know bases satisfying (B1) and (B2).

Example 1.2.3. Let \( X \) be a nonempty set and \( \mathcal{B} = \{ \{x\} : x \in X \} \). Then \( \mathcal{B} \) is a basis for a topology on \( X \).
(i) For every $x \in X$ there exists $B = \{x\} \in \mathcal{B}$ such that $x \in B$.

(ii) $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ implies there exists $B_3 = \{x\} \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Hence both (B1) and (B2) are satisfied. This implies that the collection $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for a topology on $X$.

Now let us find out $\mathcal{J}_\mathcal{B}$, the topology, generated by $\mathcal{B}$. In theorem 1.1.2 we have proved that if we define $\mathcal{J}_\mathcal{B}$ as $\mathcal{J}_\mathcal{B} = \{U \subseteq X : x \in U$ implies there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U\}$ is a topology on $X$.

In this case for any nonempty subset $U$ of $X$, $x \in U$ implies there exists $B = \{x\}$ such that $x \in B \subseteq U$. Hence by the definition of $\mathcal{J}_\mathcal{B}$, $A \in \mathcal{B}$ whenever $A$ is a nonempty subset of $X$. Also the null set $\phi \in \mathcal{J}_\mathcal{B}$ (recall the proof given in theorem 1.1.2). Hence $A \subseteq X$ implies $A \in \mathcal{J}_\mathcal{B}$ implies $\mathcal{P}(X) \subseteq \mathcal{J}_\mathcal{B}$. Also by the definition, $\mathcal{J}_\mathcal{B} \subseteq \mathcal{P}(X)$, the collection of all subsets of $X$. This implies that $\mathcal{J}_\mathcal{B}$ is same as the discrete topology $\mathcal{J}_D$ defined on $X$.

**Exercise 1.2.4.** Let $X$ be a nonempty set and let $d$ be a metric on $X$. That is $(X, d)$ is a given metric space. Then prove that the collection $\mathcal{B}$ defined as $\mathcal{B} = \{B(x, r) : x \in X, r > 0\}$ is a basis for $\mathcal{J}_d$. □

Now let us consider the special case $X = \mathbb{R}$, the set of all real numbers and $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. Then $d$ is a metric on $\mathbb{R}$. What is the collection $\mathcal{B} = \{B(x, r) : x \in X, r > 0\}$. Note that $B(x, r) = (x - r, x + r) = (a, b)$, where $a = x - r \in \mathbb{R}, b = x + r \in \mathbb{R}$ with $a < b$. That is $\mathcal{B} \subseteq \{(a, b) : a, b \in \mathbb{R}, a < b\} = \mathcal{B}'$.

![Figure 1.2](image-url)
On the other hand take a member say $B \in \mathcal{B}'$. Since $B \in \mathcal{B}'$ there exist $a, b \in \mathbb{R}, a < b$ such that $B = (a, b)$. Now let $x = \frac{a+b}{2}$ and $r = |\frac{a-b}{2}| = \frac{b-a}{2} > 0$. Then $B(x, r) = (x-r, x+r) = \left(\frac{a+b}{2} - \frac{b-a}{2}, \frac{a+b}{2} + \frac{b-a}{2}\right) = (a, b)$ implies $(a, b) = B(x, r) \in \mathcal{B}$ implies $\mathcal{B}' \subseteq \mathcal{B}$. Also we have $\mathcal{B} \subseteq \mathcal{B}'$ and hence $\mathcal{B} = \{B(x, r) : x \in \mathbb{R}, r > 0\} = \{(a, b) : a, b \in \mathbb{R}, a < b\} = \mathcal{B}'$. That is $\{(a, b) : a, b \in \mathbb{R}, a < b\}$ is basis for a topology on $\mathbb{R}$ and $\mathcal{J}_{\mathcal{B}} = \mathcal{J}_d$. This topology is called the standard or usual topology on $\mathbb{R}$, and it is denoted by $\mathcal{J}_s$.

**Exercises 1.2.5.** (i) Prove that $\mathcal{B}_Q = \{(a, b) : a, b \in \mathbb{Q}, a < b\}$, where $\mathbb{Q}$ - the set of all rational numbers is also a basis for the usual topology on $\mathbb{R}$. That is $\mathcal{J}_{\mathcal{B}_Q}$ is same as the usual topology on $\mathbb{R}$.

(ii) Is $\mathcal{B}_0 = \{B(x, \frac{1}{n}) : x \in \mathbb{Q}, n \in \mathbb{N}\}$ a basis for the usual topology on $\mathbb{R}$? Justify your answer.

(iii) It is given that $(X, d)$ is a metric space. Now prove that $\mathcal{B}' = \{B(x, \frac{1}{n}) : x \in X, n \in \mathbb{N}\}$ is a basis for a topology on $X$. Also prove that $\mathcal{J}_{\mathcal{B}'} = \mathcal{J}_d$. 

**Definition 1.2.6.** A subset $A$ of a topological space $(X, \mathcal{J})$ is said to be a **closed set** if the complement $X \setminus A = A^c$ of $A$ is an open set.

Use the DeMorgan’s law to prove the following theorem.

**Theorem 1.2.7.** In a topological space $(X, \mathcal{J})$ we have:

(i) $X$ and $\phi$ are closed.

(ii) Suppose we have a nonempty index set $J$ and to each $\alpha \in J$, $A_\alpha$ is a closed subset of $X$. Then $\bigcap_{\alpha \in J} A_\alpha$ is a closed subset of $X$. That is arbitrary intersection of closed sets is closed.

(iii) If $A_1, A_2$ are closed sets then $A_1 \cup A_2$ is also a closed set.

Use induction to prove that finite union of closed sets is closed.
Now let us prove the following theorem which tells us when a subcollection $\mathcal{B}$ of a given topology $\mathcal{J}$ on $X$ generates the topology $\mathcal{J}$.

**Theorem 1.2.8.** Let $(X, \mathcal{J})$ be a topological space and $\mathcal{B} \subseteq \mathcal{J}$. Further suppose for each $A \in \mathcal{J}$ and $x \in A$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq A$. Then $\mathcal{B}$ is a basis for a topology on $X$ and $\mathcal{J}_\mathcal{B} = \mathcal{J}$.

**Proof.** First let us prove that $\mathcal{B}$ is a basis for a topology on $X$.

(B1) Let $x \in X$. Since $X \in \mathcal{J}$, by hypothesis, there exists $B \in \mathcal{B}$ such that $x \in B$.

(B2) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. It is given that $\mathcal{B} \subseteq \mathcal{J}$. Hence $B_1, B_2 \in \mathcal{J}$ and this implies $B_1 \cap B_2 \in \mathcal{J}$. Now $x \in B_1 \cap B_2$ and $B_1 \cap B_2 \in \mathcal{J}$ implies there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. From (B1) and (B2) we see that $\mathcal{B}$ is a basis for a topology on $X$.

![Diagram of set intersections](image)

**Figure 1.3**

Now let us prove that $\mathcal{J}_\mathcal{B} = \mathcal{J}$. Let $U \in \mathcal{J}_\mathcal{B}$.

**Claim:** $U \in \mathcal{J}$. If $U = \emptyset$ then this implies that $U \in \mathcal{J}$. Otherwise let $x \in U$. By the definition of $\mathcal{J}_\mathcal{B}$ there exists $B_x \in \mathcal{B} \subseteq \mathcal{J}$ such that $x \in B_x \subseteq U$. This implies that $\bigcup_{x \in U} B_x = U$ and hence $U \in \mathcal{J}$. 

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Conversely, let $\phi \neq U \in \mathcal{J}$. Then for each $x \in U$ there exists $B \in \mathcal{B} \subseteq \mathcal{J}_\mathcal{B}$ such that $x \in B \subseteq U$. This proves that $\bigcup_{x \in U} B_x = U \in \mathcal{J}_\mathcal{B}$. Hence $\mathcal{J} \subseteq \mathcal{J}_\mathcal{B}$. Already we have proved that $\mathcal{J}_\mathcal{B} \subseteq \mathcal{J}$ and therefore $\mathcal{J} = \mathcal{J}_\mathcal{B}$.

Now it is natural to introduce the following definition.

**Definition 1.2.9.** If $(X, \mathcal{J})$ is a topological space and $\mathcal{B} \subseteq \mathcal{J}$ such that for each $A \in \mathcal{J}$ and $x \in A$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq A$, then we say that $\mathcal{B}$ is a basis for $\mathcal{J}$.

**Theorem 1.2.10.** Let $(X, \mathcal{J})$ be a topological space and $\mathcal{B} \subseteq \mathcal{J}$. Then $\mathcal{B}$ is a basis for $\mathcal{J}$ if and only if every member $A$ of $\mathcal{J}$ is the union of member of some subcollection of $\mathcal{B}$.

**Proof.** Left as an exercise.

To have a feeling of this concept do the following exercise:

**Exercise 1.2.11.** For $a_1, a_2, b_1, b_2 \in \mathbb{R}, a_1 < a_2, b_1 < b_2$ let $R = \{(x_1, x_2) \in \mathbb{R}^2 : a_1 < x_1 < a_2, b_1 < x_2 < b_2\}$. That is, $R$ is an open rectangle having sides parallel to the coordinate axes. Let $\mathcal{B}$ be the collection of all such open rectangles. Now it is easy to see that $\mathcal{B}$ is a basis for the topology $\mathcal{J}_d$ on $\mathbb{R}^2$, where $d$ is the Euclidean metric on $\mathbb{R}^2$.

**Remark 1.2.12.** Let $X$ be a set and $\mathcal{S}$ be a collection of subsets of $X$. Suppose $\bigcup_{A \in \mathcal{S}} A = X$. Then we say that $\mathcal{S}$ is a subbasis for a topology on $X$. In this case, let $\mathcal{B} = \{A \subseteq X : A = \bigcap_{B \in \mathcal{S}} B, \text{ for a finite subcollection } \mathcal{F} \text{ of } \mathcal{S}\}$. Then it is easy to prove that $\mathcal{B}$ is a basis for a topology on $X$. The topology $\mathcal{J}_\mathcal{B}$ generated by $\mathcal{B}$ is called the topology on $X$ generated by the subbasis $\mathcal{S}$.

**Exercises 1.2.13.** (i) Let $\mathcal{S}_1 = \{(a, \infty) : a \in \mathbb{R}\}$. Prove that $\mathcal{S}_1$ is a subbasis for a topology on $\mathbb{R}$. Find out the topology $\mathcal{J}_1$ generated by $\mathcal{S}_1$. 14
(ii) Let \( S_2 = \{(-\infty, a) : a \in \mathbb{R}\} \). Prove that \( S_2 \) is a subbasis for a topology on \( \mathbb{R} \). Find out the topology \( J_2 \) generated by \( S_2 \).

1.3 Interior Points, Limit Points, Boundary Points, Closure of a Set

Let \( A \) be a nonempty subset of a topological space \((X, J)\) and \( x \in A \). Then \( x \) is said to be an **interior point of** \( A \) if there exists an open set \( U \) such that \( x \in U \) and \( U \subseteq A \). Also the collection of all interior points of \( A \) denoted it by\( \text{int}(A) \) or \( A^\circ \).

For a nonempty subset \( A \) of a topological space \((X, J)\), a point \( x \in X \) is said to be a **limit point** or an **accumulation point** of \( A \) if for each open set \( U \) containing \( x \), \( U \cap (A \setminus \{x\}) \neq \emptyset \).

For \( A \subseteq X \), the **derived set of** \( A \) denoted by \( A' \) is defined as \( A' = \{x \in X : x \text{ is a limit point of } A\} \).

A point \( x \in X \) is said to be a **boundary point** of \( A \) if for each open set \( U \) containing \( x \), \( U \cap A \neq \emptyset \) and \( U \cap A^c \neq \emptyset \).

For \( A \subseteq X \), the **boundary** of \( A \), denoted by \( \text{bd}(A) \), is defined as \( \text{bd}(A) = \{x \in X : \text{for each open set } U \text{ containing } x, U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset \} \).

That is \( \text{bd}(A) \) is the collection of all boundary points of \( A \).

For \( A \subseteq X \), the **closure** of \( A \) denoted by \( \overline{A} \) or \( \text{cl}(A) \), is defined as \( \overline{A} = A \cup A' \).

**Examples 1.3.1.** (i) Let \( X = \{1, 2, 3\} \) and \( J = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} \) Is \( J \) a topology on \( X \)? Let \( A = \{1, 3\} \), \( B = \{2, 3\} \). Here \( A \in J \), \( B \in J \), but \( A \cap B = \{3\} \notin J \). Hence \( J \) is not a topology on \( X \).

(ii) Let \( X = \{1, 2, 3\} \) and \( J = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\} \) then \( J \) is a topology on \( X \). Now \( A = \{2, 3\} \) is a subset of \( X \). \( 2 \in A \) and also there is an open set \( U = \{2\} \) such that
\(2 \in U\) and \(U \subseteq A\). Hence 2 is an interior point of \(A\). But 3 is not an interior point of \(A\). How to check 3 is an interior point of \(A\) or not?

**Step 1:** First check whether \(3 \in A\) (if \(x\) is an interior point of \(A\) then it is essential that \(x \in A\)). Yes here \(3 \in \{2, 3\} = A\).

**Step 2:** Now find out all the open sets containing 3. \(X\) is the only open set containing 3. But this open set is not contained in \(A\). Hence 3 is not an interior point of \(A\).

What will happen if the given set \(A\) is an open subset of a topological space \(X\). Our aim is to check whether an element \(x \in X\) is an interior point of \(A\).

**Step 1:** It is essential that \(x \in A\).

**Step 2:** Is it necessary to find out all the open sets containing \(x\)? Of course not necessary. It is enough if we find at least an open set \(U\) such that \(x \in U\) and \(U \subseteq A\). In this case the given set \(A\) is an open set and hence there exists an open set \(U = A\) such that \(x \in U\) and \(U = A \subseteq A\). Therefore every element \(x\) of \(A\) is an interior point of \(A\). That is \(A \subseteq A^\circ\). By definition \(A^\circ \subseteq A\). Hence \(A^\circ = A\). That is if \(A\) is an open set then \(A^\circ = A\). What about the converse? Suppose for a subset \(A\) of \(X\), \(A^\circ = A\). Is \(A\) an open set? Yes, \(A\) is an open subset of \(X\). Take \(x \in A\). Then \(x \in A^\circ\). Hence by the definition of \(A^\circ\) there exists at least one open set say \(U_x\) such that \(x \in U_x\) and \(U_x \subseteq A\). This implies that \(A = \bigcup_{x \in A} U_x\). Now by the definition, \(J\) is closed under arbitrary union. Hence for each \(x \in A\), \(U_x \in J\) implies \(\bigcup_{x \in A} U_x \in J\) implies \(A \in J\). That is, \(A\) is an open set. Thus, we have proved:

**Theorem 1.3.2.** For a subset \(A\) of topological space \((X, J)\), \(A\) is open if and only if \(A^\circ = A\).

Now let us prove that for any subset \(A\) of \(X\), \(A^\circ\) is an open set and if \(B\) is an open set contained in \(A\) (\(B \subseteq A\)) then \(B \subseteq A^\circ\).
Theorem 1.3.3. For any subset \( A \) of a topological space \( (X, \mathcal{J}) \), \( A^o \) is the largest open set contained in \( A \).

**Proof.** If \( A = \emptyset \) then \( A^o = \emptyset \). For \( A \neq \emptyset \), let us prove that \( B = A^o \) is an open set. Due to theorem 1.3.2, it is enough to prove that \( (A^o)^o = A^{oo} = A^o \). If \( A^o = \emptyset \) then \( A^{oo} = \emptyset \) and we are through. Also by definition \( A^{oo} \subseteq A^o \). Let \( x \in A^o \). Then by the definition, there exists an open set \( U_x \) such that \( x \in U_x \subseteq A \). Note that for each \( y \in U_x, y \in U_x \subseteq A \). That is \( y \in A \) and there exists an open set \( U_x \) such that \( y \in U_x \) and \( U_x \subseteq A \). This implies that \( y \in A^o \). That is \( y \in U_x \) implies \( y \in A^o \) implies \( U_x \subseteq A^o \). We have the following:

- \( x \in A^o \) and,
- there exists an open set \( U_x \) such that \( x \in U_x, U_x \subseteq A^o \).

This implies that \( x \in A^{oo} \). That is \( x \in A^o \) implies \( x \in A^{oo} \) implies \( A^o \subseteq A^{oo} \). Also we have \( A^{oo} \subseteq A^o \) implies \( (A^o)^o = A^o \). From the theorem 1.3.2, \( A^o \) is an open set. Also by definition, \( A^o \subseteq A \).

To prove the second part assume that \( B \) is an open subset of \( X \) such that \( B \subseteq A \). Now we aim to prove \( B \subseteq A^o \). Which is obvious since for each \( x \in B \) there exists an open set \( B \) such that \( x \in B \) and \( B \subseteq A \). Hence by definition \( B \subseteq A^o \). □

Consider \( X = \{1, 2, 3\}, \mathcal{J} = \{\emptyset, X, \{1\}, \{1, 2\}\} \) and \( A = \{1, 2\} \). What is \( A' \), the collection of all limit points of \( A \). Is \( 1 \in A' \)? The answer is no. Since \( \{1\} \) is an open set containing 1, but \( \{1\} \cap A \setminus \{1\} = \{1\} \cap \{2\} = \emptyset \). Is \( 2 \in A' \)? Again the answer is no. Since \( \{2\} \) is an open set containing 2 and \( \{2\} \cap A \setminus \{2\} = \{2\} \cap \{1\} = \emptyset \). Is \( 3 \in A' \)? First find out all the open sets containing 3.
Here the whole space $X$ is the only open set containing 3 and $X \cap A \setminus \{3\} = \{1, 2, 3\} \cap \{1, 2\} \neq \phi$. That is for each open set $U$ containing 3, the condition namely $U \cap A \setminus \{3\} \neq \phi$ is satisfied. Hence 3 is a limit point of $A$. That is $3 \in A'$. Here $A' = \{3\}$.

What is $\overline{A}$, the closure of $A$? By definition $\overline{A} = A \cup A' = \{1, 2\} \cup \{3\} = \{1, 2, 3\}$.

Now let us prove that for any subset $A$ of a topological space $X$,

- $\overline{A}$ is a closed set and $A \subseteq \overline{A}$.
- Whenever $B$ is a closed set such that $A \subseteq B$ then $\overline{A} \subseteq B$ that is we aim to prove:

**Theorem 1.3.4.** For a subset $A$ of a topological space $X$, $\overline{A}$ is always a closed set containing $A$ and it is the smallest closed set containing $A$.

**Proof.** Let us prove that $(\overline{A})^c = X \setminus \overline{A}$ is an open set. Hence we will have to prove that interior of $(\overline{A})^c$ is itself. Let $x \in (\overline{A})^c$ then $x \notin \overline{A}$. Hence there exists an open set $U$ containing $x$ such that $U \cap A = \emptyset \Rightarrow U \subseteq A^c$. This imply that $x$ is an interior point of $A^c$, but we have to prove that $x$ is an interior point of $(\overline{A})^c$. So it is enough to prove that $U \subseteq (\overline{A})^c$.

Suppose not. Then there exists $y \in U$ such that $y \notin (\overline{A})^c$ implies $y \in (\overline{A})$. Also $U$ is an open set containing $y$. Hence $U \cap A \neq \emptyset$. This is contradiction to $U \cap A = \emptyset$. We arrived at this contradiction by assuming that $U$ is not a subset of $(\overline{A})^c$. Hence $U \subseteq (\overline{A})^c$, where $U$ is an open set containing $x$ and $x \in (\overline{A})^c$. Therefore every point of $(\overline{A})^c$ is an interior point. This implies that $(\overline{A})^c$ is an open set and hence $\overline{A}$ is a closed set.

Now let $B$ be a closed set containing $A$ then we will have to prove that $\overline{A} \subseteq B$. That is to prove $\overline{A} \cap B^c = \emptyset$. 

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Suppose not. Then there exists \( x \in \overline{A} \cap B^c \), \( B^c \) is an open set containing \( x \).

Now if \( x \in \overline{A} = A \cup A' \) is such that \( x \in A \) then \( x \in B \) (given that \( A \subseteq B \)) and we are through. On the other hand if \( x \in A' \) and \( x \notin A \) then by the definition of \( A' \), \( B^c \cap A \setminus \{x\} = B^c \cap A \neq \emptyset \) (note: \( x \notin A \Rightarrow A \setminus \{x\} = A \)) is a contradiction since \( A \subseteq B \) implies \( A \cap B^c \subseteq B \cap B^c = \emptyset \). Hence \( \overline{A} \cap B^c = \emptyset \). That is \( \overline{A} \subseteq B \). ■

1.4 Hausdorff Topological Spaces

**Definition 1.4.1.** A topological space \((X, \mathcal{J})\) is said to be a *Hausdorff topological space* (or Hausdorff space) if for \( x, \ y \in X, \ x \neq y \), there exist \( U, \ V \in \mathcal{J} \) such that

\[
\begin{align*}
&\text{(i) } x \in U, \ y \in V, \\
&\text{(ii) } U \cap V = \emptyset.
\end{align*}
\]

**Note.** In definition 1.4.1, in place of if it is also absolutely correct to use if and only if. That is, definition 1.4.1 can also be read as:

A topological space \((X, \mathcal{J})\) is said to be a Hausdorff topological space (or Hausdorff space) if and only if (iff) for \( x, \ y \in X, \ x \neq y \) there exist \( U, \ V \in \mathcal{J} \) such that

\[
\begin{align*}
&\text{(i) } x \in U, \ y \in V, \\
&\text{(ii) } U \cap V = \emptyset.
\end{align*}
\]

What is important to note here (that is while giving a definition) is one can use interchangeably “if” and “if and only if”.

* **Example 1.4.2.** If \( X = \mathbb{R} \), and \( \mathcal{J}_s \) is the standard topology on \( \mathbb{R} \), then \((\mathbb{R}, \mathcal{J}_s)\) is a Hausdorff space.

**Example 1.4.3.** Every discrete topological space \((X, \mathcal{J})\) is a Hausdorff space.

**Example 1.4.4.** If \( X \) is a set containing at least two elements and \( \mathcal{J} = \{\emptyset, X\} \) then \((X, \mathcal{J})\) is not a Hausdorff space.

**Example 1.4.5.** If \( X = \mathbb{R}, \mathcal{B} = \{(a, \infty) : a \in \mathbb{R}\} \) then \( \mathcal{B} \) is a basis for a topology \( \mathcal{J}_\mathcal{B} \) on \( \mathbb{R} \). It is easy to see that \((\mathbb{R}, \mathcal{J}_\mathcal{B})\) is not a Hausdorff space.

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Example 1.4.6. \( \mathcal{B}_l = \{[a, b) : a, b \in \mathbb{R}, a < b\} \), \( \mathcal{J}_l = \mathcal{J}_{\mathcal{B}_l} \) is known as the lower limit topology on \( \mathbb{R} \) and \( \mathcal{J}_s \subseteq \mathcal{J}_l \). Hence \((\mathbb{R}, \mathcal{J}_l)\) is a Hausdorff space.

**Note.** Weaker topology is Hausdorff implies stronger is also Hausdorff. ⋆

Let \( X \) be an infinite set and \( \mathcal{J}_f \) be the cofinite topology on \( X \). Also let \( x, y \in X \), \( x \neq y \). If \( U \in \mathcal{J}_f \) and \( x \in U \) then \( U^c \) is finite, because \( U^c \neq X \). Also \( y \in V \in \mathcal{J}_f \) implies \( V^c \) is finite. If \( U \cap V = \phi \), then \( X = (U \cap V)^c = U^c \cup V^c \) and hence \( X \) is a finite set. Which gives a contradiction. Therefore \( U \cap V \neq \phi \). Hence \( \mathcal{J}_f \) is not a Hausdorff space.

Example 1.4.7. Let \( X = \{a, b, c\} \) and \( \mathcal{J} = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} \). Then \((X, \mathcal{J})\) is not a Hausdorff space.

Example 1.4.8. \( \mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \times \cdots : \) each \( U_i \) is open in \( \mathbb{R} \), \( i = 1, 2, \ldots, n, n \in \mathbb{N} \}\} is a basis for a topology \( \mathcal{J} \) (known as product topology) on \( \mathbb{R}^w \), where \( \mathbb{R}^w = \{x = (x_n)_{n=1}^\infty : x_n \in \mathbb{R} \ \forall \ n\} \). Now \( X = \mathbb{R}^w \), \( x = (x_n) \in X \) and \( y = (y_n) \in X \) such that \( x \neq y \). Therefore there exists \( k \in \mathbb{N} \) such that \( x_k \neq y_k \).

Let \( \epsilon = \frac{|x_k - y_k|}{2} > 0 \) and \( U_k = (x_k - \epsilon, x_k + \epsilon), V_k = (y_k - \epsilon, y_k + \epsilon) \). Let \( U = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times U_k \times \mathbb{R} \times \mathbb{R} \cdots \) and \( V = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times V_k \times \mathbb{R} \times \mathbb{R} \cdots \).

Clearly, \( x \in U \), \( y \in V \) and \( U \cap V = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \phi \times \mathbb{R} \times \mathbb{R} \times \cdots = \phi \). Hence \( X = \mathbb{R}^w \) is a Hausdorff space.

**Note.** \( \prod_{n=1}^{\infty} (\frac{1}{n}, \frac{1}{n}) \) is not an open set in the product topology on \( \mathbb{R}^w \). ⋆

**Definition 1.4.9.** A sequence \( \{x_n\} \) in a topological space \((X, \mathcal{J})\) is said to **converge to a point** \( x \in X \) if for each open set \( U \) containing \( x \) there exists \( n_0 \in \mathbb{N} \) such that \( x_n \in U, \ \forall \ n \geq n_0 \). In symbol we write \( x_n \rightarrow x \) as \( n \rightarrow \infty \).

Note that \( x_n \rightarrow x \) as \( n \rightarrow \infty \) if and only if for each open set \( U \) containing \( x \) there exists \( n_0 \in \mathbb{N} \) such that \( x_n \in U, \ \forall \ n > n_0 \).
Example 1.4.10. If \( X \neq \emptyset, \mathcal{J} = \{ \emptyset, X \} \), and \( \{ x_n \} \) is a sequence in \( X \). Then \( \{ x_n \} \) converges to every element of \( X \).

Example 1.4.11. If \( X \) be an infinite set, \( \mathcal{J}_f = \{ A \subseteq X : A^c \text{ is finite or } A^c = X \} \), then \( \mathcal{J}_f \) is not Hausdorff. Let \( \{ x_n \} \) be a sequence in \( X \) and \( x \in X \). Now \( U \in \mathcal{J}_f \) and \( x \in U \Rightarrow U^c \) is finite. If \( U^c = \emptyset \) then \( U = X \). Otherwise \( U^c \) is nonempty and finite and hence \( J = \{ n : x_n \in U^c \} \) is a finite set. If \( J = \emptyset \) let \( n_0 = 1 \), otherwise let \( n_0 = \max\{ n : n \in J \} \). Then \( x_n \in U, \forall n > n_0 \). Therefore \( x_n \to x \) as \( n \to \infty \). So, \( (x_n) \) converges to every element of \( X \).

Theorem 1.4.12. Let \( (X, \mathcal{J}) \) be a Hausdorff space and let \( A \subseteq X \). Then an element \( x \in A' \) if and only if for each open set \( U \) containing \( x \), \( U \cap A \) is an infinite set.

Proof. Assume that \( x \in A' \) and suppose for some open set \( U \) containing \( x \), \( U \cap (A \setminus \{ x \}) \) is a nonempty finite set. Let \( U \cap (A \setminus \{ x \}) = \{ x_1, x_2, \ldots, x_n \} \). For each \( i, x_i \neq x \) and \( (X, \mathcal{J}) \) is a Hausdorff space implies there exist open sets \( U_i \) and \( V_i \) such that \( x_i \in U_i, x \in V_i \) and \( U_i \cap V_i = \emptyset \). Note that \( x_i \notin V_i \) for all \( i = 1, 2, \ldots, n \) and \( x \in V = \bigcap_{i=1}^{n} V_i, V \) is an open set. Also \( x \in U \). Therefore \( x \in U \cap V \). But \( (A \setminus \{ x \}) \cap (U \cap V) = \emptyset \) which is a contradiction. Hence \( U \cap A \) is an infinite set.

Conversely, if for \( x \in X, U \cap A \) is an infinite set for each open set containing \( x \) then in particular \( U \cap (A \setminus \{ x \}) \neq \emptyset \), for each open set containing \( x \). Hence \( x \) is a limit point of \( A \). That is \( x \in A' \).

Exercise 1.4.13. Let \( (X, \mathcal{J}) \) be a topological space such that for each \( x \) in \( X \), \( \{ x \} \) is closed in \( X \). Then prove that an element \( x \in A' \) if and only if for each open set \( U \) containing \( x \), \( U \cap A \) is an infinite set.

\[ \square \]

Note. If \( X \) is a Hausdorff space, and if \( A \) is a finite subset of \( X \), then \( A' = \emptyset \). \( \star \)
**Definition 1.4.14.** A topological space \((X, \mathcal{J})\) is said to be **metrizable** if there exists a metric \(d\) on \(X\) such that \(\mathcal{J}_d = \mathcal{J}\).

**Theorem 1.4.15.** Let \((X, d)\) be a Hausdorff topological space. Then \(\{x\}\) is closed for each \(x \in X\).

**Proof.** If \(\{x\}^c = \emptyset\), then \(X = \{x\}\) is a closed set. Suppose \(\{x\}^c \neq \emptyset\), this implies that for each \(y \in \{x\}^c, y \neq x\). Now \((X, \mathcal{J})\) is a Hausdorff space implies there exist open sets \(U_x, V_y\) in \(X\) such that \(x \in U_x, y \in U_y\) and \(U_x \cap V_y = \emptyset\). Therefore \(V_y \subseteq \{x\}^c\) implies \(y \in (\{x\}^c)^c\). This implies that \(\{x\}\) is closed. ■

**Theorem 1.4.16.** Let \((X, \mathcal{J})\) be a topological space, \(Y \subseteq X\) and let \(\mathcal{J}_Y = \{A \cap Y : A \in \mathcal{J}\}\) then \(\mathcal{J}_Y\) is a topology on \(Y\).

**Proof.** (i) \(\emptyset \in \mathcal{J}_Y\). Now \(\emptyset \in \mathcal{J}_X\) implies \(\emptyset \cap Y = \emptyset \in \mathcal{J}_Y\).

(ii) Since \(X \in \mathcal{J}_X\) implies \(X \cap Y = Y \in \mathcal{J}_Y\).

Let \(A_i \in \mathcal{J}_Y\) for \(i \in I\). Now \(A_i \in \mathcal{J}_Y\) implies there exists \(B_i \in \mathcal{J} = \mathcal{J}_X\) such that \(A_i = B_i \cap Y\). Now \(B_i \in \mathcal{J}\) for each \(i \in I\), \(\mathcal{J}\) is a topology on \(X\) implies \(\bigcup_{i \in I} B_i \in \mathcal{J}\).

Hence \(\bigcup_{i \in I} A_i = \left(\bigcup_{i \in I} B_i\right) \cap Y\) is open in \(\mathcal{J}_Y\).

(iii) Let \(A_1, A_2, \ldots, A_n \in \mathcal{J}_Y\). Then there exists \(B_i \in \mathcal{J}_X\) such that \(A_i = B_i \cap Y\). Therefore \(\bigcap_{i=1}^n B_i \in \mathcal{J}_X\). Therefore \(A_1 \cap A_2 \cap \cdots \cap A_n = (B_1 \cap Y) \cap (B_2 \cap Y) \cap \cdots \cap (B_n \cap Y) = (\bigcap_{i=1}^n B_i) \cap Y \in \mathcal{J}_Y\), and this implies that \(A_1 \cap A_2 \cap \cdots \cap A_n \in \mathcal{J}_Y\). From (i), (ii), and (iii) \(\mathcal{J}_Y\) is a topology on \(Y\). ■

**Definition 1.4.17.** Let \((X, \mathcal{J})\) be a topological space, and let \(\mathcal{J}_Y = \{A \cap Y : A \in \mathcal{J}\}\) then \(\mathcal{J}_Y\) is a topology on \(Y\). This topology \(\mathcal{J}_Y\) is called the **relative topology** on \(Y\) induced by \(\mathcal{J}\).
Note. Let $A \subseteq Y \subseteq X$. We use $\overline{A}$ to denote the closure of $A$ in $X$, and $\overline{A}_Y$ to denote the closure of $A$ in $Y$.

**Result 1.4.18.** For $A \subseteq Y$, $\overline{A}_Y = \overline{A} \cap Y$.

**Proof.** It is always true that $\overline{A}_Y \subseteq \overline{A}$. Now $\overline{A}_Y = $ closure of $A$ in $Y$, $\overline{A}_Y \subseteq Y$ and hence $\overline{A}_Y \subseteq \overline{A} \cap Y$. Let $x \in \overline{A} \cap Y$. This implies that $x \in \overline{A}$ and $x \in Y$. Then for each open set $U$ containing $x$, $U \cap A \neq \emptyset$. Hence $(U \cap Y) \cap A = U \cap A \neq \emptyset$. Thus $x \in \overline{A}_Y$. Therefore $\overline{A} \cap Y = \overline{A}_Y$. ■

**Theorem 1.4.19.** Let $(X, \mathcal{J})$ be a topological space and $A, B$ be subsets of $X$. Then

(i) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, (ii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, (iii) $\overline{A} = \overline{A}$, (iv) $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$, (v) $(A \cap B)^\circ = A^\circ \cap B^\circ$, (vi) $(A^\circ)^\circ = A^\circ$.

**Proof.** (i) An element $x \in \overline{A}$ if and only if for each open set $U$ containing $x$ such that $U \cap A \neq \emptyset$. Let $x \in \overline{A \cup B}$. This implies that for every neighbourhood $U$ containing $x$,

$$U \cap (A \cup B) \neq \emptyset \Rightarrow (U \cap A) \cup (U \cap B) \neq \emptyset.$$  \hspace{1cm} (1.4)

Suppose $x \notin \overline{A}$ and $x \notin \overline{B}$. Then for some open sets $U, V$ containing $x$ such that $U \cap A = \emptyset$ and $V \cap B = \emptyset$. Let $U_0 = U \cap V$. Then $U_0 \cap (A \cup B) = (U_0 \cap A) \cup (U_0 \cap B) = \emptyset$, a contradiction to Eq. (1.4). Therefore $x \in \overline{A}$ or $x \in \overline{B}$. This proves that

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}. \hspace{1cm} (1.5)$$

Now let $x \in \overline{A \cup B}$. This implies that $x \in \overline{A}$ or $\overline{B}$ or both. Hence for each neighbourhood $U$ of $x$, $U \cap A \neq \emptyset$ or $U \cap B \neq \emptyset$. This implies that $(U \cap A) \cup (U \cap B) \neq \emptyset$ implies $U \cap (A \cup B) \neq \emptyset$. This shows that $x \in \overline{A \cup B}$. Therefore

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}. \hspace{1cm} (1.6)$$
From Eqs. (1.5) and (1.6) \( A \cup B = \overline{A} \cup \overline{B} \).

(ii) Let \( x \in \overline{A \cap B} \). Then for every open set \( U \) containing \( x \), \( U \cap (A \cap B) \neq \emptyset \). Hence \( U \cap A \neq \emptyset \) and \( U \cap B \neq \emptyset \). This implies \( x \in \overline{A} \) and \( x \in \overline{B} \). Hence \( \overline{A \cap B} \subseteq \overline{A} \cap \overline{B} \).

(iii) \( \overline{A} \) is the smallest closed set containing \( A \) implies \( \overline{A} \subseteq \overline{A} \). Also \( \overline{A} \) is a closed set containing \( \overline{A} \). But \( \overline{A} \) is the smallest closed set containing \( A \). Hence \( \overline{A} \subseteq \overline{A} \). So we have \( \overline{A} = \overline{A} \).

(iv) Now \( x \in A^o \cup B^o \Rightarrow x \in A^o \) or \( x \in B^o \) or both. Without loss of generality assume that \( x \in A^o \). Then there exists \( U \in J \) such that \( x \in U \subseteq A \). This implies that \( x \in U \subseteq (A \cup B) \). Hence \( x \in (A \cup B)^o \). That is \( x \in A^o \cup B^o \) implies \( x \in (A \cup B)^o \). Hence \( (A \cup B)^o \subseteq A^o \cup B^o \).

(v) Let \( x \in (A \cap B)^o \). Then there exists \( U \in J \) such that \( x \in U \subseteq A \cap B \). This implies \( x \in A^o \) and \( x \in B^o \). Hence \( A^o \cap B^o \subseteq (A \cap B)^o \).

Now let \( x \in A^o \cap B^o \). Then \( x \in A^o \) and \( x \in B^o \). Hence there exists \( U \in J \) such that \( x \in U \subseteq A \) and \( V \in J \) such that \( x \in V \subseteq B \), hence \( x \in U \cap V \subseteq A \cap B \). Hence \( x \in (A \cap B)^o \). This implies \( (A \cap B)^o = A^o \cap B^o \).

(vi) Now \( A^{oo} \) is the largest open set contained in \( A^o \) implies \( A^{oo} \subseteq A^o \). Also \( A^o \) is an open set contained in \( A^o \). Hence \( A^o \subseteq A^{oo} \). So we have proved that \( A^o = A^{oo} \). ■

**Example 1.4.20.** For \( A = \mathbb{Q}, B = \mathbb{Q}^c, A \cup B = \mathbb{R} \) and \( (A \cup B)^o = \mathbb{R}, A^o = \emptyset, B^o = \emptyset, \mathbb{R} \neq \emptyset \). Hence \( (A \cup B)^o \neq A^o \cup B^o \).

\( \overline{A} = \mathbb{R}, \overline{B} = \mathbb{R}, \overline{A \cap B} = \emptyset, A \cap B = \mathbb{R} \). Hence \( \overline{A} \cap \overline{B} \neq \overline{A \cap B} \).

**Definition 1.4.21.** Let \( (X, J) \) be a topological space. Then a set \( B \subseteq Y \) is open in \( Y \) if and only if \( B = A \cap Y \), for some \( A \in J \).

**Example 1.4.22.** The set \([0, 1)\) is open in \( Y = [0, \infty) \). Note that \( A = (-1, 1) \cap [0, \infty) = [0, 1) \). Now \( A^o = Y \setminus A = [1, \infty) \) is open in \((Y, J_Y)\) if and only if for each \( x \in [1, \infty) \)
there exists $U \in \mathcal{J}_Y$ such that $x \in U \subseteq [1, \infty)$. But 1 is not an interior point of $A^c$. Hence $A^c$ is not open and therefore $A$ is not closed. Whereas $[1, \infty)$ is open in $[1, \infty) \cup \mathbb{Z}$.

**Exercises 1.4.23.** (i) Let $X = \mathbb{R}$ and $\mathcal{J}_f$ be the cofinite topology on $\mathbb{R}$. Let $Y = \mathbb{Q}$ what is $\mathcal{J}_f/Y$? (ii) Prove that for each $x \in \mathbb{R}$, the sequence $(x_n) = (\frac{1}{n}) \to x$ in $(\mathbb{R}, \mathcal{J}_f)$. □

**Result 1.4.24.** Let $(X, \mathcal{J})$ be a topological space and $\mathcal{B}$ is a basis for $\mathcal{J}$ then for each $Y \subseteq X$, $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{J}_Y$.

**Proof.** Let $U \in \mathcal{J}_Y$ and $x \in U$. $U \in \mathcal{J}_Y$ implies $U = V \cap Y$ for some $V \in \mathcal{J}$. Now $x \in V, V \in \mathcal{J}$ and $\mathcal{B}$ is a basis for $\mathcal{J}$ this implies that there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Therefore $B \cap Y \subseteq V \cap Y = U$. Now $B \cap Y \in \mathcal{B}_Y$ such that $x \in B \cap Y \subseteq U$. Therefore $\mathcal{B}_Y$ is a basis for $\mathcal{J}_Y$. ■

**Note.** $A \subseteq X$ and $\mathcal{B}$ is a basis for $\mathcal{J}$, then $x \in A^c$ if and only if there exists $B \in \mathcal{B}$ such that $x \in B \subseteq A$. *

1.5 Continuous Functions

**Definition 1.5.1.** Let $(X, \mathcal{J})$ and $(Y, \mathcal{J}')$ be topological spaces and let $f : (X, \mathcal{J}) \to (Y, \mathcal{J}')$. Then $f$ is said to be **continuous at a point** $x \in X$ if for each open set $V$ containing $f(x)$ there exists an open set $U$ containing $x$ such that $y \in U$ implies $f(y) \in V$. If $f$ is continuous at each $x \in X$ then we say that $f$ is a continuous function.

**Theorem 1.5.2.** Let $(X, \mathcal{J}), \ (Y, \mathcal{J}')$ be topological spaces. Then a function $f : X \to Y$ is continuous if and only if for each open set $V$ in $Y$, $f^{-1}(V)$ is open in $X$. 25
Proof. Let \( f \) be continuous and \( V \) be an open set in \( Y \).

Claim: \( f^{-1}(V) = \{ x \in X : f(x) \in V \} \) is open in \( X \).

If \( f^{-1}(V) = \emptyset \) then \( f^{-1}(V) \) is open in \( X \). If \( f^{-1}(V) \neq \emptyset \), let \( x \in f^{-1}(V) \). Then \( f(x) \in V \). Now \( f \) is continuous at \( x \), \( V \) is open containing \( f(x) \) implies there exists an open set \( U \) such that \( x \in U \) and \( x' \in U \) implies \( f(x') \in V \). That is \( f(U) \subseteq V \). This implies \( U \subseteq f^{-1}(f(U)) \). Therefore \( x \) is an interior point of \( f^{-1}(V) \). Hence \( f^{-1}(V) \) is open in \( X \).

To prove the converse, assume that \( f^{-1}(V) \) is open in \( X \) whenever \( V \) is open in \( Y \). Now take \( x \in X \) and an open set \( V \) in \( Y \) such that \( f(x) \in V \). Now \( V \) is open in \( Y \) implies \( f^{-1}(V) \) is open in \( X \). Also \( f(x) \in V \) implies \( x \in f^{-1}(V) = U \). That is \( U \) is an open set in \( X \) containing \( x \) such that \( y \in U \) implies \( f(y) \in f(f^{-1}(V)) \subseteq V \). Hence \( f \) is continuous at each \( x \in X \).

Theorem 1.5.3. A function \( f : X \to Y \) continuous if and only if \( f^{-1}(A) \) is closed in \( X \) whenever \( A \) is closed in \( Y \).

Proof. Assume that \( f : X \to Y \) is a continuous function. Take a closed set \( A \) in \( Y \). Since \( A \) is a closed set in \( Y \), \( A^c \) is an open set in \( Y \). Therefore \( f \) is a continuous function implies \( f^{-1}(A^c) = [f^{-1}(A)]^c \) is an open set in \( X \). This proves that \( f^{-1}(A) \) is a closed set in \( X \).

To prove the converse, assume that \( f^{-1}(A) \) is closed in \( X \) whenever \( A \) is closed in \( Y \). Take an open set \( V \) in \( Y \). Now \( V \) is an open set in \( Y \) implies \( f^{-1}(V^c) = [f^{-1}(V)]^c \) is a closed set in \( X \). Therefore \( f^{-1}(V^c) \) is a closed set in \( X \), and hence \( f^{-1}(V) \) is an open set in \( X \). This gives that \( f \) is a continuous function.
Example 1.5.4. Let \( X = \mathbb{R}^w = \{(x_1, x_2, \ldots, x_n \ldots) : x_n \in \mathbb{R}, n \in \mathbb{N}\} \) and let \( \mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_k \times \mathbb{R} \times \mathbb{R} \times \cdots : k \in \mathbb{N}, \text{each } U_i \text{ is open in } \mathbb{R}, i = 1, 2, \ldots, k\}. \) For \( A, B \in \mathcal{B}, A \cap B = (U_1 \cap V_1) \times \cdots \times (U_k \cap V_k) \times \mathbb{R} \times \mathbb{R} \times \cdots \in \mathcal{B} \). Then \( \mathcal{B} \) is a basis for a topology on \( \mathbb{R}^w \). The topology \( \mathcal{J} \) on \( \mathbb{R}^w \) induced by \( \mathcal{B} \) is called the product topology on \( \mathbb{R}^w \). \( \mathcal{B}_b = \{U_1 \times U_2 \times \cdots \times U_k \times U_{k+1} \times \cdots \} \) each \( U_k \) is open in \( \mathbb{R}, \forall k \in \mathbb{N}\} \) = \( \left\{ \prod_{k=1}^{\infty} U_k : U_k \text{ is open in } \mathbb{R} \forall k \right\} \). Then \( \mathcal{B}_b \) is also a basis for a topology on \( \mathbb{R}^w \). Let \( \mathcal{J}_b \) be the topology on \( \mathbb{R}^w \) induced by \( \mathcal{B}_b \). This topology on \( \mathbb{R}^w \) is called the box topology on \( \mathbb{R}^w \).

Example 1.5.5. Define \( f : \mathbb{R} \to (\mathbb{R}^w, \mathcal{J}_b) \) by \( f(t) = (t, t, t, \ldots) \) and \( U = (-1, 1) \times \left(\frac{-1}{2}, \frac{1}{2}\right) \times \left(\frac{-1}{3}, \frac{1}{3}\right) \times \cdots = \prod_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n}\right) \). Then \( U \in \mathcal{J}_b, f^{-1}(U) = \{t \in \mathbb{R} : f(t) \in U\} = \{t \in \mathbb{R} : (t, t, \ldots) \in \prod_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n}\right) = U\} = \{t \in \mathbb{R} : |t| < \frac{1}{n}, \forall n \in \mathbb{N}\} = \{0\}, \) and \( \{0\} \) is not an open set in \( \mathbb{R} \). Hence \( f \) is not a continuous function. But the same \( f : \mathbb{R} \to (\mathbb{R}^w, \mathcal{J}) \) is a continuous function, when we consider the product topology \( \mathcal{J} \) on \( \mathbb{R}^w \).

Theorem 1.5.6. A function \( f : X \to Y \) is continuous if and only if for every subset \( A \) of \( X \), \( f(A) \subseteq \overline{f(A)} \) (where it is understood that \( X, Y \) are topological spaces).

Proof. Now assume that \( f : X \to Y \) is continuous. To prove for \( A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)} \). Now \( \overline{f(A)} \) is a closed set in \( Y \) and \( f : X \to Y \) is a continuous function implies \( f^{-1}(\overline{f(A)}) \) is a closed set in \( X \). Also \( A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}) \). That is \( f^{-1}(\overline{f(A)}) \) is a closed set containing \( A \). Hence \( \overline{A} \subseteq f^{-1}(\overline{f(A)}) \). This gives that \( f(\overline{A}) \subseteq f(f^{-1}(f(A))) \subseteq f(A) \).

Conversely assume that for \( A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)} \). Let \( F \) be a closed set in \( Y \) and \( A = f^{-1}(F) \). Now \( f(\overline{A}) \subseteq \overline{f(A)} \) implies \( f(f^{-1}(F)) \subseteq \overline{f(f^{-1}(F))} \subseteq \overline{F} = F \). Hence \( f^{-1}(f(\overline{f^{-1}(F)})) \subseteq f^{-1}(F) \). This gives that \( \overline{f^{-1}(F)} \subseteq f^{-1}(F) \). This proves that
\( f^{-1}(F) \) is a closed set in \( X \) whenever \( F \) is a closed set in \( Y \). Therefore \( f : X \to Y \) is a continuous function.

**Remark 1.5.7.** Intuitively what do we mean by a continuous function? In the above theorem, for any subset \( A \) of \( X \), if a point \( x \) is closer to \( A \) then the image \( f(x) \) is closer to \( f(A) \). Here \( x \in \overline{A} \) means \( x \) is closer to \( A \) and hence \( f(x) \in f(\overline{A}) \). Now we want that \( f(x) \) is closer to \( f(A) \). That is \( f(x) \in f(A) \). So a function \( f : X \to Y \) is continuous if and only if for every subset \( A \) of \( X \), \( x \) is closer to \( A \) implies \( f(x) \) is closer to \( f(A) \).

**Definition 1.5.8.** Let \( X, Y \) be topological spaces. Then a function \( f : X \to Y \) is said to be a **homeomorphism** if and only if

(i) \( f \) is bijective
(ii) \( f : X \to Y, f^{-1} : Y \to X \) are continuous.

**Example 1.5.9.** (i) \( f : [0,1] \to [a,b] \) defined by \( f(t) = (1-t)a + tb \) is a homeomorphism.
(ii) \( f : (0,1) \to (1,\infty) \) defined by \( f(t) = \frac{1}{t} \) is a homeomorphism.
(iii) \( f : (0,1) \to (0,\infty) \) defined by \( f(t) = \frac{1}{1-t} \) is a homeomorphism.
(iv) Let \( X = (\mathbb{R}, \mathcal{J}_s), Y = (\mathbb{R}, \mathcal{J}_f) \). Let \( F \neq \emptyset \) be a closed in \( Y \). Then \( F \) is a finite set or \( F = \mathbb{R} \). In any case \( f^{-1}(F) = F \) is closed in \( X \). Hence \( f \) is continuous but the identity map \( f^{-1} : Y \to X \) is not continuous.

**Example 1.5.10.** Let \( X = [(-1, -1), (1, -1)] \) be the line segment joining the points \((-1, -1)\) and \((1, -1)\) in \( \mathbb{R}^2 \) and \( Y = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1, y \geq 0, x^2 + y^2 = 1\} \). Then \( X, Y \) are subspaces of the Euclidean space \( \mathbb{R}^2 \).

Define \( f : X \to Y \) as \( f((x,y)) = f(x, -1) = (x, \sqrt{1-x^2}) \) then \( f \) is a homeomorphism. That is \( f \) is bijective and \( U \) is open in \( X \) if and only if \( f(U) \) is open in \( Y \).
Exercise 1.5.11. Let $A$ and $B$ be two distinct points in $\mathbb{R}^2$ and $\gamma$ be a curve joining $A$ and $B$ as shown below: That is $\gamma : [0, 1] \to \mathbb{R}^2$ is a one-one continuous function.

Then prove that $\gamma : [0, 1] \to \{\gamma(t) : t \in [0, 1]\}$ is a homeomorphism. That is $[0,1]$ and $\{\gamma(t) : t \in [0, 1]\}$ are equivalent topological spaces. That is, there is a homeomorphism between these two topological spaces. \hfill \Box

Exercise 1.5.12. Prove that $f : X \to Y$ is a homeomorphism if and only if

(i) $f$ is bijective

(ii) $f(A) = \overline{f(A)}$, for $A \subseteq X$. \hfill \Box
EXERCISES

1. Prove that the half-open interval \([0, 1)\) is neither open nor closed in \(\mathbb{R}\), but is both a union of closed sets and an intersection of open sets.

2. Prove that the set \(A = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}\) is closed in \(\mathbb{R}\).

3. Find the collection of all interior points, limit points and boundary points of
   (i) \(\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}\)
   (ii) \(\left\{ \frac{m}{n} : m \in \mathbb{N}, 1 \leq n \leq m \right\}\)
   (iii) \(\left\{ \frac{n}{m} : n, m \in \mathbb{N} \right\}\)
   (iv) \(\left\{ m + \frac{1}{m} : m \in \mathbb{N} \right\}\)
   (v) \(\left\{ 1 + \frac{n}{m} : n, m \in \mathbb{N} \right\}\)
   (vi) \(\left\{ \frac{1}{m} : n, m \in \mathbb{N} \right\}\)
   (vii) \(\{(x, y) : 0 \leq x \leq 2, 1 < y < 2 \}\)
   (viii) \(\{(x, y) : xy = 1, x > 0, x \in \mathbb{Q} \}\).

4. Prove that \(A^\circ = \bigcup_{U \in S} U\), where \(S = \{U \subseteq A : U \text{ is open} \}\).

5. Prove that \(\overline{A} = \bigcap_{U \in S} U\), where \(S = \{U : A \subseteq U \text{ and } U \text{ is closed} \}\).

6. Let \(X = \{a, b, c, d\}\) be four points. Show that the collection \(J := \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a, c, d\}, \{a, b, c\}\}\) defines a topology on \(X\). Find the collection of all limit points, closures, interiors, and boundaries of all subsets of \(X\).

7. Prove that for any set \(A\) in a topological space \(\text{bd}(\overline{A}) = \text{bd}(A)\) and \(\text{bd}(\text{int}A) \subseteq \text{bd}(A)\). Give an example when all these three sets are different.

8. Find examples of sets \((\neq \emptyset, X)\) in a topological space that are both open and closed, neither open, nor closed.

9. Let \(X\) be an uncountable set with co-countable topology. Which of the following sets are closed, open. Justify your answer.
   (i) \(A\) is a finite set (ii) \(A\) is a countable set (iii) \(A\) is an uncountable set
   (iv) \(A\) is a proper subset of \(X\) such that both \(A\) and \(A^c\) are uncountable.

10. Let \(X\) be the set of all real numbers with lower limit topology. Let \(\alpha, \beta \in X\), \(\alpha < \beta\). Find the interior and closure of (i) \([\alpha, \beta]\) (ii) \((\alpha, \beta]\) (iii) \([\alpha, \beta)\).
11. Let $X$ be the set of all real numbers with $K$-topology. That is the topology generated by the basis $\mathcal{B} := \{(a, b), (a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Find the closure and interior of the set $E = \{\frac{1}{n} : n \in \mathbb{N}\}$.

12. Let $X$ be the set of all natural numbers with the usual topology (i.e., $\mathbb{N}$ is considered as a subset of $\mathbb{R}$). Find all the open and closed sets in $X$.

13. Let $A, B$ are nonempty subsets of a topological space $X$. Assume that $\text{bd}(A) \cap \text{bd}(B) = \phi$. Prove that $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$.

14. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two topologies on $X$. Prove that identity map $\text{id} : (X, \mathcal{J}_1) \to (X, \mathcal{J}_2)$ is continuous if and only if $\mathcal{J}_1$ is finer than $\mathcal{J}_2$ (that is $\mathcal{J}_2 \subseteq \mathcal{J}_1$).

15. Give an example of a continuous map $f$ from a topological space $X$ to another topological space $Y$, such that $f(A)$ is not open (respectively not closed) for a open (closed) subset $A$ in $X$.

16. Let $X$ be a topological space. Prove that a map $f : X \to \mathbb{R}$ is continuous if and only if for every $a \in \mathbb{R}$ the sets $f^{-1}(-\infty, a) := \{x \in \mathbb{R} : f(x) < a\}$ and $f^{-1}(a, \infty) := \{x \in \mathbb{R} : f(x) > a\}$ are open.

17. Let $(X, \mathcal{J}_X)$, $(Y, \mathcal{J}_Y)$ and $(Z, \mathcal{J}_Z)$ be topological spaces. If functions $f : X \to Y$ and $g : Y \to Z$ are continuous, then show that the composition $g \circ f : X \to Z$ is continuous. Is the converse true? Justify your answer.

18. Given a function $f : X \to Y$ and a basis $\mathcal{B}$ for $Y$, then $f$ is continuous if and only if $f^{-1}(U)$ is open in $X$ for each $U \in \mathcal{B}$.

19. Given a function $f : X \to Y$ and a subbasis $\mathcal{S}$ which generates the topology on $Y$, then $f$ is continuous if and only if $f^{-1}(U)$ is open for $X$ for each $U \in \mathcal{S}$.

20. Suppose $f : X \to Y$ is continuous, $A \subset X$. Show that $f |_A$ is continuous, where $g = f |_A : A \to Y$ is defined as $g(x) = f(x)$ for all $x \in A$.  

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21. Give an example of a function \( f : \mathbb{R} \rightarrow \mathbb{R} \), such that

(a) \( f \) is continuous at exactly 10 points.
(b) \( f \) is discontinuous only at \( 2n \) number of points, \( n \in \mathbb{N} \).

22. Say true or false and justify your answer. Let \( \mathcal{J}_1, \mathcal{J}_2 \) be topologies on \( \mathbb{R} \).
Suppose \( f : (\mathbb{R}, \mathcal{J}_1) \rightarrow (\mathbb{R}, \mathcal{J}_2) \) is defined as \( f(x) = |x| \).

(a) if \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are usual and lower limit topologies respectively, then \( f \) is continuous,
(b) if \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are lower limit and usual topologies respectively, then \( f \) is continuous,
(c) if \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) both are lower limit topologies, then \( f \) is continuous.

23. Say true or false and justify your answer. Let \( \mathcal{J}_1, \mathcal{J}_2 \) be two topologies on \( \mathbb{R} \).
Suppose \( f : (\mathbb{R}, \mathcal{J}_1) \rightarrow (\mathbb{R}, \mathcal{J}_2) \) is defined as \( f(x) := \begin{cases} x - 1 & \text{if } x < 0, \\ x + 1 & \text{if } x \geq 0 \end{cases} \)

(a) if \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are usual and lower limit topologies respectively, then \( f \) is continuous,
(b) if \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are lower limit and usual topologies respectively, then \( f \) is continuous,
(c) if \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) both are lower limit topologies, then \( f \) is continuous.

24. Let \( (X, \mathcal{J}_X) \) and \( (Y, \mathcal{J}_Y) \) be topological spaces and let \( A, B \) be nonempty subsets of \( X \) with \( A \cup B = X \). Suppose \( f : X \rightarrow Y \) is a function. Then prove or disprove

(a) if \( f \upharpoonright_A \) and \( f \upharpoonright_B \) are continuous, then \( f \) is continuous,
(b) if \( f \upharpoonright_A \) and \( f \upharpoonright_B \) are continuous and \( bd(A) \cap bd(B) = \emptyset \), then \( f \) is continuous.

25. Let \( f, g : (\mathbb{R}, \mathcal{J}_s) \rightarrow (\mathbb{R}, \mathcal{J}_s) \), (where \( \mathcal{J}_s \) is the usual (standard) topology on \( \mathbb{R} \)) be continuous. Prove or disprove:
(a) the set \( \{ x \in \mathbb{R} : f(x) \leq g(x) \} \) is closed,

(b) the function \( h : \mathbb{R} \to \mathbb{R} \) defined as \( h(x) := \min\{f(x), g(x)\} \) for \( x \in \mathbb{R} \) is continuous,

(c) the function \( h : \mathbb{R} \to \mathbb{R} \), defined as \( h(x) := \max\{f(x), g(x)\} \) for \( x \in \mathbb{R} \) is continuous.
Chapter 2

Product and Quotient Spaces

Let \((X_i, \mathcal{J}_i), i = 1, 2, \ldots, n\) be given topological spaces. Also note that it is possible that \((X_i, \mathcal{J}_i) = (X_j, \mathcal{J}_j)\) even when \(i \neq j\). What do we mean by the cartesian product \(X = X_1 \times X_2 \times \cdots \times X_n\)? We define the cartesian product as
\[
\prod_{j=1}^n X_j = X_1 \times X_2 \times \cdots \times X_n = \{f : J \to \bigcup_{j \in J} X_j : f(i) = x_i \in X_i \ \forall \ i \in J\},
\]
where \(J = \{1, 2, \ldots, n\}\).

Here we identify an \(f \in X\) with \((x_1, x_2, \ldots, x_n)\), where \(x_i = f(i)\) for all \(i = 1, 2, \ldots, n\).

What is the advantage of defining the cartesian product in this way? Let \(J\) be a nonempty set (finite or infinite) and for each \(\alpha \in J\) we have a topological space \((X_\alpha, \mathcal{J}_\alpha)\). Now we define the cartesian product \(\prod_{\alpha \in J} X_\alpha = X\) as
\[
X = \{f : J \to \bigcup_{\alpha \in J} X_\alpha : f(\alpha) = x_\alpha \in X_\alpha, \ \forall \ \alpha \in J\}.
\]

2.1 Product Space

Definition 2.1.1. For each \(\alpha \in J\), define a function \(p_\alpha : X \to X_\alpha\), known as \(\alpha\text{-th projection or coordinate function}\), as \(p_\alpha(f) = f(\alpha) = x_\alpha\).

Our aim is to define a topology \(\mathcal{J}\) on \(\prod_{\alpha \in J} X_\alpha\) which will have the following properties:

- Each projection function \(p_\alpha : (X, \mathcal{J}) \to (X_\alpha, \mathcal{J}_\alpha)\) is continuous.
- Whenever \(\mathcal{J}'\) is a topology on \(X\) such that each \(p_\alpha : (X, \mathcal{J}') \to (X_\alpha, \mathcal{J}_\alpha)\) is continuous then \(\mathcal{J} \subseteq \mathcal{J}'\).
That is $J$ is the smallest (or weakest topology) on $X$ that makes each $p_\alpha$ continuous. For $\alpha \in J$, let $S_\alpha = \{ p_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{J}_\alpha \}$. Then we require that $p_\alpha^{-1}(U_\alpha)$ is open in our proposed topological space $(X, J)$. Hence we require that $\bigcup_{\alpha \in J} S_\alpha \subseteq J$.

Note that $p_\alpha^{-1}(U_\alpha) = \{ f \in X : p_\alpha(f) = f(\alpha) = x_\alpha \in X_\alpha \}$. Hence if we fix an $\alpha \in J$, then $p_\alpha^{-1}(U_\alpha) = \prod_{\beta \in J} A_\beta$, where $A_\alpha = U_\alpha$ and $A_\beta = X_\beta$ when $\beta \neq \alpha$. If $\alpha_1, \alpha_2, \ldots, \alpha_n, n \in \mathbb{N}$ and $U_{\alpha_i} \in \mathcal{J}_{\alpha_i}$, $i = 1, 2, \ldots, n$, then $p_{\alpha_i}^{-1}(U_{\alpha_i}) \in J$. Also $J$ is closed under finite intersections means that $\bigcap_{i=1}^n p_{\alpha_i}^{-1}(U_{\alpha_i}) = \prod_{\alpha \in J} A_\alpha \in J$, where $A_{\alpha_i} = U_{\alpha_i}, i = 1, 2, \ldots, n$ and $A_\alpha = X_\alpha$ when $\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_n$. Now it is easy to see that $\mathcal{B} = \{ \bigwedge_{\alpha \in J} U_\alpha : U_\alpha \in \mathcal{J}_\alpha$ for all $\alpha \in J$ and $U_\alpha = X_\alpha$, except for finitely many $\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_n \in J \}$ is a basis for a topology on $X$. The topology $J$ induced by $\mathcal{B}$ is called the product topology on $X = \prod_{\alpha \in J} X_\alpha$ and the topological space $(X, J)$ is called the product topological space (also known as product space) induced by the topological spaces $(X_\alpha, \mathcal{J}_\alpha), \alpha \in J$.

**Remark 2.1.2.** What will happen when $J = \{1, 2, \ldots, n\}$ for some natural number $n$? When $n = 1$, $X = \{ f : \{1\} \to X_1 : f(1) = x_1 \in X_1 \} = X_1$ and $X = \{ f : \{1, 2, \ldots, n\} \to \bigcup_{i=1}^n X_i : f(i) = x_i \in X_i \}$. That is $f = (f(1), f(2), \ldots, f(n)) = (x_1, x_2, \ldots, x_n) \in X = \prod_{i=1}^n X_i$. Hence $X = \{(x_1, x_2, \ldots, x_n) : x_i \in X_i, i = 1, 2, \ldots, n\}$

$= X_1 \times X_2 \times \cdots \times X_n$.

In this case, that is when $J = \{1, 2, \ldots, n\}$ is a finite index set containing $n$ elements, $\mathcal{B} = \{ \prod_{i=1}^n U_i : each \ U_i \ is \ open \ in \ X_i, \ i = 1, 2, \ldots, n \}$ is a basis for the product topology $J$ on $X = \prod_{i=1}^n X_i$. ♦

Now let us prove the following theorem:
**Theorem 2.1.3.** Let \( J \neq \emptyset \) be an index set and \((X_\alpha, J_\alpha), \alpha \in J\) be a collection of Hausdorff topological spaces. Then the product space \( \prod_{\alpha \in J} X_\alpha, J \) is also a Hausdorff topological space.

**Proof.** Our aim is to prove that the product topological space \( \prod_{\alpha \in J} X_\alpha, J \) is a Hausdorff topological space. So take two distinct elements \( f, g \) in \( \prod_{\alpha \in J} X_\alpha \). Now \( f, g \in \prod_{\alpha \in J} X_\alpha \) implies \( f : J \rightarrow \bigcup_{\alpha \in J} X_\alpha \) and \( g : J \rightarrow \bigcup_{\alpha \in J} X_\alpha \) such that \( f(\alpha) = x_\alpha \in X_\alpha, g(\alpha) = y_\alpha \in X_\alpha \), for each \( \alpha \in J \). Also \( f \neq g \Rightarrow \) there exists \( \alpha_0 \in J \) such that \( x_{\alpha_0} = f(\alpha_0) \neq g(\alpha_0) = y_{\alpha_0} \). We have \( x_{\alpha_0}, y_{\alpha_0} \in X_{\alpha_0} \) and \( x_{\alpha_0} \neq y_{\alpha_0} \). Hence \( (X_{\alpha_0}, J_{\alpha_0}) \) is a Hausdorff topological space implies that there exist \( U_{\alpha_0}, V_{\alpha_0} \in J_{\alpha_0} \) satisfying:

(i) \( x_{\alpha_0} \in U_{\alpha_0}, y_{\alpha_0} \in V_{\alpha_0} \) and (ii) \( U_{\alpha_0} \cap V_{\alpha_0} = \emptyset \).

Now use (i) and (ii) to construct basic open sets \( U, V \) in the product space satisfying \( f \in U, g \in V, \) and \( U \cap V = \emptyset \). So, let \( U_\alpha = X_\alpha, V_\alpha = X_\alpha \), whenever \( \alpha \neq \alpha_0 \). We already have \( U_{\alpha_0}, V_{\alpha_0} \) which are open sets in \( (X_{\alpha_0}, J_{\alpha_0}) \). Define \( U, V \) as \( U = \prod_{\alpha \in J} U_\alpha, V = \prod_{\alpha \in J} V_\alpha \), where \( U, V \) are defined as above. We have \( f(\alpha) \in X_\alpha, g(\alpha) \in X_\alpha \) for all \( \alpha \in J \) and hence \( f \in U, g \in V \) (why?). Also \( U \cap V = (\prod_{\alpha \in J} U_\alpha) \cap (\prod_{\alpha \in J} V_\alpha) = \prod_{\alpha \in J} U_\alpha \cap V_\alpha = \emptyset \), since \( U_{\alpha_0} \cap V_{\alpha_0} = \emptyset \). That is for \( f, g \in \prod_{\alpha \in J} X_\alpha \) with \( f \neq g \) there exist basic open sets \( U, V \) in the product space such that \( f \in U, g \in V, \) and \( U \cap V = \emptyset \). This implies that the product space \( \prod_{\alpha \in J} X_\alpha, J \) is a Hausdorff space. □

**Note.** Let \((X, J)\) be a topological space and \( \mathcal{B} \) be a basis for \((X, J)\) (or say \( \mathcal{B} \) is a basis for \( J \)). Then for a subset \( A \) of \( X \), \( x \in \overline{A} \) if and only if for each \( U \in \mathcal{B} \) with \( x \in U, U \cap A \neq \emptyset \). That is \( x \in \overline{A} \) if and only if for each basic open set \( U \) containing \( x \), \( U \cap A \neq \emptyset \). *
Theorem 2.1.4. Let \((X_\alpha, J_\alpha)\), \(\alpha \in J\) be a collection of topological spaces and \(A_\alpha \subseteq X_\alpha\) for each \(\alpha \in J\) then \(\prod_{\alpha \in J} \overline{A_\alpha} = \prod_{\alpha \in J} \overline{A_\alpha}\), with respect to the product space \((\prod_{\alpha \in J} X_\alpha, J)\).

Proof. First let us prove \(\prod_{\alpha \in J} A_\alpha \subseteq \prod_{\alpha \in J} \overline{A_\alpha}\). Let \(f \in \prod_{\alpha \in J} A_\alpha\). Then \(f : J \to \bigcup_{\alpha \in J} \overline{A_\alpha}\) such that \(f(\alpha) = x_\alpha \in \overline{A_\alpha}\) for all \(\alpha \in J\). We aim to prove that \(f\) is in the closure of \(\prod_{\alpha \in J} A_\alpha\) in the product topological space \(\prod_{\alpha \in J} X_\alpha\). So take a basic open set \(B\) in the product space \((\prod_{\alpha \in J} X_\alpha, J)\) containing \(f\), say \(B = \prod_{\alpha \in J} U_\alpha\). It is given that \(f \in \prod_{\alpha \in J} \overline{A_\alpha}\). Hence \(f(\alpha) = x_\alpha \in \overline{A_\alpha}\) for each \(\alpha \in J\). Now \(f \in B = \prod_{\alpha \in J} U_\alpha\) implies \(f(\alpha) \in U_\alpha\) for all \(\alpha \in J\). That is \(U_\alpha\) is an open set containing \(x_\alpha\) and \(x_\alpha \in \overline{A_\alpha}\). This implies that \(U_\alpha \cap A_\alpha \neq \emptyset\). Let \(z_\alpha \in U_\alpha \cap A_\alpha\) for all \(\alpha \in J\). Define \(g : J \to \bigcup_{\alpha \in J} A_\alpha\) as \(g(\alpha) = z_\alpha \in A_\alpha\) then \(g \in B \cap \prod_{\alpha \in J} A_\alpha\). This implies that for each basic open set \(B\) containing \(f\), \(B \cap \prod_{\alpha \in J} A_\alpha \neq \emptyset\). This implies \(f \in \prod_{\alpha \in J} \overline{A_\alpha}\). That is \(f \in \prod_{\alpha \in J} \overline{A_\alpha}\) implies \(f \in \prod_{\alpha \in J} A_\alpha\) and this proves the assertion

\[
\prod_{\alpha \in J} A_\alpha \subseteq \prod_{\alpha \in J} \overline{A_\alpha}. \tag{2.1}
\]

Now let us prove the converse part namely \(\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \prod_{\alpha \in J} A_\alpha\). So let \(f \in \prod_{\alpha \in J} \overline{A_\alpha}\). Then \(f \in \prod_{\alpha \in J} A_\alpha\) and \(f \in \prod_{\alpha \in J} X_\alpha\). Our aim is to prove: \(f \in \prod_{\alpha \in J} A_\alpha\) if and only if \(f(\alpha) \in X_\alpha\) for each \(\alpha \in J\) and \(f \in \prod_{\alpha \in J} \overline{A_\alpha}\) if and only if \(f(\alpha) \in \overline{A_\alpha}\) for each \(\alpha \in J\). For a fixed \(\alpha_0 \in J\) take an open set \(U_{\alpha_0}\) containing \(f(\alpha_0) = x_{\alpha_0}\). We will have to use the fact that \(f \in \prod_{\alpha \in J} \overline{A_\alpha}\). To use this fact we will have to construct a basic open set containing \(f\). Keeping this in mind, we define \(B = \prod_{\alpha \in J} U_\alpha\), where \(U_\alpha = X_\alpha\), when \(\alpha \neq \alpha_0\) and \(U_{\alpha_0}\) is as given above. Now this \(B\) is a basic open set containing \(f\) and hence \(f \in \prod_{\alpha \in J} A_\alpha\) implies \(B \cap \prod_{\alpha \in J} A_\alpha \neq \emptyset\). That is \((\prod_{\alpha \in J} U_\alpha) \cap (\prod_{\alpha \in J} A_\alpha) = \prod_{\alpha \in J} U_\alpha \cap A_\alpha \neq \emptyset\). This implies that each \(U_\alpha \cap A_\alpha \neq \emptyset\). In particular \(U_{\alpha_0} \cap A_\alpha_0 \neq \emptyset\) implies \(x_{\alpha_0} \in \overline{A_{\alpha_0}}\). Note that
though our $\alpha_0 \in J$ is a fixed element, there is no restriction on $\alpha_0 \in J$ and the proof will go through for any $\alpha \in J$. This gives that $f(\alpha) = x_\alpha \in \overline{A}_\alpha$ for all $\alpha \in J$ and this implies that $f \in \prod_{\alpha \in J} \overline{A}_\alpha$. Hence $f \in \prod_{\alpha \in J} \overline{A}_\alpha \Rightarrow f \in \prod_{\alpha \in J} A_\alpha$. This implies

$$
\prod_{\alpha \in J} A_\alpha \subseteq \prod_{\alpha \in J} \overline{A}_\alpha.
$$

(2.2)

Now combining Eqs. (2.1) and (2.2) we have $\prod_{\alpha \in J} A_\alpha = \prod_{\alpha \in J} \overline{A}_\alpha$. ■

2.2 The Box Topology

**Definition 2.2.1.** Let $(X_\alpha, J_\alpha), \alpha \in J,$ be a collection of topological spaces and $B_b = \{ \prod_{\alpha \in J} U_\alpha : U_\alpha \in J_\alpha \text{ for } \alpha \in J \}$. Then $B_b$ is a basis for a topology on $X = \prod_{\alpha \in J} X_\alpha$ and $J_b$, the topology induced by $B_b$, is called the **box topology on** $X$.

**Remark 2.2.2.** From the definitions of product and box topologies, it is clear that $B \subseteq B_b$, where the product topology $J$ on $X$ is induced by $B$ (refer the definition of product topology) and the box topology $J_b$ is induced by $B_b$. Now $B \subseteq B_b$ implies $J_B = J \subseteq J_b$. That is the product topology on $\prod_{\alpha \in J} X_\alpha = X$ is weaker than the box topology $J_b$ on $X$. ♦

It is to be noted that if a subset $A$ of $X$ is open with respect to the product topology on $X$ then it is also open with respect to the box topology on $X$. Note that the set $A = \prod_{i=1}^{\infty} \left( \left( -\frac{1}{n}, \frac{1}{n} \right) \right)$ is an open set in $\mathbb{R}^w = \mathbb{R} \times \mathbb{R} \times \cdots$ with respect to the box topology but not open with respect to the product topology on $\mathbb{R}^w$. Also we have proved that if $(X_\alpha, J_\alpha), \alpha \in J,$ is a collection of Hausdorff topological spaces then the product space $\left( \prod_{\alpha \in J} X_\alpha, J_\alpha \right)$ is also a Hausdorff space. Since $J \subseteq J_b$ it is clear that if $(X_\alpha, J_\alpha), \alpha \in J$ is a collection of Hausdorff topological spaces then $\left( \prod_{\alpha \in J} X_\alpha, J_b \right)$ is also a Hausdorff space.

Note that if $J$ is a nonempty finite index set then $J = J_b$ on $\prod_{\alpha \in J} X_\alpha$. 

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Theorem 2.2.3. Let \((X_\alpha, \mathcal{J}_\alpha), \alpha \in J,\) be a collection of topological spaces and for each \(\alpha \in J,\) let \(A_\alpha \subseteq X_\alpha.\) Then \(\prod_{\alpha \in J} \overline{A_\alpha} = \prod_{\alpha \in J} A_\alpha,\) where \(\overline{A_\alpha}\) denotes the closure of \(A_\alpha\) in \((X_\alpha, \mathcal{J}_\alpha)\) and \(\prod_{\alpha \in J} A_\alpha\) denotes the closure of \(\prod_{\alpha \in J} A_\alpha\) in \((\prod_{\alpha \in J} X_\alpha, \mathcal{J}_b).\)

Proof. Proof of this theorem is similar to that of theorem 2.1.4. ■

Theorem 2.2.4. Let \((X, \mathcal{J}), (Y, \mathcal{J}'), (Z, \mathcal{J}'')\) be topological spaces and \(f : (X, \mathcal{J}) \to (Y, \mathcal{J}'), g : (Y, \mathcal{J}') \to (Z, \mathcal{J}'')\) be continuous functions then the composite function \(g \circ f : (X, \mathcal{J}) \to (Z, \mathcal{J}'')\) defined as \((g \circ f)(x) = g(f(x))\) is also a continuous function.

Proof. We aim to prove \(g \circ f : (X, \mathcal{J}) \to (Z, \mathcal{J}'')\) is a continuous function. So start with an open set \(W\) in \((Z, \mathcal{J}'')\). Now \(W\) is an open set in \(Z\) (means \(W \in \mathcal{J}'')\) and \(g : (Y, \mathcal{J}') \to (Z, \mathcal{J}'')\) is a continuous function implies \(g^{-1}(W)\) is an open set \(Y.\) Now \(f : (X, \mathcal{J}) \to (Y, \mathcal{J}')\) is also a continuous function. Hence \(f^{-1}(g^{-1}(W))\) is an open set in \(X.\) We define for \(A \subseteq Y,\) \(f^{-1}(A) = \{x \in X : f(x) \in A\}.\) That is \(x \in f^{-1}(A)\) if and only if \(f(x) \in A.\) Hence \(f^{-1}(g^{-1}(W)) = \{x \in X : f(x) \in g^{-1}(W)\}\) = \(\{x \in X : g(f(x)) \in W\} = (g \circ f)^{-1}(W).\) That is we have proved: \(W\) is an open set in \((Z, \mathcal{J}'')\) implies \((g \circ f)^{-1}(W)\) is an open in \((X, \mathcal{J})\) implies \(g \circ f : (X, \mathcal{J}) \to (Z, \mathcal{J}'')\) is a continuous function. ■

Definition 2.2.5. A sequence \(\{x_n\}\) is a topological space \((X, \mathcal{J})\) is said to converge an element \(x\) in \(X\) if for each open set \(U\) containing \(x,\) there exists a natural number \(n_0\) (that is \(n_0 \in \mathbb{N}\)) such that \(x_n \in U\) for all \(n \geq n_0.\)

The product topology \(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \prod_{n=1}^{\infty} \mathbb{R}_{\alpha} = \mathbb{R}^w,\) where \(\mathbb{R}_{\alpha} = \mathbb{R},\) \(n = 1, 2, 3 \ldots\) is metrizable. That is, we will have to define a metric say \(d_1\) of \(\mathbb{R}^w\) such that \(\mathcal{J}_{d_1} = \mathcal{J},\) the product topology on \(\mathbb{R}^w.\) For \(x = (x_n)_{n=1}^{\infty} = (x_n) \in \mathbb{R}^w,\)
y = (y_n) ∈ R^w, let d_1(x, y) = \sup_{n \geq 1} \left\{ \frac{d(x_n, y_n)}{n} \right\}, \text{ where } \overline{d}(x_n, y_n) = \min\{1, |x_n - y_n|\}.

(Exercise. Let (X, d) be a metric space and \overline{d}(x, y) = \min\{1, d(x, y)\} for all x, y ∈ X. Prove that (i) \overline{d} is a metric on X, (ii) J_\overline{d} = J_d.)

It is easy to prove that d_1 is a metric on R^w. First let us prove that J_{d_1} ⊆ J. So, let U ∈ J_{d_1}. We aim to prove that each point of U is an interior point of U with respect to the product topology J on R^w. Take x ∈ U. Now x ∈ U, U is an open set in the metric space (R^w, d_1) implies there exists r > 0 such that B_{d_1}(x, r) ⊆ U. Now choose n_0 ∈ N such that \frac{1}{n_0} < r and B = (x_1 - \frac{\epsilon}{n_0} + \epsilon) \times \cdots \times (x_n - \frac{\epsilon}{n_0} + \epsilon) \times R \times R \times \cdots

then B is a basic open set in (R^w, J) containing x = (x_n). Now we leave it as an exercise to prove that B ⊆ B_{d_i}(x, \epsilon). Hence for each x ∈ U, there exists a basic open set B in (R^w, J) such that x ∈ B ⊆ U. This proves that U ∈ J that is

J_{d_1} ⊆ J. \hspace{1cm} (2.3)

Now let us prove that J ⊆ J_{d_1}. To prove this statement it is enough to prove that every basic open subset V of (R^w, J) is in J_{d_1}. Now V is a basic open set in the product topology implies there exists k ∈ N such that V = V_1 \times V_2 \times \cdots \times V_k \times R \times R \times \cdots. Let x = (x_n)_{n=1}^\infty ∈ V. Hence there exist \epsilon_1, \epsilon_2, \ldots, \epsilon_k, \ 0 < \epsilon_i < 1 \text{ for } i = 1, 2, \ldots, k \text{ such that } (x_i - \epsilon_i, x_i + \epsilon_i) \subseteq V_i. Now let \epsilon = \min\{\frac{\epsilon}{i} : i = 1, 2, \ldots, k\}. (note: we have U_i = R for all i > k and hence it is enough to consider \epsilon_1, \ldots, \epsilon_k) and we claim that B_{d_1}(x, \epsilon) \subseteq V. So, let y ∈ B_{d_1}(x, \epsilon) then d_1(x, y) < \epsilon implies \sup_{n \geq 1} \left\{ \frac{\overline{d}(x_n, y_n)}{n} \right\} < \epsilon implies \overline{d}(x_n, y_n) < \epsilon for all n = 1, 2, \ldots, k implies \frac{1}{n} \min\{1, |x_n - y_n|\} < \epsilon for all n = 1, 2, \ldots, k implies \min\{1, |x_n - y_n| < n\epsilon < \epsilon_n < 1 \text{ for all } n = 1, 2, \ldots, k \text{ implies } |x_n - y_n| < \epsilon_n \text{ for all } n = 1, 2, \ldots, k \text{ implies } y = (y_n) ∈ V_1 \times \cdots \times V_k \times R \times \cdots = V.

This proves that B_{d_1}(x, \epsilon) \subseteq V. That is for each x ∈ V there exists \epsilon > 0 such that
$B_d(x, \epsilon)$. Hence every point of $V$ is an interior point of $V$ with respect to $(\mathbb{R}^w, J_{d_1})$.

Hence

$$V \in J_{d_1}. \tag{2.4}$$

Now if $U \in J$ then there exists $k \in \mathbb{N}$ and $B_1, B_2, \ldots, B_k \in \mathcal{B}$ ($\mathcal{B} = \{ \prod_{n=1}^{\infty} U_n: \text{each } U_n \text{ is open in } \mathbb{R} \text{ and } U_n = \mathbb{R} \text{ for except finitely many } n's \}$ is our standard basis for the product topology $J$ on $\mathbb{R}^w$) such that $U = B_1 \cap \cdots \cap B_k$. We have proved that each basic open set $B$ of $J$ belongs to $J_{d_1}$ (i.e $B \in J_{d_1}$) (refer Eq. (2.4)). Now $B_1, B_2, \ldots, B_k \in J_{d_1}$ and $J_{d_1}$ is a topology implies $B_1 \cap B_2 \cap \cdots \cap B_k \in J_{d_1}$. This proves that $U \in J_{d_1}$.

That is

$$U \in J \Rightarrow U \in J_{d_1} \Rightarrow J \subseteq J_{d_1}. \tag{2.5}$$

From Eqs. (2.3) and (2.5) we see that $J = J_{d_1}$. Hence the product space $\mathbb{R}^w = \mathbb{R} \times \mathbb{R} \times \cdots$ with product topology $J$ is metrizable. It is interesting to note that if we consider the box topology say $J_6$ on $\mathbb{R}^w$, then $(\mathbb{R}^w, J_b)$ is not a metrizable topological space.

How to prove that a given topological space $(X, J)$ is not metrizable. If $(X, J)$ is metrizable then we will have to find a metric (finding such a metric is not at all an easy task and this statement will become meaningful if we have patience to wait and see the proof of the Urysohn metrization theorem) say $d$ on $X$ such that $J_d = J$. We have just proved that $(\mathbb{R}^w, J)$ (with product topology) is metrizable and in this case we could define a metric $d$ on $\mathbb{R}^w$ such that $J_d = J$.

To prove that a topological space is not metrizable space is comparatively easier. For example, if the given topological space $(X, J)$ is not a Hausdorff topological space then it is clear that there cannot exist any metric $d$ on $X$ such
that \( J_d = J \). We know that if \( d \) is a metric on \( X \), then \((X, J_d)\) is a Hausdorff space. So, what we need here is to find a property a metric space has whereas the given topological space does not have that particular property. Now let us come back and prove that \((\mathbb{R}^w, J_b)\) is not metrizable. Suppose there exists a metric say \( d \) on \( \mathbb{R}^w \) such that 

\[
J_d = J_b.
\]

Then we know that for \( A \subseteq X, x \in \overline{A} \) if and only if there exists a sequence \((x_n)\) in \( A \) such that \( x_n \to x \) as \( n \to \infty \) (to prove this statement, observe the following: For \( x \in A, B(x, \frac{1}{n}) \cap A \neq \emptyset \), for each \( n \in \mathbb{N} \) and hence we have 

\[
\{B(x, \frac{1}{n}) \cap A\}_{n=1}^{\infty} = \{A_n\}_{n=1}^{\infty} = \{A_n\}_{n \in \mathbb{N}} \text{ a collection of nonempty sets.}
\]

Now by axiom of choice there exists a choice function say \( f : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n \) such that \( x_n = f(n) \in A_n \). So using axiom of choice we have got a sequence \((x_n)\) in \( A \) and now it is easy to see that \( x_n \to x \) as \( n \to \infty \).

Note that normally we just say that \( B(x, \frac{1}{n}) \cap A \neq \emptyset \) implies there exists \( x_n \in B(x, \frac{1}{n}) \cap A = A_n \). In such case it is to be understood that in fact we use axiom of choice to define such a sequence \((x_n)\). Now let us prove that if \( A = \{(x_1, x_2, x_3, \ldots) : x_k > 0 \text{ for all } k \in \mathbb{N}\} \). Then \( 0 = (0,0,0,\ldots) \in \overline{A} \), but there does not exist any sequence \((x^{(n)})\) in \( A \) such that \( x^{(n)} \to (0,0,\ldots) \) in \((\mathbb{R}^w, J_b)\).

**Step 1**: Prove that \( 0 \in \overline{A} \).

So take an open set say \( U \) containing 0 then there exists a basic open set \( B = \prod_{n=1}^{\infty} B_n = B_1 \times B_2 \times \cdots \) such that \((0,0,\ldots) \in B_1 \times B_2 \times \cdots \subseteq U \). (Here each \( B_k \) is an open set in \( \mathbb{R} \) containing \( 0 \in \mathbb{R} \) \( 0 \in B_k, k = 1,2,3,\ldots \) implies there exist \( a_k, b_k \in \mathbb{R}, a_k < b_k \) such that \( 0 \in (a_k, b_k) \subseteq B_k, b_k > 0 \) and hence \( b_k^2 > 0 \) implies \( b = (\frac{b_k}{2})_{k=1}^{\infty} \in A \cap B \) implies \( A \cap B \neq \emptyset \) implies \( A \cap U \neq \emptyset \) implies \( 0 = (0,0,\ldots) \in \overline{A} \). Now we claim that there cannot exist any sequence \( x^{(n)} \) in \( A \) such that \( x^{(n)} \to (0,0,0,\ldots) \). Let \( x^{(n)} = (a_{1n}, a_{2n}, \ldots) \in A \). Then each \( a_{in} > 0 \) for all \( i = 1,2,\ldots \). In particular,
\(a_{kk} > 0\) for all \(k = 1, 2, \ldots\). Let \(U = \left(\frac{(-a_{11})}{2}, \frac{a_{11}}{2}\right) \times \left(\frac{(-a_{22})}{2}, \frac{a_{22}}{2}\right) \times \ldots\) then \(U\) is an open set in \((\mathbb{R}^w, J_b)\) containing 0.

What will happen if \(U \cap A \neq \emptyset\). Note that for each \(n\), \(a_{nn} \notin \left(\frac{(-a_{nn})}{2}, \frac{a_{nn}}{2}\right)\) and hence \(x^{(n)} = (a_{1n}, a_{2n}, \ldots) \notin U\). If \(x^{(n)} \to (0, 0, \ldots)\) then there exists \(n_0 \in \mathbb{N}\) such that \(x^{(n)} \in U\) for all \(n \geq n_0\). But here \(x^{(n)} \notin U\) for every \(n\). Hence \(x^{(n)}\) does not converge to \((0,0,0, \ldots)\). So \((0,0,0, \ldots) \in \overline{A}\) but there cannot exist any sequence in \(A\) which converges to \((0,0, \ldots)\) with respect to \(J_b\). This proves that \((\mathbb{R}^w, J_b)\) is not a metrizable topological space.

2.3 Quotient (Identification) Spaces

We start with a given topological space \((X, J)\). By identifying some of the points of \(X\) we can produce a new topology on a new set say \(X^*\). For example if we consider the closed unit ball in \(\mathbb{R}^2\), then our given topological space is \((X, J)\), where \(X\) is the closed unit ball in \(\mathbb{R}^2\). Here we consider \((X, J)\) as a subspace of the Euclidean space \(\mathbb{R}^2\). Now we get a new set \(X^* = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \cup \{S'\}\), where \(S'\) is the unit circle (boundary) of the closed disc \(X\). By defining a suitable topology \(J^*\) on \(X^*\) we can show that \((X^*, J^*)\) is homeomorphic to the 2-sphere \(S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}\). It is to be noted that here we are considering \(S^2\) as a subspace of \(\mathbb{R}^3\) (also note that if no topology on \(\mathbb{R}^n, n \geq 1\) is mentioned then it is understood that we have the usual topology on \(\mathbb{R}^n\)).

Now let us see how to construct the quotient topology. Let \((X, J)\) be a topological space and \(X^*\) be a nonempty set. Let \(p : X \to X^*\) be a surjective map.
Then \( J^* = \{ A \subseteq Y : p^{-1}(A) \text{ is open in } (X, \mathcal{J}) \} \) is a topology on \( X^* \). This topology \( J^* \) on \( X^* \) is called the quotient topology on \( X^* \) induced by \( p \).

It is easy to prove that \( J^* \) is a topology on \( X \) and we leave it as an exercise.

**Definition 2.3.1.** Let \((X, \mathcal{J})\) be a topological space and \( X^* \) be a partition of \( X \) into disjoint subsets whose union is \( X \). Let \( p : X \to X^* \) be the natural map satisfying the condition namely \( x \in p(x) \), for each \( x \in X \). Suppose for a given \( x \in X \) there exist \( A, B \in X^* \) such that \( x \in A \) and \( x \in B \). Then \( x \in A \cap B \). This implies \( B = A \). Hence for each \( x \in X \) there exists a unique \( A \in X^* \) such that \( x \in A \) and this \( A \) is our \( p(x) \). Also \( \bigcup_{A \in X^*} A = X \) implies that \( p \) is onto. The quotient topology \( J^* \) on \( X^* \) is induced by \( p \) and we say that \((X^*, J^*)\) is a quotient topology of \((X, \mathcal{J})\).

Let \((X, \mathcal{J})\) be a topological space and \( X^* \) be a partition of \( X \) into disjoint subsets whose union is \( X \). Define a relation \( R \) on \( X \) as follows:

\[
R = \{(x, y) \in X \times X : x, y \in A \text{ for some } A \in X^* \}
\]

then (i) \( xRx \), that is \((x, x) \in R \) for all \( x \in X \), (ii) for \( x, y \in X \), \( xRy \) implies there exists \( A \in A^* \) such that \( x, y \in A \). Hence \( y, x \in A \) and this gives \( yRx \) that is for \( x, y \in X \) \( xRy \Rightarrow yRx \), (iii) for \( x, y, z \in X \), \( xRy \) and \( yRz \) implies there exist \( A, B \in X^* \) such that \( x, y \in A \) and \( y, z \in B \). Therefore \( y \in A \cap B \) and this implies that \( A = B \). From this we have \( x, z \in A \). Hence \( xRz \). That is \( xRy \) and \( yRz \) implies \( xRz \). From (i), (ii) and (ii) we see that \( R \) is an equivalence relation on \( X \) and hence this relation \( R \) will partition \( X \) into disjoint equivalence classes.

For each \( x \in X \), the equivalence classes determined by \( x \) is given by \( \overline{x} = \{ y \in X : yRx \} \). Hence if \( x \in A \), for some \( A \in X^* \) then \( \overline{x} = A \). Now it is easy to see that for \( U \subseteq X^* \), \( U \in J^* \) if and only if \( \bigcup_{A \in U} A \) is an open subset of \( X \).

Let \((X, \mathcal{J})\) be a topological space and \( X^* \) be a family of disjoint nonempty subsets
of $X$ such that $X = \bigcup_{A \in X^*} A$. Define $q : X \to X^*$ as $q(x) = A$, where $A \in X^*$ is such that $x \in A$. Then the topology $\mathcal{J}_q$ on $X^*$ is the largest topology on $X^*$ which makes $q : (X, \mathcal{J}) \to (X^*, \mathcal{J}_q)$ a continuous function is called the quotient topology (or identification) topology on $X^*$ induced by $q$.

**Theorem 2.3.2.** Let $(X^*, \mathcal{J}_q)$ be an identification space (i.e $\mathcal{J}_q$ is the identification topology on $X^*$ with respect to $q$) defined as above and $(Y, \mathcal{J}')$ be an arbitrary topological space. Then a function $f : (X^*, \mathcal{J}_q) \to (Y, \mathcal{J}')$ is continuous if and only if $f \circ q : (X^*, \mathcal{J}_q) \to (Y, \mathcal{J}')$ is continuous.

**Proof.** Let $f : (X^*, \mathcal{J}_q) \to (Y, \mathcal{J}')$ be a continuous function. We know that by the definition of identification space, $q : (X, \mathcal{J}) \to (X, \mathcal{J}_q)$ is a continuous function. Hence the composite function $f \circ q : (X^*, \mathcal{J}_q) \to (Y, \mathcal{J}')$ is a continuous function. We will have to prove that $f : (X^*, \mathcal{J}_q) \to (Y, \mathcal{J}')$ is continuous. So start with an open set $U$ in $Y$. That is we will have to prove that $f^{-1}(U)$ is open in $(X^*, \mathcal{J}_q)$. But the subset $f^{-1}(U)$ is open in the identification space if and only if $q^{-1}(f^{-1}(U))$ is open in $(X, \mathcal{J})$. But $q^{-1}(f^{-1}(U)) = (f \circ q)^{-1}(U)$ an open set in $(X, \mathcal{J})$ (since $f \circ q : (X, \mathcal{J}) \to (Y, \mathcal{J}')$ is a continuous function). This is what we wanted to prove and hence $f : (X, \mathcal{J}_q) \to (Y, \mathcal{J}')$ is a continuous function.

Let $(X, \mathcal{J})$ be a topological space and $Y$ be a nonempty set. Let $f : X \to Y$ be an onto map. Then $X^* = \{f^{-1}(y) : y \in Y\}$ is a family of disjoint subsets of $X$ such that $\bigcup_{y \in Y} f^{-1}(y) = X$. That is $X^*$ is a partition of $X$. Let $q : X \to X^*$ be the map, known as identification map, defined as above. Let $\mathcal{J}'$ be the largest topology on $Y$ for which $f : (X, \mathcal{J}) \to (Y, \mathcal{J}')$ is continuous. Then it is easy to prove the following:
**Theorem 2.3.3.** Let \((X, \mathcal{J}), (Y, \mathcal{J}')\) be topological spaces and \(f : (X, \mathcal{J}) \xrightarrow{onto} (Y, \mathcal{J}')\) be a homeomorphism. Further suppose \((Z, \mathcal{J}_1)\) is any topological space. Then a function \(g : (Y, \mathcal{J}') \to (Z, \mathcal{J}_1)\) is continuous if and only if \(g \circ f : (X, \mathcal{J}) \to (Z, \mathcal{J}_1)\) is continuous.

**Proof.** To prove \((Y, \mathcal{J}')\) and \((X^*, \mathcal{J}_q)\) are homeomorphic we will have to define a map say \(h : X^* \to Y\) and prove that this map is a homeomorphism.

Let \(z \in X^* = \{f^{-1}(y) : y \in Y\}\) be any element. Then \(z = f^{-1}(y)\), for some \(y \in Y\). So let \(h(z) = h(f^{-1}(y)) = y\). The defined map \(h : X \to Y\) is such that \((h \circ q)(x) = h(q(x)) = h(f^{-1}(y))\) (where \(y \in Y\) is such that \(x \in f^{-1}(y)\) = \(y = f(x)\)). That is \(h \circ q = f\). Let us prove that \(h\) is continuous. Let \(V\) be an open set in \(Y\).

The given topology \(\mathcal{J}'\) on \(Y\) is the largest topology on \(Y\) for which \(f : (X, \mathcal{J}) \to (Y, \mathcal{J}')\) is continuous. Hence \(V\) is an open set on \(Y\) implies \(f^{-1}(V)\) is an open set in \(X\) and hence \((h \circ q)^{-1}(V) = q^{-1}(h^{-1}(V))\) is an open set in \(X\). Therefore \(h^{-1}(V)\) is an open set in \(X^*\). That is \(V\) is an open set in \(Y\) implies \(h^{-1}(V)\) is an open set in \(X^*\). This implies that \(h\) is a continuous map. \(\blacksquare\)
The Torus

Let $X = [0, 1] \times [0, 1]$ with the topology $\mathcal{J}$ on $X$ induced by the standard topology on $\mathbb{R}^2$ (that is $\mathcal{J}$ is the topology on $X$ induced by the Euclidean metric).

Partition $X$ into the subsets of the type:

- the set $A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ consisting of the four corner points,
- all the sets of the form $A_x = \{(x, 0), (x, 1)\}$ for $0 < x < 1$,
- all the sets of the form $A_y = \{(0, y), (1, y)\}$ for $0 < y < 1$,
- all singleton sets of the form $\{(x, y)\}$, $0 < x < 1, 0 < y < 1$. Then the resulting identification space is the torus.

Exercise 2.3.4. Let $(X, \mathcal{J})$, $(Y, \mathcal{J}')$ be topological spaces and $f : X \to Y$ be an onto map. If $f$ maps open sets in $X$ to open sets in $Y$ (that is $f$ is an open map) then prove that $\mathcal{J}'$ is the quotient topology on $Y$ induced by $f$. □
Exercises

1. A function \( f : Z \to X \times Y \) is continuous if and only if its component functions \( f_1 : Z \to X \) and \( f_2 : Z \to Y \) are both continuous, where \( f(z) = (x,y) = (f_1(z), f_2(z)) \).

2. Let \( X \) and \( Y \) be topological spaces. If \( \mathcal{B}_1 \) is basis for the topology on \( X \) and \( \mathcal{B}_2 \) is a basis for the topology on \( Y \), then show that \( \mathcal{B} := \{ B \times C : B \in \mathcal{B}_1, C \in \mathcal{B}_2 \} \) is a basis for a topology on \( X \times Y \).

3. Let \( X \) denote the set of all real numbers with lower limit topology and \( Y \) denote the set of all real numbers with usual topology. Then
   
   (a) find the topologies on (i) \( X \times X \), (ii) \( X \times Y \), (iii) \( Y \times X \), (iv) \( Y \times Y \),
   
   (b) show that each of the above topologies is Hausdorff,
   
   (c) compare the topologies.

4. Prove or disprove the following:

   (a) The diagonal \( \Delta = \{ x \times x : x \in \mathbb{R} \} \) is closed with respect to the usual topology on \( \mathbb{R}^2 \).
   
   (b) The diagonal \( \Delta = \{ x \times x : x \in \mathbb{R} \} \) is closed with respect to the lower limit topology on \( \mathbb{R}^2 = \mathbb{R}_l \times \mathbb{R}_l \).

5. Show that a topological space \( X \) is Hausdorff (also known as \( T_2 \)-space) if and only if the diagonal \( \Delta = \{ x \times x : x \in X \} \) is closed in \( X \times X \).

6. Let \( X_1, X_2, \ldots, X_n \) be topological spaces and let \( A_i \subset X_i, i = 1, 2, \ldots, n \), \( X = \prod_{i=1}^{n} X_i \) with product topology. Then show that \( \text{int}(\prod_{i=1}^{n} A_i) = \prod(\text{int}A_i) \).
7. Let $P_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection of $\mathbb{R}^2$ onto the $x$-axis. Show that $P_1$ is open but not closed.

8. Find the interior points, limit points and boundary points of each of the following subsets of $\mathbb{R}^2$, with respect to each of topology given in question (3a).

(a) $A = \{(x, y) : x \in \mathbb{R}, y = 0\}$.
(b) $B = \{(x, y) : x > 0, y \neq 0\}$.
(c) $C = A \cup B$.
(d) $D = \{(x, y) : x \in \mathbb{Q}, y \in \mathbb{R}\}$.
(e) $E = \{(x, y) : x, y \in \mathbb{Q}\}$.
(f) $F = \{(x, y) : x \in \mathbb{Q}, y \in \mathbb{Q}^c\}$.
(g) $G = \{(x, y) : x \neq 0, y \leq \frac{1}{2}\}$.
(h) $H = \{(x, y) : 0 < x^2 + y^2 \leq 1\}$.
(i) $I = \{(x, y) : 0 < x^2 - y^2 \leq 1\}$.

9. Let $\mathbb{R}^w : \{(x_1, x_2, x_3, \cdots) : x_i \in \mathbb{R}, \text{for all } i \in \mathbb{N}\}$ and let $\mathcal{J}$ and $\mathcal{J}_b$ be the product and box topologies on $\mathbb{R}^w$ respectively. Suppose $f$ is a map on $\mathbb{R}^w$, defined as $f(x_1, x_2, x_3, \cdots) = (a_1 x_1, a_2 x_2, a_3 x_3, \cdots)$, where $a_i > 0$ for all $i \in \mathbb{N}$.

Then prove or disprove:

(a) $f : (\mathbb{R}^w, \mathcal{J}) \to (\mathbb{R}^w, \mathcal{J})$ is continuous.
(b) $f^{-1} : (\mathbb{R}^w, \mathcal{J}) \to (\mathbb{R}^w, \mathcal{J})$ is continuous.
(c) $f : (\mathbb{R}^w, \mathcal{J}) \to (\mathbb{R}^w, \mathcal{J}_b)$ is continuous.
(d) $f^{-1} : (\mathbb{R}^w, \mathcal{J}) \to (\mathbb{R}^w, \mathcal{J}_b)$ is continuous.
(e) $f : (\mathbb{R}^w, \mathcal{J}_1) \to (\mathbb{R}^w, \mathcal{J})$ is continuous.
(f) $f^{-1} : (\mathbb{R}^w, \mathcal{J}_1) \to (\mathbb{R}^w, \mathcal{J})$ is continuous.
(g) $f : (\mathbb{R}^w, \mathcal{J}_1) \to (\mathbb{R}^w, \mathcal{J}_b)$ is continuous.
(h) \( f^{-1} : (\mathbb{R}^w, J_1) \to (\mathbb{R}^w, J_b) \) is continuous.

10. Let \((X_n, d_n), n \in \mathbb{N}\) be a countable collection of metric spaces and \(X = \prod_{n=1}^{\infty} X_n\). For \(x = (x_n) \in X, y = (y_n) \in Y\). Let \(d(x, y) = \sum_{n=1}^{\infty} \alpha_n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}\). Prove that (i) \(d\) is a metric on \(X\), (ii) \(J_d\) is the product topology on \(X\) and \(\{\alpha_n\}_{n=1}^{\infty}\) is a sequence of positive real numbers such that \(\sum_{n=1}^{\infty} \alpha_n\) is a convergent series.

11. For \(i = 1, 2\) let \(f_i : X_i \to Y_i\) be maps between topological spaces. The map \(g : X_1 \times X_2 \to Y_1 \times Y_2\) is defined by \(g(x_1, x_2) = (f_1(x_1), f_2(x_2))\). Show that \(g\) is continuous if and only if \(f_1\) and \(f_2\) are continuous.

12. Let \(X = \mathbb{R}\) and \(X^* = \{\{x\}, \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} : x \in \mathbb{R}\}\). Define \(f : X \to X^*\) as \(f(x) = \{x\}\) for all \(x \in \mathbb{R}\) and \(f(x) = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}\) for all \(x \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}\). Let \(J_K\) be the \(K\)-topology on \(\mathbb{R}\) and \(J^*\) be the quotient topology on \(X^*\) induced by \(f\). Prove that \((X^*, J^*)\) is a \(T_1\)-space. Is \((X^*, J^*)\) a Hausdorff space? Justify your answer.
Chapter 3

Connected Topological Spaces

3.1 Connected Spaces

Definition 3.1.1. A topological space \((X, \mathcal{J})\) is said to be a \textit{disconnected topological space} if there exist nonempty open sets \(A\) and \(B\) of \(X\) such that (i) \(A \cap B = \emptyset\), (ii) \(X = A \cup B\).

In such a case \(B = A^c\) and \(A = B^c\) and hence \(A\) and \(B\) are closed sets. Also \(X\) contains a nonempty proper subset \(A\) (that is \(A \neq \emptyset, X\)) which is both open and closed in \(X\).

A topological space \((X, \mathcal{J})\) is said to be \textit{connected} if there cannot exist nonempty closed (open) subsets \(A\) and \(B\) of \(X\) such that (i) \(A \cap B = \emptyset\), (ii) \(X = A \cup B\). Equivalently, \((X, \mathcal{J})\) is connected if and only if \(\emptyset\) and \(X\) are the only subsets of \(X\) which are both open and closed in \(X\).

Examples 3.1.2. (i) Let \(X\) be a set containing at least two elements and \(A \subseteq X\), \(A \neq \emptyset, X\). Then \(\mathcal{J} = \{\emptyset, X, A, A^c\}\) is a topological space. Here \(A\) is such that \(A \neq \emptyset\), \(A \neq X\) and \(A\) is both open and closed. Hence \((X, \mathcal{J})\) is not a connected topological space.

(ii) Let \(X\) be a nonempty set and \(\mathcal{J}_f = \{A \subseteq X, X \setminus A = A^c\ \text{finite or } A^c = X\}\) then \((X, \mathcal{J}_f)\) is a topology on \(X\), known as cofinite topology on \(X\).
Note. If $X$ is a finite set containing at least two elements then $\mathcal{J}_f$ is the discrete topology on $X$. In this case $(X, \mathcal{J}_f)$ is not a connected topological space.

What will happen when $X$ is an infinite set and $\mathcal{J}_f$ is the cofinite topology on $X$. Is $(X, \mathcal{J}_f)$ not connected? In other words can we find a subset $A$ of $X$ such that $A \neq \phi$ and $A \neq X$ but $A$ is both open and closed? $A \neq X$, $A$ is closed implies $A$ is a finite set. Also $A \neq \phi$ implies such a nonempty finite set cannot be an open set. Therefore $\phi$ and $X$ are the only sets which are both open and closed. This implies $(X, \mathcal{J}_f)$ is a connected topological space whenever $X$ is an infinite set. 

3.2 Connected Subsets of the Real Line

Keeping our intuition alive, let us prove that intervals are connected subsets of $\mathbb{R}$ and they are the only connected subsets of $\mathbb{R}$. Recall that a subset $J$ of $\mathbb{R}$ is an interval if and only if whenever $a, b \in J$ and $a < c < b$, we have $c \in J$. Note that the null set $\phi$ and singleton sets are also intervals. For example for $x \in \mathbb{R}$, $\phi = (x, x) = (1, 1)$ and $\{x\} = [x, x]$, the singleton set containing $x$.

We say that a subset $Y$ of a topological space $X$ is connected if $(Y, \mathcal{J}_Y)$ is connected.

First let us prove that if $A \subseteq \mathbb{R}$ is not an interval then $A$ is not connected. Here $\mathcal{J}_A = \{A \cap U : U$ is an open set in $\mathbb{R}\}$. So, we will have to prove that the topological space $(A, \mathcal{J}_A)$ is not connected. The given set $A$ is not an interval implies there exist $x, y \in A$ and $z \in \mathbb{R}$ such that $x < z < y$ and $z \notin A$. We know that $(-\infty, z), (z, \infty)$ are open sets in $\mathbb{R}$. This implies that $(-\infty, z) \cap A$ and $(z, \infty) \cap A$ are open sets in $(A, \mathcal{J}_A)$. Also $x \in (-\infty, z) \cap A = C$ and $y \in (z, \infty) \cap A = D$ and $A = C \cup D$. That is $C, D$ are nonempty open sets in $(A, \mathcal{J}_A)$ such that $C \cap D = \phi$.
and \( A = C \cup D \). Hence the topological space \((A, J_A)\) cannot be connected. That is the given subset \( A \) of \( \mathbb{R} \) (which is not an interval) is not connected.

Now let us prove:

**Theorem 3.2.1.** Every interval in \( \mathbb{R} \) is connected.

**Proof.** Let \( J \) be an interval and let us assume that \( J \) contains at least two elements. For if \( J = \emptyset \) or a singleton set then the null set \( \emptyset \) and the whole space \( J \) are the only sets which are both open and closed in \( J \) and hence \( J \) is connected. Now let us suppose that there exist nonempty closed sets \( A, B \) in \( J \) (that is \( A, B \subseteq J \) and \( A, B \) are closed sets in \((J, J_J)\), where \( J_J = \{U \cap J : U \text{ is open in } \mathbb{R}\}\)) such that \( J = A \cup B \). Fix \( a \in A, b \in B \) and without loss of generality let us say \( a < b \). Note that \( a \in A \subseteq J, b \in B \subseteq J \).

![Figure 3.1](image)

Since \( J \) is an interval \([a, b] \subseteq J \). Let \( y = \sup(A \cap [a, b]) \). Now let us prove that \( y \in A \cap B \). First let us prove that \( y \in A \). Let \( U \) be an open set in \( J \) containing \( y \) \((y \in [a, b] \subseteq J)\). Then there exists an open set \( V \) in \( \mathbb{R} \) such that \( V \cap J = U \). Hence there exists \( \epsilon > 0 \) such that \((y - \epsilon, y + \epsilon) \cap J \subseteq V \cap J = U \). (If \( y = a \), we are through, otherwise take \( 0 < \epsilon < y - a \) and \( \epsilon < b - y \) when \( y \neq b \).) Now \( y - \epsilon \) is not an upper bound for \([a, b] \cap A \) implies there exists \( x_0 \in [a, b] \cap A \) such that \( y - \epsilon < x_0 \). This implies \( x_0 \in (y - \epsilon, y + \epsilon) \cap A \subseteq (y - \epsilon, y + \epsilon) \cap J \subseteq U \). Hence \( x_0 \in U \cap A \).

That is, whenever \( U \) is an open set in \( J \) containing \( y \), then \( U \cap A \neq \emptyset \). This proves that \( y \in \overline{A} = A, \overline{A} \) is the closure of \( A \) in \( J \). Also if \( y = b \), then \( y \in B \). Suppose \( y < b \), \( y \) is the supremum of \( C = A \cap [a, b] \) and \( b \) is an upper bound of \( C \) then take
\( y_0 \in (y, y + \varepsilon) \). Now \( y_0 \in U \cap B \). Hence \( y \in \overline{B_J} = B \). Hence we have \( y \in A \cap B \). Therefore there cannot exist nonempty closed subsets of \( A, B \) in the subspace \((J, J_J)\) such that \( A \cap B = \phi \), \( A \cup B = J \). This proves that \( J \) is connected. ■

### 3.3 Some Properties of Connected Spaces

Now let us prove that continuous image of a connected topological space is connected.

**Theorem 3.3.1.** Let \((X, J)\) be a connected topological space and \((Y, J')\) be any topological space. Suppose \( f : (X, J) \to (Y, J') \) is a surjective continuous map then the image \( f(X) = Y \) is a connected topological space.

**Proof.** (By contradiction). Let \( A, B \) be nonempty open subsets of \( Y \) such that \( A \cap B = \phi \), \( A \cup B = Y \). Now \( f \) is a continuous map, and \( A, B \) are open sets in \( Y \) implies that \( f^{-1}(A), f^{-1}(B) \) are open sets in \( X \). Also

\[
f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \phi \quad (3.1)
\]

and

\[
X = f^{-1}(Y) = f^{-1}(A \cup B) \quad \text{(from 3.1)}
\]

\[
= f^{-1}(A) \cup f^{-1}(B). \quad (3.2)
\]

Since \( A, B \neq \phi \), let \( y \in A \) and \( y' \in B \), and \( f \) is a surjective map implies that there exist \( x, x' \in X \) such that \( f(x) = y \) and \( f(x') = y' \) implies \( f(x) \in A \) and \( f(x') \in B \) implies \( x \in f^{-1}(A) \) and \( x' \in f^{-1}(B) \). Hence \( f^{-1}(A), f^{-1}(B) \) are nonempty open subsets of the connected topological space \((X, J)\) satisfying Eqs. (3.1) and (3.2). This means \((X, J)\) cannot be a connected topological space. That is we have proved: \( f(X) = Y \) is not connected implies \( X \) is also not connected and this gives a contradiction. Hence
our assumption that \((Y, J')\) is not connected is not valid. Therefore \((Y, J')\) is a connected topological space.

It is to be noted that if \(X\) is a connected topological space and \(f : X \to Y\) is a continuous function, where \(Y\) is any topological space, then the image \(f(X)\) is also a connected topological space. Here we will have to consider \(f(X)\) as a subspace of the given topological space \(Y\). Also if \(f : X \to Y\) is continuous then \(f : X \to f(X)\) is also continuous. Hence this result will follow from the previous result.

**Definition 3.3.2.** A subspace \(Y\) of a topological space \((X, J)\) is said to have a separation if and only if there exist nonempty subsets \(A, B\) of \(X\) such that

(i) \(Y = A \cup B\), (ii) \(\overline{A} \cap B = \phi = A \cap B\). Here \(\overline{A}\) is the closure of \(A\) in \(X\).

**Example 3.3.3.** Let \(X = \mathbb{R}\) and \(Y = [0, 2) \cap (2, 5)\). Then \(Y\) has a separation.

Take \(A = [0, 2), B = (2, 5)\), then (i) \(Y = A \cup B\) is satisfied. Now (ii) \(\overline{A} \cap B = [0, 2] \cap (2, 5) = \phi\), and \(A \cap \overline{B} = [0, 2) \cap [2, 5] = \phi\). Hence from (i) and (ii) we see that \(Y\) has a separation.

**Theorem 3.3.4.** A subspace \(Y\) of a topological space \((X, J)\) is connected if and only if there does not exist any separation for \(Y\).

**Proof.** First let us assume that the subspace \(Y\) is connected. This means the subspace \((Y, J_Y)\) is a connected topological space. So, we will have to prove that \(Y\) does not admit any separation. Suppose \(Y\) has a separation. Hence there exist nonempty subsets \(A, B\) of \(X\) such that \(Y = A \cup B\), \(\overline{A} \cap B = \phi = A \cap \overline{B}\). Now \(\overline{A}_Y = \overline{A} \cap Y = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = A \cup \phi = A\). This implies that \(A\) is a closed subset of \((Y, J_Y)\). Similarly \(\overline{B}_Y = \overline{B} \cap Y = (\overline{B} \cap (A \cup B)) = (\overline{B} \cap A) \cup (\overline{B} \cap B) = B\). This implies that \(B\) is a closed set in \((Y, J_Y)\). Also \(B_Y^c = Y \setminus B = Y \cap B^c = (A \cup B) \cap B^c = (A \cap B^c) \cup (B^c \cap B) = A \cap B^c = A\)
(since \(A \cap B = \phi\)). This means complement of \(B\) with respect to \(Y\) is \(A\). Hence \(B\) is such that \(B \neq \phi\), \(B \neq Y\) (since \(Y = A \cap B\) and \(A \neq \phi\)) and \(B\) is both open and closed in \(Y\). This implies that the subspace \((Y, J_Y)\) is a disconnected space. This is a contradiction and we arrived at this contradiction by assuming that \(Y\) has a separation. Hence there does not exist any separation for \(Y\).

Conversely, now assume that there does not exist any separation for \(Y\). Now suppose that the subspace \((Y, J_Y)\) is a disconnected space. Then there exist nonempty closed subsets \(A, B\) in \((Y, J_Y)\) such that \(Y = A \cup B\) and \(A \cap B = \phi\). Now \(\overline{A} \cap B = \overline{A} \cap (Y \cap B) = (\overline{A} \cap Y) \cap B = \overline{A_Y} \cap B = A \cap B = \phi\). (\(A\) is closed in \(Y\) implies \(\overline{A_Y} = A\)) Similarly, \(A \cap \overline{B} = A \cap Y \cap \overline{B} = A \cap B_Y = A \cap B = \phi\). (\(B\) is closed in \(Y\) implies \(B_Y = B\).) We have nonempty subsets \(A, B\) in \(X\) such that (i) \(Y = A \cup B\), (ii) \(\overline{A} \cap B = \phi = A \cap \overline{B}\). This means \(Y\) has a separation and this gives a contradiction. We arrived at this contradiction by assuming that the subspace \((Y, J_Y)\) is a disconnected space. Hence the assumption is wrong. That means \((Y, J_Y)\) is connected.

\[\square\]

**Theorem 3.3.5.** Let \((X, J)\) be a disconnected topological space and \(A\) be a subset of \(X\) such that (i) \(A \neq \phi, X\) (ii) \(A\) is both open and closed in \(X\). Suppose \(Y\) is a nonempty connected subspace of \(X\). Then either \(Y \subseteq A\) or \(Y \subseteq A^c\).

**Proof.** \(X = A \cup B\), where \(B = A^c\) implies \(Y = X \cap Y = (A \cup B) \cap Y = (A \cap Y) \cup (B \cap Y)\). Also \((A \cap Y) \cap (B \cap Y) \subseteq A \cap B = A \cap B = \phi\) (since \(B = A^c\) is a closed set in \(X\)) implies \((A \cap Y) \cap (B \cap Y) = \phi\). Similarly, \((A \cap Y) \cap (B \cap Y) \subseteq \overline{A} \cap B = A \cap B = \phi\) implies \((A \cap Y) \cap (B \cap Y) = \phi\). It is given that \(Y\) is a connected subspace of \(X\). Hence it cannot happen that \(A \cap Y \neq \phi\) and \(B \cap Y \neq \phi\). Since \(Y\) is a connected...
subspace of \( X, Y \) cannot admit any separation. Hence \( A \cap Y = \emptyset \) or \( B \cap Y = \emptyset \) implies \( Y \subseteq A^c \) or \( Y \subseteq B^c = A \). ■

**Alternate Proof.** Now \( A \cap Y, B \cap Y \) (where \( B = A^c \)) are open sets in the subspace \( Y \) such that (i) \( Y = (A \cap Y) \cup (B \cap Y) \) and (ii) \( (A \cap Y) \cap (B \cap Y) \subseteq A \cap B = \emptyset \) implies \( A \cap Y = \emptyset \) or \( B \cap Y = \emptyset \) implies \( Y \subseteq A^c \) or \( Y \subseteq A \). Let us prove the following:

**Theorem 3.3.6.** Let \( (X, J) \) be a topological space and \( Y_1, Y_2 \) be connected subspaces of \( X \). Further suppose \( Y_1 \cap Y_2 \neq \emptyset \). Then \( Y_1 \cup Y_2 \) is a connected subspace of \( X \).

**Proof.** We will have to prove that the subspace \( Y_1 \cup Y_2 \) cannot admit any separation. Suppose \( A, B \) are subsets of \( X \) such that (i) \( Y_1 \cup Y_2 = A \cup B \), (ii) \( \overline{A} \cap B = \emptyset = A \cap \overline{B} \) (here \( \overline{A} \) is the closure of \( A \) in \( X \)).

From (i)

\[
Y_1 = (Y_1 \cap A) \cup (Y_1 \cap B),
\]

and from (ii)

\[
\overline{(Y_1 \cap A)} \cap (Y_1 \cap B) = \emptyset \quad \text{and} \quad (Y_1 \cap A) \cap \overline{(Y \cap B)} = \emptyset.
\]

It is given that \( Y_1 \) is a connected subspace. Hence \( Y_1 \) cannot admit a separation. Therefore from Eqs. (3.3) and (3.4) we conclude that \( Y_1 \cap A = \emptyset \) or \( Y_1 \cap B = \emptyset \). Let us say that \( Y_1 \cap A = \emptyset \). This implies that

\[
Y_1 \subseteq A^c.
\]

Similarly, using the fact that \( Y_2 \) is a connected subspace we conclude that \( Y_2 \cap A = \emptyset \) or \( Y_2 \cap B = \emptyset \). If \( Y_2 \cap B = \emptyset \) then

\[
Y_2 \subseteq B^c.
\]
From Eqs. (3.5) and (3.6) we have

$$Y_1 \cap Y_2 \subseteq A^c \cap B^c = (A \cup B)^c. \quad (3.7)$$

Also $Y_1 \cap Y_2 \neq \emptyset$. Hence there exists an element say $x_0 \in Y_1 \cap Y_2$. Then from Eq. (3.7), $x_0 \notin A \cup B$, a contradiction to $Y_1 \cup Y_2 = A \cup B$. Therefore both $Y_1 \subseteq A^c$ and $Y_2 \subseteq A^c$ implies $Y_1 \cup Y_2 \subseteq A^c$ implies $A = \emptyset$. (If there exists $x \in A \subseteq Y_1 \cup Y_2$ then $x \in Y_1 \cup Y_2 \subseteq A^c$.) What we have proved is the following: If (i) and (ii) happens then that leads to $A = \emptyset$ or $B = \emptyset$. ($Y_1 \cap A = \emptyset$ or $Y_1 \cap B = \emptyset$ implies $Y_1 \subseteq A^c$ or $Y_1 \subseteq B^c$. So if we assume $Y_1 \subseteq B^c$ then we would have arrived at $B = \emptyset$.) That is we have proved that $Y_1 \cup Y_2$ cannot admit any separation and this implies $Y_1 \cup Y_2$ is a connected subspace of $X$. \hfill \blacksquare

**Remark 3.3.7.** If $X$ is a topological space and $Y_1, Y_2, \ldots, Y_n$ for some $n \in \mathbb{N}$ is a finite collection of connected topological spaces such that $\bigcap_{i=1}^{n} Y_i \neq \emptyset$ then by using induction we can prove that $Y_1 \cup Y_2 \cup \cdots \cup Y_n$ is a connected subset of $X$. \hfill \blacklozenge

In fact using exactly the same idea of proving that the subspace $Y_1 \cup Y_2$ is connected whenever $Y_1$, $Y_2$ are connected subspaces with the added condition that $Y_1 \cap Y_2 \neq \emptyset$, we prove the following theorem.

**Theorem 3.3.8.** If $\{Y_{\alpha}\}_{\alpha \in J}$ is a collection of connected subspaces of a topological space $X$ and further there exists an $x_0 \in X$ such that $x_0 \in Y_{\alpha}$, for each $\alpha \in J$, then $\bigcup_{\alpha \in J} Y_{\alpha}$ is a connected subspace of $X$.

**Proof.** Here again we will prove that $\bigcup_{\alpha \in J} Y_{\alpha}$ cannot admit any separation. Suppose $A, B$ are subsets of $X$ such that (i) $\bigcup_{\alpha \in J} Y_{\alpha} = A \cup B$, (ii) $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Then our aim is to prove $A = \emptyset$ or $B = \emptyset$. For each fixed $\alpha_0 \in J$, $Y_{\alpha_0} = (Y_{\alpha_0} \cap A) \cup (Y_{\alpha_0} \cap B)$
and \((Y_{\alpha_0} \cap A) \cap (Y_{\alpha_0} \cap B) = \emptyset = (Y_{\alpha_0} \cap A) \cap (Y_{\alpha_0} \cap B)\). Hence \(Y_{\alpha_0}\) is a connected subspace of \(X\) implies \(Y_{\alpha_0} \cap A = \emptyset\) or \(Y_{\alpha_0} \cap B = \emptyset\). Let us say \(Y_{\alpha_0} \cap A = \emptyset\). This will imply that \(Y_{\alpha_0} \subseteq A^c\). Note that \(\alpha_0 \in J\) is an arbitrary element. Hence for each \(\alpha \in J\), \(Y_{\alpha} \subseteq A^c\) or \(Y_{\alpha} \subseteq B^c\). But if \(Y_{\alpha_1} \subset A^c\) for some \(\alpha_1 \in J\) and \(Y_{\alpha_2} \subset B^c\) for some \(\alpha_2 \in J\) then \(x_0 \in Y_{\alpha_1} \cap Y_{\alpha_2} \subseteq A^c \cap B^c = (A \cup B)^c = \left(\bigcup_{\alpha \in J} Y_{\alpha}\right)^c\). This means \(x_0 \notin Y_{\alpha}\) for some \(\alpha \in J\) and that gives a contradiction. Therefore \(Y_{\alpha} \subseteq A^c\) for all \(\alpha \in J\) or \(Y_{\alpha} \subseteq B^c\) for all \(\alpha \in J\) implies \(\bigcup_{\alpha \in J} Y_{\alpha} \subseteq A^c\) or \(\bigcup_{\alpha \in J} Y_{\alpha} \subseteq B^c\) implies \(A = \emptyset\) or \(B = \emptyset\). This means \(\bigcup_{\alpha \in J} Y_{\alpha}\) cannot admit any separation and that is what we wanted to prove.

\[\text{Theorem 3.3.9. Let} \ X \ \text{be a topological space and} \ Y_1, Y_2, \ldots, Y_n, \ldots \ \text{be a collection of connected topological spaces. Further suppose} \ Y_k \cap Y_{k+1} \neq \emptyset \ \text{for all} \ k \in \mathbb{N}. \ \text{Then} \ \bigcup_{k=1}^{\infty} Y_k \ \text{is also a connected space.}\]

\[\textbf{Proof.}\] Now \(Y_1, Y_2\) are connected subspaces of \(X\) and \(Y_1 \cap Y_2 \neq \emptyset\) implies \(E_2 = Y_1 \cup Y_2\) is a connected subspace. Now \(E_2\) is a connected subspace of \(X\) and \(Y_3\) is a connected subspace of \(X\). Further \(Y_2 \cap Y_3 \subseteq (Y_1 \cup Y_2) \cap Y_3 = E_2 \cap Y_3\). Hence \(Y_2 \cap Y_3 \neq \emptyset\) implies \(E_2 \cap Y_3 \neq \emptyset\) implies \(E_3 = E_2 \cup Y_3 = Y_1 \cup Y_2 \cup Y_3\) is a connected subspace of \(X\). Now use induction to prove that, for each \(k \in \mathbb{N}\), \(E_k = Y_1 \cup Y_2 \cup \cdots \cup Y_k\) is a connected subspace of \(X\). We have a collection \(\{E_k\}_{k=1}^{\infty}\) of connected subspaces of \(X\) such that \(Y_1 = E_1 \subseteq \bigcap_{k=1}^{\infty} E_k\). Also \(\bigcap_{k=1}^{\infty} E_k \neq \emptyset\). Since \(Y_1 \neq \emptyset\), Hence \(\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} Y_k\) is a connected subspace.

Now let us prove that if \(Y\) is a nonempty connected subspace of a topological space \(X\) and then it remains connected after adding some (or all) of its limits points to \(E\).
Theorem 3.3.10. Let $Y$ be a nonempty connected subspace of a topological space $X$ and $E$ be a subset of $X$ such that $Y \subseteq E \subseteq Y$. Then $E$ is also a connected subspace of $X$.

**Proof.** Let $A, B$ be subsets of $X$ such that (i) $E = A \cup B$ (ii) $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ then $Y = Y \cap E = (A \cap Y) \cup (B \cap Y)$. Also $(A \cap Y) \cap (B \cap Y) = \emptyset = (A \cap Y) \cap (B \cap Y)$. Hence $Y$ is a connected subspace of $X$ implies $A \cap Y = \emptyset$ or $B \cap Y = \emptyset$. Let us say $A \cap Y = \emptyset$. This implies $Y = B \cap Y \Rightarrow Y \subseteq B \Rightarrow Y \subseteq B$. Since, $E \subseteq Y$, $E = A \cup B$ and $A \cap B = A \cap \overline{B} = \emptyset$ we get $A \subseteq E \subseteq B$ and $A = A \cap B = \emptyset$. That is we have proved that for subsets $A, B$ of $X$, $E = A \cup B$, $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. This implies $A = \emptyset$. We arrived at this conclusion by assuming that $A \cap Y = \emptyset$. If we had assumed $B \cap Y = \emptyset$, then we would have proved that $B = \emptyset$. Therefore $E$ does not admit any separation and hence $E$ is a connected subspace of $X$. ■

Theorem 3.3.11. If $(X_1, \mathcal{J}_1)$, $(X_2, \mathcal{J}_2)$ are connected topological spaces, then the product space $X_1 \times X_2$ is also a connected space.

**Proof.** Fix $a_1 \in X_1$, $a_2 \in X_2$. (Note. $X_1 = \emptyset$ or $X_2 = \emptyset \Rightarrow X_1 \times X_2 = \emptyset$ and in this case $X_1 \times X_2$ is a connected space.) For each $x_1 \in X_1$, $f_{x_1} : X_2 \to X_1 \times X_2$ defined as $f_{x_1}(x_2) = (x_1, x_2)$ is a continuous function. Also we know that continuous image of a connected space is connected. In this case, the continuous image is $f_{x_1}(X_2) = x_1 \times X_2$. That is $x_1 \times X_2 = \{(x_1, x_2) : x_2 \in X_2\}$ is a connected subspace of the product space $X_1 \times X_2$ (refer the vertical line passing through $x_1$). Similarly, $X_1 \times a_2$ is a connected subspace of the product space. Also $(x_1, a_2) \in (x_1 \times X_2) \cap (X_1 \times a_2)$. Hence $(x_1 \times X_2) \cup (X_1 \times a_2)$ is a connected subspace of the product space. Let $T_{x_1} = (x_1 \times X_2) \cup (X_1 \times a_2)$. Also note that $(a_1, a_2) \in T_{x_1}$, for each $x_1 \in X_1$. That is 62
\{T_{x_1}\} is a collection of connected subspaces of the product space \(X_1 \times X_2\). Further \((a_1, a_2) \in T_{x_1}\) for all \(x_1 \in X_1\). Hence \(\bigcup_{x_1 \in X_1} T_{x_1} = X_1 \times X_2\) is a connected space.

Now we use mathematical induction to prove: if \((X_1, J_1), (X_2, J_2), \ldots, (X_n, J_n)\) are a finite collection of connected topological spaces then the product space \(X_1 \times X_2 \times \cdots \times X_n\) is also a connected space. But it is to be noted that we cannot use (say why?) mathematical induction to prove: If \((X_n, J_n), n \in \mathbb{N}\) is a collection of connected topological spaces then the product space \(X_1 \times X_2 \times \cdots = \prod_{n=1}^{\infty} X_n\) is also a connected space. However, we prove the following theorem when we have a collection \((X_\alpha, J_\alpha), \alpha \in J\) (where \(J\) is a nonempty index set) of connected topological spaces.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.2.png}
\caption{Figure 3.2}
\end{figure}

\textbf{Theorem 3.3.12.} Let \((X_\alpha, J_\alpha), \alpha \in J\) be a collection of connected topological spaces. Then the product space \(X = \prod_{\alpha \in J} X_\alpha\) is also a connected space.

\textbf{Proof.} When \(J\) is a finite set we have already proved this result. So let us assume that \(J\) is an infinite set. Also let us assume that each \(X_\alpha \neq \phi\). From each \(X_\alpha\), fix an
element say \( a_a \in X_a \). That is we have a function \( f : J \to \bigcup_{a \in J} X_a \) such that 
\[ f(\alpha) = a_a \in X_a \text{ for all } \alpha \in J. \]
That is \( f \in \prod_{a \in J} X_a \). We normally write \( f = (a_a)_{a \in J} \) and just say that \( (a_a)_{a \in J} \in \prod_{a \in J} X_a \). If \( \alpha_1, \alpha_2, \ldots, \alpha_k \in J \) then \( X(\alpha_1, \alpha_2, \ldots, \alpha_k) = \{ x = (x_\alpha)_{a \in J} : x_\alpha = a_a, \text{ when } \alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_k \} \). If our \( J = \mathbb{N} \) and \( \alpha_j = j, j = 1, 2, \ldots, k \) then \( X(1, 2, \ldots, k) = \{ x_1, x_2, \ldots, x_k, a_{k+1}, a_{k+2}, \ldots \} = X_1 \times X_2 \times \cdots \times X_k \times a_{k+1} \times a_{k+2} \times \cdots \). Note that \( g(x_1, x_2, \ldots, x_k) = (x_1, x_2, \ldots, x_k, a_{k+1}, a_{k+2}, \ldots) \) is a continuous function from the connected space \( X_1 \times X_2 \times \cdots \times X_k \to \prod_{n=1}^{\infty} X_n \) and its image is \( X_1 \times X_2 \times \cdots \times X_k \times a_{k+1} \times a_{k+2} \times \cdots \). Hence \( X(1, 2, \ldots, k) = X_1 \times X_2 \times \cdots \times X_k \times a_{k+1} \times a_{k+2} \times \cdots \) is a connected subspace of \( \prod_{n \in \mathbb{N}} X_n \) and \( X(\alpha_1, \alpha_2, \ldots, \alpha_k) \) is a connected subspace of the product space. Also our fixed \( a = (a_a)_{a \in J} \in X(\alpha_1, \alpha_2, \ldots, \alpha_k) \). Therefore \( Y = \bigcup_{\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq J} X(\alpha_1, \alpha_2, \ldots, \alpha_k) \) (that is \( x \in Y \) if and only if there exists \( k \in \mathbb{N} \) and \( \alpha_1, \alpha_2, \ldots, \alpha_k \in J \) such that \( x \in X(\alpha_1, \alpha_2, \ldots, \alpha_k) \)) is a connected subspace of the product space.

Again \( Y \) is a connected subspace of the product space implies \( \overline{Y} \) is also a connected space. Now let us prove that \( \overline{Y} = \prod_{a \in J} X_a = X \). So take an element say \( x = (x_\alpha)_{a \in J} \in X \) our aim is to prove that \( x \in \overline{Y} \). Start with a basic open set say \( U = \prod_{a \in J} U_a \) containing the point \( x = (x_\alpha)_{a \in J} \). \( U \) is a basic open set in the product space implies there exists \( k \in \mathbb{N} \) and \( \alpha_1, \alpha_2, \ldots, \alpha_k \in J \) such that \( U_a = X_a \) for \( \alpha \in J \) and \( \alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_k \) and \( U_{\alpha_j} \in J_{\alpha_j} \) for all \( j = 1, 2, \ldots, k \). Let \( y = (y_\alpha)_{a \in J} \) be such that \( y_{\alpha_j} = x_{\alpha_j} \) for all \( j = 1, 2, \ldots, k \) and \( y_\alpha = a_\alpha \) when \( \alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_k \). Then \( y \in U \cap X(\alpha_1, \alpha_2, \ldots, \alpha_k) \subseteq U \cap Y \) and hence \( U \cap Y \neq \emptyset \) for every basic open set \( U \) containing \( x \). This implies \( x \in \overline{Y} \). Hence \( X \subseteq \overline{Y} \). This implies \( X = \overline{Y} \). Now \( Y \) is a connected subspace of \( X \) implies \( \overline{Y} \) is also a connected subspace of \( X \) and hence \( X = \overline{Y} \) is a connected space. \( \blacksquare \)
3.4 Connected Components

Now let us define a relation say $R$ on a given topological space $(X, \mathcal{J})$. We know that a relation on $X$ is a subset of $X \times X$. Here $R = \{(x, y) \in X \times X : x, y \in E_{xy} \}$ for a connected subset $E_{xy}$ of $X$. If $(x, y) \in R$ then we write $xRy$ (read as $x$ is related to $y$). So we say that $xRy$ if and only if there is a connected set containing $x, y$. Now it is easy to see that this relation is an equivalence relation. For each $x \in X$, singleton set $\{x\}$ is a connected subset of $X$. That is $E_{xx} = \{x\}$ is a connected set containing $x, x$. Hence $xRx$ for all $x \in X$. Therefore $R$ is reflexive.

Now suppose for $x, y \in X$, $xRy$ this implies there is a connected subset say $E_{xy}$ of $X$ such that $x, y \in E_{xy}$. But $x, y \in E_{xy}$ implies $y, x \in E_{xy}$ this implies $yRx$. That is for $x, y \in X$, $xRy \Rightarrow yRx$ and hence $R$ is symmetric.

Now suppose for $x, y, z \in X$ $xRy$ and $yRz$. Now $xRy$ implies there exists a connected subset say $E_{xy}$ of $X$ such that $x, y \in E_{xy}$. Similarly, $yRz$ implies there exists a connected subset say $E_{yz}$ of $X$ such that $y, z \in E_{yz}$. Now $E_{xy}, E_{yz}$ are connected subsets of $X$ such that $y \in E_{xy} \cap E_{yz}$. This gives us $E_{xy} \cup E_{yz}$ is a connected set and further $y, z \in E_{xy} \cup E_{yz}$. Hence there exists a connected set containing $y, z$. This implies $yRz$. That is we have proved that $xRy$ and $yRz$ implies $xRz$ that is $R$ is transitive.

Thus $R$ is an equivalence relation on $X$. Hence this relation will partition the set $X$ into disjoint equivalence classes.

It is to be noted that for each $x \in X$ there is exactly one equivalence class containing $x$. Let us denote the equivalence class containing $x$ by $[x]$. That is for $x \in X$, $[x] = \{y \in X : yRx\}$ is the equivalence class containing $x$. Now suppose
$x \in [x]$, which is true since $xRx$ and also $x \in [y]$, for some $y \in X$. What is $[y]$? Note that $[y] = \{z \in X : zRy\}$. If such a situation arises then we aim to prove that $[x] = [y]$. So let $z \in [x]$. This implies $zRx$. Also we have $xRy$. Now our relation $R$ is a transitive relation. Hence $zRy$ and $xRy$ together implies that $zRy$ this implies that $z \in [y]$. That is $z \in [x]$ implies $z \in [y]$ implies $[x] \subseteq [y]$. Exactly in the same way we can prove that $[y] \subseteq [x]$. Hence $[x] = [y]$, whenever $x \in [y]$.

For $x, y \in X$, either $[x] \cap [y] = \emptyset$ or $[x] = [y]$. For $x \in X$ there is exactly one equivalence class containing $x$. Such an equivalence class which we denoted by $[x]$, is called a connected component (also known as a component or a maximal connected set containing $x$. Why should we call $[x]$ as a maximal connected set containing $x$? So we will have to prove that if $E$ is a connected subset of $X$ containing $x \in E$, then $E \subseteq [x]$. That is $[x]$ is the largest connected set containing $x$. Have we proved that $[x]$ is a connected subset of $X$?

Note that for each $y \in [x]$, there is a connected subset say $E_y$ ($E_y$ is just a notation) of $X$ containing $x, y$. That is $\{E_y\}_{y \in [x]}$ is a collection of connected sets and $x \in E_y$ for all $y \in [x]$. This implies that $\bigcup_{y \in [x]} E_y$ is also connected set. It is simple exercise to see that $[x] = \bigcup_{y \in [x]} E_y$. Hence $[x]$ is a connected set containing $x$. If $E$ is a connected set containing $x$ then for each $y \in E$, $yRx$. Hence $E \subseteq [x]$.

Also we know that if $A$ is a connected subset of a topological space $X$ then $\overline{A}$ is also a connected subset of $X$. Hence $\overline{[x]}$, the closure of the set $[x]$, is also connected set containing $x$. But we have just proved that $[x]$ is the maximal connected set containing $x$. Hence $\overline{[x]} \subseteq [x]$ implies $\overline{[x]} = [x]$ and $[x]$ is a closed set. Therefore our maximal connected set containing $x$ is a closed set.
Theorem 3.4.1. **Intermediate value theorem.** Let \((X, \mathcal{J})\) be a connected topological space and \(f : (X, \mathcal{J}) \to \mathbb{R}\) be a continuous function. If \(x, y\) are points of \(X\) and \(\alpha\) is a real number such that \(\alpha\) lies between \(f(x)\) and \(f(y)\). So let us say that \(f(x) < \alpha < f(y)\). Then there exists \(z \in X\) such that \(f(z) = \alpha\).

**Proof.** Now \(f\) is a continuous function and \(X\) is a connected topological space implies \(f(X)\) is a connected subset of \(\mathbb{R}\). But we know that every connected subset of \(\mathbb{R}\) is an interval. Hence \(f(X)\) is an interval in \(\mathbb{R}\). Now \(f(x), f(y) \in f(X)\) such that \(f(x) < f(y)\), \(f(X)\) is an interval implies \([f(x), f(y)] \subseteq f(X)\). Suppose \(\alpha \in \mathbb{R}\) such that \(f(x) < \alpha < f(y)\).

\[
\begin{array}{c}
f(x) \quad \alpha \quad f(y)
\end{array}
\]

**Figure 3.3**

In particular \(\alpha \in [f(x), f(y)] \subseteq f(X)\). That is \(\alpha \in f(X)\). This implies that there exists \(z \in X\) such that \(f(z) = x\). ■

Using the fact that \([a, b](a, b \in \mathbb{R}, a < b)\) is a connected subspace of \(\mathbb{R}\), we prove the following result:

**Result 3.4.2.** Let \(f : [a, b] \to [a, b]\) be a continuous function, then there exists \(x_0\) in \([a, b]\) such that \(f(x_0) = x_0\).

**Proof.** (Proof by contradiction:)

Suppose \(f(x) \neq x\) for each \(x \in [a, b]\). Let \(A = \{x \in [a, b] : f(x) < x\}\), \(B = \{x \in [a, b] : f(x) > x\}\). Since \(f\) is a continuous function implies that both \(A\)
and $B$ are open subsets of $[a, b]$. Also $A \cap B = \emptyset$ and $[a, b] = A \cup B$. Hence $[a, b]$ is a connected topological space implies either $A = \emptyset$ or $B = \emptyset$.

Suppose $A = \emptyset$ then we get that $B = [a, b]$. That is $[a, b] = \{ x \in [a, b] : f(x) > x \}$. In particular $b \in [a, b]$ implies $f(b) > b$ and this gives a contradiction to our assumption that $f : [a, b] \to [a, b]$. (Note: If $B = \emptyset$ then $A = [a, b]$ and $a \in [a, b]$ implies $f(a) < a$ again we will get a contradiction.) We get a contradiction if we assume that $f(x) \neq x$, for every $x \in [a, b]$. Hence there exists at least one $x_0 \in [a, b]$ such that $f(x_0) = x_0$.

**Note.** In the above result such an $x_0 \in [a, b]$ (satisfying $f(x_0) = x_0$) is called a fixed point of $f$.

**Remark 3.4.3.** In the proof of the above result we have used the fact that $[a, b]$ is a connected topological subspace (in addition to the fact that $f : [a, b] \to [a, b]$ is a continuous map). What is to be noted here is we have not used the intermediate value theorem to prove the above result. Now let us use the intermediate value theorem to observe the following:

Define $g : [a, b] \to \mathbb{R}$ as $g(x) = f(x) - x$ then $g$ is a continuous map. Also $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$. If $f(a) = a$ or $f(b) = b$ then we are through. If not $g(b) = f(b) - b < 0 < f(a) - a = g(a)$. That is $g(b) < 0 < g(a)$. Hence by intermediate value theorem there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$.

**Definition 3.4.4.** A topological space $(X, \mathcal{J})$ is said to be **totally disconnected** if and only if the connected components of $X$ are singletons. That is if $A$ is a nonempty connected subset of $X$ then $A$ is a singleton set.

**Exercises 3.4.5.** Prove that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}^c$ are totally disconnected topological spaces.
Now let us introduce a concept known as pathwise connected.

**Definition 3.4.6.** A topological space \((X, J)\) is said to be **pathwise connected** space if for \(x, y \in X\) there exists a continuous function say \(f : [0, 1] \to X\) such that \(f(0) = x, f(1) = y\). That is, \(X\) is pathwise connected if and only if for \(x, y \in X\) there exists a curve joining \(x\) and \(y\).

**Note.** For \(a, b \in \mathbb{R}, a < b\), \([0, 1]\) is homeomorphic to \([a, b]\). That is there exists a bijective continuous function \(f : [0, 1] \to [a, b]\) such that \(f^{-1} : [a, b] \to [0, 1]\) is also continuous. For let \(f(t) = (1 - t)a + tb = a + (b - a)t, 0 \leq t \leq 1\). Then \(f\) is bijective and \(f, f^{-1}\) are continuous functions.

Now let us prove the following:

**Theorem 3.4.7.** A topological space \((X, J)\) is pathwise connected if and only if for \(x, y \in X\) and \(a, b \in \mathbb{R}, a < b\) there exists a continuous function \(g : [a, b] \to X\) such that \(g(a) = x\) and \(g(b) = y\).

**Proof.** Given \(x, y \in X\) and \(a, b \in \mathbb{R}, a < b\). Now \(X\) is a pathwise connected space implies there exists a continuous function say \(f_1 : [0, 1] \to X\) such that \(f_1(0) = x\) and \(f_1(1) = y\). Now let \(f_2(x) = \frac{x - a}{b - a}, x \in [a, b]\). Now \(f_2\) is a continuous function such that \(f_2(a) = 0\) and \(f_2(b) = 1\). Hence \(g : f_1 \circ f_2 : [a, b] \to X\) is a continuous function such that \(g(a) = f_1(f_2(a)) = f_1(0) = x\) and \(g(b) = f_1(f_2(b)) = f_1(1) = y\).

**Theorem 3.4.8.** Every pathwise connected topological space \(X\) is a connected space.

**Proof.** Suppose \(X\) is disconnected. Then there exist nonempty closed subsets \(A, B\) of \(X\) such that (i) \(X = A \cup B\) and (ii) \(A \cap B = \emptyset\). Take \(x \in A, y \in B\). Now \(X\) is a pathwise connected space implies there exists a continuous function
say \( f : [0, 1] \to X \) such that \( f(0) = x, \ f(1) = y \). Hence \( X = A \cup B \) implies
\[
X = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).
\]

Also \( x = f(0) \in A \) implies \( 0 \in f^{-1}(A) \), \( y = f(1) \in B \) implies \( 1 \in f^{-1}(B) \) and \( f^{-1}(A), f^{-1}(B) \) are closed subsets of \([0, 1]\) \((f : [0, 1] \to X \) is continuous \(A, B\) are closed sets in \(X\) implies \(f^{-1}(A), f^{-1}(B)\) are closed sets in \([0, 1]\)). Now \( f^{-1}(A), f^{-1}(B) \) are nonempty closed subsets of \([0, 1]\) such that \( [0, 1] = f^{-1}(X) = f^{-1}(A) \cup f^{-1}(B) \) and \( f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset \) implies \([0, 1]\) is not connected and this gives a contradiction. Hence our initial assumption namely \( X \) is disconnected is not valid. Therefore \( X \) is a connected space. ■

It is easy to prove that the continuous image of a pathwise connected space is pathwise connected.

**Result 3.4.9.** If \( X \) is a nonempty convex subset of \( \mathbb{R}^n \ (n \in \mathbb{N}) \) then \( X \) is pathwise connected and hence \( X \) is a connected space.

**Proof.** For \( x, y \in X \), define \( f : [0, 1] \to X \) as \( f(t) = (1 - t)x + ty \) for all \( t \in [0, 1] \).

Then for \( t_n \in [0, 1], \ t_n \to t \in [0, 1] \) implies \( f(t_n) \to f(t) \). Hence \( f \) is a continuous function such that \( f(0) = x \) and \( f(1) = y \). Therefore \( X \) is pathwise connected. ■

The following result is known as pasting lemma is very useful in the study of connected spaces.

**Lemma 3.4.10.** (Pasting lemma) Let \( A, B \) be nonempty closed subsets of a topological space \( X \). Suppose \( f : A \to Y, \ g : B \to Y \) are continuous functions (where \( Y \) is a topological space) such that \( f(x) = g(x) \), whenever \( x \in A \cap B \). Then
\[ h : A \cup B \rightarrow Y \] is defined as

\[
h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}
\]
is also a continuous function.

**Proof.** Left as an exercise to the reader.

(Hint. For a nonempty closed subset \( C \) of \( Y \), prove that \( h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C) \).

\[ \square \]

**Exercises 3.4.11.** (i) In the above lemma assume that \( A, B \) are nonempty open sets and arrive at the conclusion that \( h \) defined as above is a continuous function.

(ii) Given that \((X, \mathcal{J})\) is a topological space. For \( x, y \in X \) define \( xRy \) (read as \( x \) is related to \( y \)) if there is a path (or say curve) joining \( x \) and \( y \). That is there exists a continuous function \( f : [0, 1] \rightarrow X \) such that \( f(0) = x \) and \( f(1) = y \).

\[ \square \]

**Hint.** Now \( xRy \) implies there exists a continuous function \( f : [0, \frac{1}{2}] \rightarrow X \) such that \( f(0) = x, f \left( \frac{1}{2} \right) = y \). Also \( yRz \) implies there exists a continuous function \( g : \left[ \frac{1}{2}, 1 \right] \rightarrow X \) such that \( g \left( \frac{1}{2} \right) = y, g(1) = z \). Now use pasting lemma to prove \( xRz \).

That is it is easy to prove that \( R \) is an equivalence relation on \( X \).

**Definition 3.4.12.** A topological space \((X, \mathcal{J})\) is **locally connected at a point** \( x \) in \( X \) if and only if for each open set \( U \) containing \( x \), there is a connected open set \( V \) such that \( x \in V \) and \( V \subseteq U \). If \((X, \mathcal{J})\) is locally connected at each \( x \) in \( X \) then we say that \((X, \mathcal{J})\) is a locally connected topological space.

Note that neither every connected topological space is a locally connected space nor every locally connected topological space is a connected space.
Example 3.4.13. Let $B = \{(x, y) : 0 < x \leq 1, y = \sin \left( \frac{1}{x} \right) \}$ and $X = \overline{B}$. Define $f : (0, 1] \to \mathbb{R}^2$ as $f(x) = (x, \sin \left( \frac{1}{x} \right))$ then $f$ is continuous. Hence $(0, 1]$ is a connected space implies the image $f((0, 1]) = B$ is a connected subspace of $\mathbb{R}^2$. This implies that $\overline{B} = X$ is also a connected space and $X = B \cup (0 \times [-1, 1])$. This space $X$ is called the topologist’s sine curve and $X$ is neither locally connected nor pathwise connected. Now $U = B \left( (0, \frac{1}{2}), \frac{1}{4} \right) \cap X$ is an open set containing $(0, \frac{1}{2})$.

Now we leave it as an exercise to prove that there cannot exist any connected open set $V$ satisfying $(0, \frac{1}{2}) \in V$ and $V \subseteq U$.

Example 3.4.14. $X = (0, 1) \cup (2, 3)$ is locally connected but it is not connected.

Theorem 3.4.15. Let $Y$ be a connected open subset of $\mathbb{R}^n$ (with Euclidean metric). Then $Y$ is pathwise connected.

**Proof.** Fix $x \in Y$. Let us prove that for each $y \in Y$ there exists a path joining $x$ and $y$. So let $A = \{y \in Y : \text{there is a path joining } y \text{ and } x \}$. As $x \in A$, $A \neq \emptyset$. We aim to prove that $A = Y$. Let $y \in A$. This implies $y \in Y$, and hence there exists $r > 0$ such that $B(y, r) \subseteq Y$. For $z \in B(y, r)$. Define $f : [0, 1] \to Y$ as $f(t) = (1 - t)z + ty$. Then $(1 - t)z + ty \in B(y, r)$, $f$ is a continuous function satisfying the condition that $f(0) = z$ and $f(1) = y$. So $f$ is a path joining $z$ and $y$. Also $y \in A$ implies there exists a path joining $y$ and $x$. That is there exists a continuous function say $g : [0, 1] \to Y$ such that $g(0) = y, g(1) = x$.

Now define $h : [0, 1] \to Y$ as $h(t) = f(2t)$ if $0 \leq t \leq \frac{1}{2}$ and $h(t) = g(2t - 1)$ if $\frac{1}{2} \leq t \leq 1$. (Note: $0 \leq t \leq \frac{1}{2} \iff 0 \leq 2t \leq 1$ and $\frac{1}{2} \leq t \leq 1 \iff 0 \leq 2t - 1 \leq 1$.) Here $t \to f(2t)$ is a continuous function on $[0, \frac{1}{2}]$ and $t \to g(2t - 1)$ is a continuous function on $[\frac{1}{2}, 1]$ and $f(2t) = g(2t - 1)$ when $t = \frac{1}{2}$. Hence by pasting lemma $h : [0, 1] \to Y$ is a continuous function such that $h(0) = z$ and $h(1) = x$. So we have a path $h$ joining $x$
and \( z \) and hence \( z \in A \). This gives that \( B(y, r) \subseteq A \) and hence each point \( y \) of \( A \) is an interior point of \( A \). This proves that \( A \) is an open set in the subspace \( Y \). Now in the same way we can prove that \( Y \setminus A \) is also an open set in \( Y \). That is \( Y = A \cup (Y \setminus A) \), where \( A, Y \setminus A \) are both open sets in \( Y \). As \( x \in A, A \neq \emptyset \). It is given that \( Y \) is connected and hence \( Y \setminus A = \emptyset \). Therefore \( Y = A \). Now \( A \) is pathwise connected and hence \( Y \) is pathwise connected. (For \( y, z \in A = Y \), there is a path joining \( y \) and \( x \) also there is a path joining \( z \) and \( x \). If there is path joining \( z \) and \( x \) then there is also a path joining \( x \) and \( z \). Again there is a path joining \( y \) and \( x \) also a path joining \( x \) and \( z \) implies there is a path joining \( y \) and \( z \).)

\[ \blacksquare \]

**Example of a topological space which is connected but not pathwise connected.**

Let \( Y_1 = \{(0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\} \), \( Y_2 = \{(x, \sin(\frac{\pi}{x})) \in \mathbb{R}^2 : 0 < x \leq 1\} \) and \( X = Y_1 \cup Y_2 \). Let \( f : (0, 1] \rightarrow Y_2 \) defined as \( f(x) = (x, \sin(\frac{\pi}{x})) \) is a continuous function, \( f(0, 1] = Y_2 \) (that is \( f \) is surjective) and \( (0,1] \) is a connected space implies the continuous image \( f(0,1] = Y_2 \) is a connected space. Also \( Y_2' = Y_1 \) and hence \( Y_2 = Y_1 \cup Y_2 = X \). Now \( Y_2 \) is a connected space implies \( Y_2 = X \) is also a connected space. Now let us prove that \( X \) is not pathwise connected. Suppose there exists a continuous function say \( g : [0, 1] \rightarrow X \) such that \( g(0) = (0,1) \) (that is we want to prove that there is no path joining \((1, 0)\) and a point of \( Y_2 \)), \( Y_1 \) is a closed subset of \( \mathbb{R}^2 \) implies \( Y_1 \) is also a closed subset of \( X \) (\( X \) is a subspace of \( \mathbb{R}^2 \)) and hence \( g^{-1}(Y_1) \) is a closed subset of \([0, 1]\). Also \( g(0) = (0,1) \in Y_1 \) implies \( 0 \in g^{-1}(Y_1) \). Now let us prove that \( g^{-1}(Y_1) \) is also an open subset of \([0,1]\). So, let \( t \in g^{-1}(Y_1) \) then \( g(t) \in Y_1 \).

Now \( B(g(t), \frac{1}{2}) \cap X \) is an open set in \( X \) and \( g \) is a continuous function implies \( g^{-1}(B(g(t), \frac{1}{2}) \cap X) \) is an open set in \([0,1]\). Also \( t \in g^{-1}(B(g(t), \frac{1}{2}) \cap X) \) implies there
exists \( r > 0 \) such that \((t-r, t+r) \cap [0, 1] \subseteq g^{-1}(B(g(t), \frac{1}{2}) \cap X)\). Hence \( g((t-r, t+r) \cap [0, 1]) \subseteq B(g(t), \frac{1}{2}) \cap X \), \( B(g(t), \frac{1}{2}) \cap X \) consists of an interval on the \( y \)-axis, together with segments of the curve \( y = \sin(\frac{\pi}{x}) \), each of which is homomorphic to an interval.

Further any two of these sets are separated from one another in \( B(g(t), \frac{1}{2}) \cap Y_1 \). Hence \( B(g(t), \frac{1}{2}) \cap Y_1 \) is a connected component of \( B(g(t), \frac{1}{2}) \cap X \) containing \( g(t) \). Also \( g(t-r, t+r) \subseteq B(g(t), \frac{1}{2}) \cap Y_1 \). That is \((t-r, t+r) \subseteq g^{-1}(B(g(t), \frac{1}{2}) \cap Y_1) \subseteq g^{-1}(Y_1)\).

This proves that \( g^{-1}(Y_1) \) is also an open set. That is \( g^{-1}(Y_1) \) is a nonempty set which is both open and closed in \([0, 1]\). Hence \([0, 1]\) is a connected space implies \( g^{-1}(Y_1) = [0, 1] \). That is \( g([0, 1]) \subseteq g(g^{-1}(Y_1)) \subseteq Y_1 \). Therefore there cannot exist any continuous function \( g : [0, 1] \to X \) such that \( g(0) = (0, 1) \). This proves that \( X \) is not pathwise connected.
EXERCISES

1. Prove that every open subset of the real line is a union of disjoint open intervals.

2. Let \((X, \mathcal{J})\) be a locally connected topological space and \((Y, \mathcal{J}')\) be another topological space. Suppose \(f : (X, \mathcal{J}) \rightarrow (Y, \mathcal{J}')\) be a homeomorphism (that is \((X, \mathcal{J})\) is a homeomorphic to \((Y, \mathcal{J}')\)) then prove that \((Y, \mathcal{J}')\) is a locally connected topological space.

3. Say true or false (Justify your answer)
   (a) If \(A\) is a connected subset of a topological space then
       (i) \(\text{int}(A) = A^o\), the interior of \(A\) is also a connected subset of \(X\).
       (ii) \(\text{bd}(A)\), the boundary of \(A\), is also a connected subset of \(A\).
       (iii) \(X \setminus A = A^c\), the complement of \(A\) is also a connected subset of \(A\).
   (b) Subset of a connected set is connected.
   (c) If \(A, B\) are connected subsets of a topological space \(X\) then
       (i) \(A \cup B\) is a connected subset of \(X\).
       (ii) \(A \cap B\) is a connected subset of \(X\).
       (iii) \(A \triangle B = (A \setminus B) \cup (B \setminus A)\) is a connected subset of \(X\).
       (iv) \(A \setminus B\) is a connected subset of \(X\).
   (d) \(\{x = (x_1, x_2) : x_1^2 + x_2^2 = 1\}\) is connected subset of \(\mathbb{R}^2\).
   (e) There exists no homeomorphism between \(\mathbb{R}\) and \(\mathbb{R}^2\).
   (f) There exists a homeomorphism between \((0, 1)\) and \((a, b)\), where \(a, b \in \mathbb{R}\), with \(a < b\).

4. Let \((X, \mathcal{J})\) be a connected topological space and \(f : (X, \mathcal{J}) \rightarrow (\mathbb{R}, \mathcal{J}_s)\) be a non-constant continuous function. Then prove that \(X\) is an uncountable set.

5. Prove that there cannot exist any non-constant continuous function \(f : \mathbb{R} \rightarrow \mathbb{Q}\).

6. Is \(\{(x, \frac{1}{x}) : x > 0\}\) a connected subset of \(\mathbb{R}^2\)? Justify your answer.
7. Prove that there cannot exist a homeomorphism between \( \mathbb{R} \) and \( S' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \).

8. Let \( X \) be a connected topological space and \( Y \) be a topological space and \( f : X \to Y \) be a continuous function. Prove that \( G = \{(x, f(x)) : x \in X\} \) is a connected subspace of the product space \( X \times Y \).

9. In \( \mathbb{R} \), find the maximal connected subset containing \( \mathbb{N} \) (the set of all natural numbers). Justify your answer.

10. Let \( (X, \mathcal{J}) \) be a Hausdorff topological space. If \( (X, \mathcal{J}) \) has a base \( \mathcal{B} \) consisting of open and closed sets then prove that \( (X, \mathcal{J}) \) is totally disconnected.
Chapter 4

Compact Topological Spaces

4.1 Compact Spaces and Related Results

Definition 4.1.1. A subset $K$ of a topological space $(X, \mathcal{J})$ is said to be a compact set if $\mathcal{A}$ is a collection of open sets in $X$ such that $K \subseteq \bigcup_{A \in \mathcal{A}} A$ then there exists $n \in \mathbb{N}$ and $A_1, A_2, A_3, \ldots, A_n \in \mathcal{A}$ such that $K \subseteq \bigcup_{i=1}^{n} A_i$. That is $K$ is a compact subset of a topological space $(X, \mathcal{J})$ if and only if $\mathcal{A}$ is any open cover for $K$ implies $\mathcal{A}$ has a finite subcollection say $\mathcal{A}_f$ that will also cover $K$.

Note. If $\mathcal{A}$ is a collection of open sets in $(X, \mathcal{J})$ and $K \subseteq X$ is such that $K \subseteq \bigcup_{A \in \mathcal{A}} A$, then we say that $\mathcal{A}$ is an open cover for $K$.

Example 4.1.2. Let $X$ be nonempty set and $\mathcal{J}$ be a topology on $X$. Let $K$ be a finite subset of $X$.

Case 1: $K = \phi$.

Then verify that $K$ is a compact set (exercise).

Case 2: $K$ is a nonempty finite set.

In this case, there exists $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$ such that $K = \{x_1, x_2, \ldots, x_n\}$.

Now suppose $\mathcal{A}$ is a collection of open sets in $X$ and $\{x_1, x_2, \ldots, x_n\} = K \subseteq \bigcup_{A \in \mathcal{A}} A$.

Then for each $i \in \{1, 2, \ldots, n\}$, $x_i \in A_i$ for some $A_i \in \mathcal{A}$. (Note that $i \neq j$ need not imply $A_i \neq A_j$.) Now $\mathcal{A}_f = \{A_1, A_2, \ldots, A_n\}$ is a finite subcollection of $\mathcal{A}$ such that $K = \{x_1, x_2, \ldots, x_n\} \subseteq A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{A \in \mathcal{A}_f} A$. That is, we started with an open
cover $\mathcal{A}$ for $K$ and we could get a finite subcollection $\mathcal{A}_f$ of $\mathcal{A}$ that also covers $K$. Hence by the definition, $K$ is a compact subset of $(X, \mathcal{J})$.

**Example 4.1.3.** Let $X$ be a nonempty set and $\mathcal{J}_f = \{ A \subseteq X : A^c = X \setminus A \text{ is a finite set or } A^c = X \}$. That is, $\mathcal{J}_f$ is the cofinite topology on $X$. We have proved in example 4.1.2 that every finite subset of any topological space is compact.

So, let us assume that $K$ is an infinite subset of $X$. Now consider a collection $\mathcal{A}$ of open sets such that $K \subseteq \bigcup_{A \in \mathcal{A}} A$. Since we have assumed $K$ is an infinite set, $K \neq \emptyset$. Take an element say $x_0 \in K$. Now $x_0 \in K \subseteq \bigcup_{A \in \mathcal{A}} A$. This implies there exists $A_0 \in \mathcal{A}$ such that $x_0 \in A_0$. Now $A_0$ is a nonempty open set in the cofinite topological space $(X, \mathcal{J}_f)$ implies $X \setminus A_0 = A_0^c$ is a nonempty finite set (or $A_0^c = \emptyset \Rightarrow A_0 = X \Rightarrow \mathcal{A}_f = \{ A_0 \}$). Also $K \subseteq A_0 \Rightarrow \mathcal{A}_f = \{ A_0 \}$. Let $K \cap A_0^c = \{ x_1, x_2, \ldots, x_n \}$. Since $K \subseteq \bigcup_{A \in \mathcal{A}} A$ each $x_i \in A_i$, $i = 1, 2, \ldots, n$. Now $K \subseteq X = A_0 \cup A_0^c \Rightarrow K \subseteq (A_0 \cup A_0^c) \cap K = (A_0 \cap K) \cup (A_0^c \cap K) \subseteq A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n$. That is, $\mathcal{A}_f = \{ A_0, A_1, A_2, \ldots, A_n \}$ or $\mathcal{A}_f = \{ A_0 \}$ is a finite subcollection of $\mathcal{A}$ that also covers $K$. Hence $K$ is a compact subset of $(X, \mathcal{J}_f)$. That is, in a cofinite topological space every subset is a compact set.

**Remark 4.1.4.** In a cofinite topological space $(X, \mathcal{J}_f)$, if $A$ is a nonempty open set then $A$ will almost cover any $K \subseteq X$. That is, maximum finitely many elements may not be in $A$ and hence every subset $K$ becomes a compact set.

**Example 4.1.5.** Let $X$ be any set and $\mathcal{J}$ be a topology on $X$. Note that $\mathcal{J} \subseteq \mathcal{P}(X)$, the collection of all subsets of $X$. If $\mathcal{J}$ is a finite set then every subset $K$ of $X$ is compact in $(X, \mathcal{J})$.

Also note that in a cofinite topological space $(X, \mathcal{J}_f)$ every finite subset of $X$ is closed and if $F \neq X$, $F$ is not a finite set then $F$ is not closed. Now let us
prove that \( \mathbb{R} \) with usual topology \( J_s \) is not a compact space. That is \((\mathbb{R}, J_s)\) is not a compact space. Now we want to prove that the subset \( \mathbb{R} \) of the topological space \((\mathbb{R}, J_s)\) is not compact. Note that \( \mathbb{R} \subseteq \bigcup_{n=1}^{\infty} (-n, n) \). That is, \( \mathcal{A} = \{(-n, n) : n \in \mathbb{N}\} \) is an open cover for \( \mathbb{R} \). Suppose this open cover has a finite subcollection say \( \mathcal{A}^f = \{A_1, A_2, \ldots, A_k\} \) such that \( \mathbb{R} \subseteq \bigcup_{i=1}^{k} A_i \). If \( A_i \in \mathcal{A} \) this implies there exists \( n_i \in \mathbb{N} \) such that \( A_i = (-n_i, n_i) \). So, \( \mathbb{R} \subseteq (-n_1, n_1) \cup (-n_2, n_2) \cup \cdots \cup (-n_k, n_k) \). Let \( n_0 = \max\{n_1, n_2, \ldots, n_k\} \). Then \( \mathbb{R} \subseteq (-n_0, n_0) \), a contradiction. Note that \( n_0 + 1 \in \mathbb{R} \) but \( n_0 + 1 \notin (-n_0, n_0) \). We could arrive at this contradiction by assuming that \( \mathcal{A} \) has a finite subcollection that also covers \( \mathbb{R} \). Hence such an assumption is wrong. That is, this particular collection \( \mathcal{A} = \{(-n, n) : n \in \mathbb{N}\} \) is an open cover for \( \mathbb{R} \). But this cannot have any finite subcover. Therefore \((\mathbb{R}, J_s)\) is not a compact space. Note that \( \mathbb{R} \) with cofinite topology \( J_f \) is a compact space. That is, \((\mathbb{R}, J_f)\) is a compact topological space but \((\mathbb{R}, J_s)\) is not a compact topological space.

Also note that if \( A \) is any unbounded subset of \( \mathbb{R} \) then \( A \) is not a compact subset of the topological space \((\mathbb{R}, J_s)\). Note that \( \mathbb{R} \subseteq \bigcup_{n=1}^{\infty} (-n, n) \) but \( A \) is not a bounded set (that is \( A \) is unbounded set) implies there cannot exist any \( n_0 \in \mathbb{N} \) such that \( A \subseteq (-n_0, n_0) \). (Recall: \( A \) is a bounded subset of \( \mathbb{R} \) if and only if there exists \( n_0 \in \mathbb{N} \) such that \( |x| < n_0 \forall x \in A \). That is \( A \) is bounded if and only if \( A \subseteq (-n_0, n_0) \) for some \( n_0 \in \mathbb{N} \).)

Now let us prove that a closed subset of a compact topological space is compact.

**Theorem 4.1.6.** If \( A \) is a closed subset of a compact topological space \((X, J)\) then \( A \) is a compact set in \((X, J)\).
**Proof.** Let \( \mathcal{A} \) be a collection of open sets in \( X \) such that \( A \subseteq \bigcup_{B \in \mathcal{A}} B \). Now \( \mathcal{A}' = \mathcal{A} \cup \{A^c\} \) is a collection of open sets such that \( X = A \cup A^c \subseteq A^c \cup \bigcup_{B \in \mathcal{A}} B \). That is, \( \mathcal{A}' \) is an open cover for the compact space \((X, J)\) and hence there exists \( n \in \mathbb{N} \) and \( A_1, A_2, \ldots, A_n \in \mathcal{A}' \) such that \( X \subseteq A_1 \cup A_2 \cup \cdots \cup A_n \). If one of \( A_i = A^c \), then \( \{A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n\} \subseteq \mathcal{A}' \) such that \( A \subseteq \bigcup_{i=1}^{n} A_i \). Hence in any case every open cover \( \mathcal{A} \) of \( A \) has a finite subcollection that also covers \( A \). This implies that \( A \) is a compact subset of \((X, J)\). \( \blacksquare \)

Recall that a topological space \((X, J)\) is said to be a Hausdorff topological space if \( x, y \in X, x \neq y \) then there exist open sets \( U, V \) in \( X \) such that \( x \in U, y \in V \) and \( U \cap V = \phi \).

In a cofinite topological space we have proved that every subset is compact. In particular \((X, J_f)\) is a compact topological space for any set \( X \). But if \( A \neq X \) is an infinite subset of \( X \) then \( A \) is a compact set but it is not a closed set. Now let us prove that such a thing cannot happen in a Hausdorff topological space.

**Theorem 4.1.7.** If \( K \) is a compact subset of a Hausdorff topological space then \( K \) is a closed set.

**Proof.** Let us prove that \( K^c = X \setminus K \) is an open set. So take \( x \in K^c \). Our aim is to prove that \( x \) is an interior point of \( K^c \). That is, we will have to find an open set \( U_x \) in \( X \) such that \( x \in U_x \subseteq K^c \). Now \( x \neq y \) for each \( y \in K \). Hence \((X, J)\) is a Hausdorff space implies there exist open sets \( U_y, V_y \) such that

\[
x \in U_y, \quad y \in V_y, \quad U_y \cap V_y = \phi.
\]

(4.1)
Now \( \{ V_y : y \in K \} \) is an open cover for the compact set \( K \). Hence this implies there exist \( y_1, y_2, \ldots, y_n \in K \) such that \( K \subseteq \bigcup_{i=1}^{n} V_{y_i} \). Let \( U_x = \bigcap_{i=1}^{n} U_{y_i} \) (refer Eq. (4.1)) then \( U_x \) is an open set containing \( x \) and \( U_x \cap K \subseteq U_x \cap \bigcup_{i=1}^{n} V_{y_i} = \bigcup_{i=1}^{n} (U_x \cap V_{y_i}) \subseteq \bigcup_{i=1}^{n} (U_{y_i} \cap V_{y_i}) = \phi \). This implies \( U_x \cap K = \phi \Rightarrow U_x \subseteq K^c \). Therefore, each \( x \in K^c \) is an interior point of \( K^c \). Hence \( K^c \) is an open set and therefore \( K \) is a closed set. 

**Note.** Let \( (X, \mathcal{J}) \) be a topological space and \( Y \subseteq X \). Then \( \mathcal{J}_Y = \{ A \cap Y : A \in \mathcal{J} \} \) is a topology on \( Y \).

So, now it is easy to prove:

**Theorem 4.1.8.** A subset \( Y \) of a topological space \( (X, \mathcal{J}) \) is compact if and only if whenever \( \mathcal{A} \) is a collection of open sets in \( (Y, \mathcal{J}_Y) \) such that \( Y = \bigcup_{A \in \mathcal{A}} A \) then there exists \( n \in \mathbb{N} \) and \( A_1, A_2, \ldots, A_n \in \mathcal{A} \) such that \( Y = \bigcup_{i=1}^{n} A_i \).

**Proof.** Let us assume the given hypothesis. That is, assume that whenever \( \mathcal{A} \) is a collection of open sets in the topological space \( (Y, \mathcal{J}_Y) \) then this open cover \( \mathcal{A} \) for \( Y \) has a finite subcover. Now we will have to prove that the given subset \( Y \) of the topological space \( (X, \mathcal{J}) \) is compact. So start with a collection say \( \mathcal{B} \) of open sets in \( (X, \mathcal{J}) \) satisfying the condition that \( Y \subseteq \bigcup_{B \in \mathcal{B}} B \) (recall the definition). Now \( B \in \mathcal{B} \subseteq \mathcal{J} \Rightarrow B \cap Y \in \mathcal{J}_Y \). Hence \( Y \subseteq \bigcup_{B \in \mathcal{B}} B \) implies \( Y = \bigcup_{B \in \mathcal{B}} B \cap Y \). That is, \( \mathcal{A} = \{ B \cap Y : B \in \mathcal{B} \} \) is a collection of open sets in \( (Y, \mathcal{J}_Y) \) which also covers \( Y \). Hence by the given hypothesis there exists \( n \in \mathbb{N} \) and \( B_1, B_2, \ldots, B_n \in \mathcal{B} \) such that \( Y = \bigcup_{i=1}^{n} (B_i \cap Y) \). This implies that \( Y \subseteq \bigcup_{i=1}^{n} B_i \). Now we have proved: whenever \( \mathcal{B} \) is a collection of open sets in \( (X, \mathcal{J}) \) which covers \( Y \) then there exists \( n \in \mathbb{N} \) and \( B_1, B_2, \ldots, B_n \in \mathcal{B} \) such that \( Y \subseteq \bigcup_{i=1}^{n} B_i \). Therefore by our definition the given subset \( Y \) of \( (X, \mathcal{J}) \) is a compact set. The proof of \( Y \) is a compact subset of \( (X, \mathcal{J}) \) implies
that the given hypothesis is satisfied follows in similar lines and hence the proof is
left as an exercise.

From what we have proved, we observe that a subset $Y$ of a topological space $(X, J)$ is compact if and only if, with respect to the induced topology $J_Y$, the topological space $(Y, J_Y)$ is compact.

Now let us prove that continuous image of a compact space is compact.

**Theorem 4.1.9.** Let $(X, J)$ be a compact topological space and $(Y, J')$ be any other topological space. Let $f : (X, J) \to (Y, J')$ be a continuous function. Then the image $f(X)$ is a compact subset of $Y$.

**Proof.** To prove the subset $f(X)$ of $(Y, J')$ is a compact set, we start with a collection say $\mathcal{A}$ of open sets in $(Y, J')$ which satisfies

$$f(X) \subseteq \bigcup_{A \in \mathcal{A}} A. \quad (4.2)$$

Now $A \in \mathcal{A}$ implies $A$ is open in $(Y, J')$. Hence $f : (X, J) \to (Y, J')$ is a continuous function implies that $f^{-1}(A)$ is open in $(X, J)$. From Eq. (4.2) $X = f^{-1}(f(X)) \subseteq f^{-1}\left(\bigcup_{A \in \mathcal{A}} A\right) = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$. This implies $\mathcal{A}' = \{f^{-1}(A) : A \in \mathcal{A}\}$ is an open cover for the compact topological space $(X, J)$. Hence there exists $n \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n \in \mathcal{A}$ such that $X \subseteq \bigcup_{i=1}^{n} f^{-1}(A_i)$ (here $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$). This implies that $f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(A_i)\right) = \bigcup_{i=1}^{n} f\left(f^{-1}(A_i)\right) \subseteq \bigcup_{i=1}^{n} A_i$. We have proved: any arbitrary open cover $\mathcal{A}$ of $f(X)$ has a finite subcover. Hence by the definition, $f(X)$ is a compact subset of $(Y, J')$.

Using the above theorem and the result that every compact subset of a Hausdorff space is closed we prove:
Theorem 4.1.10. Let \((X, \mathcal{J})\) be a compact topological space and \((Y, \mathcal{J'})\) be a Hausdorff topological space. Let \(f : (X, \mathcal{J}) \to (Y, \mathcal{J'})\) be a bijective continuous map. Then the inverse map \(f^{-1} : (Y, \mathcal{J'}) \to (X, \mathcal{J})\) is also a continuous map. That is \(f\) is a homeomorphism.

**Proof.** Take an open set \(A\) in \((X, \mathcal{J})\). Now \(f\) is a bijective map implies that

\[
(f^{-1})^{-1}(A^c) = f(A^c)
\]  

(4.3)

Note that \(A^c\) is a closed subset of the compact space implies \(A^c\) is a compact set implies \(f(A^c)\) is a compact subset of the Hausdorff space. This implies \(f(A^c)\) is a closed set in \(Y\). Hence from Eq. (4.3) \((f^{-1})^{-1}(A^c)\) is a closed set implies \(f(A) = (f^{-1})^{-1}(A) = Y \setminus (f^{-1})^{-1}(A^c)\) is an open set.

We have proved: \(A\) is an open set in \((X, \mathcal{J})\) implies \((f^{-1})^{-1}(A)\) is an open set in \(Y\). Hence \(f^{-1} : (Y, \mathcal{J'}) \to (X, \mathcal{J})\) is a continuous map. \(\blacksquare\)

**Remark 4.1.11.** To prove the above theorem it is also enough to prove that if \(B\) is a closed subset of \(X\) then \(f(B)\) is a closed subset of \(Y\). \(\diamondsuit\)

**Definition 4.1.12.** A collection \(\mathcal{F}\) of subsets of a given set \(X\) is said to have **finite intersection property** (f.i.p) if for any \(n \in \mathbb{N}\) and \(F_1, F_2, \ldots, F_n \in \mathcal{F}\) then \(\bigcap_{i=1}^{n} F_i \neq \emptyset\).

**Theorem 4.1.13.** A topological space \((X, \mathcal{J})\) is a compact space if and only if whenever \(\mathcal{F}\) is a collection of closed subsets of \(X\) which has f.i.p then \(\bigcap_{F \in \mathcal{F}} F \neq \emptyset\).

**Proof.** Assume that \((X, \mathcal{J})\) is a compact topological space. Now start with a collection \(\mathcal{F}\) of closed subsets of \(X\) which has the f.i.p. Our aim is to prove \(\bigcap_{F \in \mathcal{F}} F \neq \emptyset\). To achieve this, let us use the method of proof by contradiction.
Suppose \( \bigcap_{F \in \mathcal{F}} F = \phi \). Then by the DeMorgan’s law \( \left( \bigcap_{F \in \mathcal{F}} F \right)^c = \bigcup_{F \in \mathcal{F}} F^c = X \). This implies \( \{F^c : F \in \mathcal{F}\} \) is an open cover for the compact space. Hence there exists \( n \in \mathbb{N} \) and \( F_1, F_2, \ldots, F_n \in \mathcal{F} \) such that \( X = \bigcup_{i=1}^{n} F_i^c \Rightarrow X^c = \left( \bigcup_{i=1}^{n} F_i^c \right)^c = \bigcap_{i=1}^{n} F_i \). Therefore \( \bigcap_{i=1}^{n} F_i = \phi \), a contradiction to the fact that \( \mathcal{F} \) has the finite intersection property. We arrived at this contradiction by assuming that \( \bigcap_{F \in \mathcal{F}} F = \phi \). Hence this is not a valid assumption. This implies \( \bigcap_{F \in \mathcal{F}} F \neq \phi \). Let us leave the converse part as an exercise. ■

Now we prove that real valued continuous function on a compact topological space attains its maximum.

**Theorem 4.1.14.** Let \((X, \mathcal{J})\) be a compact topological space and \(\mathcal{J}_s\) be the usual topology on \(\mathbb{R}\). Let \(f: (X, \mathcal{J}) \rightarrow (\mathbb{R}, \mathcal{J}_s)\) be a continuous function. Then there exists \(x_0 \in X\) such that \(f(x) \leq f(x_0)\) for all \(x \in X\). That is \(f\) attains its maximum at \(x_0\).

**Proof.** Let us use the method of proof by contradiction. Then for a given \(a \in X\), \(f\) cannot attains its maximum at \(a\). Hence there exists \(a' \in X\) such that \(f(a) < f(a')\). This means that \(f(a) \in (-\infty, f(a'))\). (Fix any one \(a' \in X\) satisfying \(f(a) < f(a')\).)

Hence \(f(X) \subseteq \bigcup_{a \in X} (-\infty, f(a'))\).

**Figure 4.1**

This implies that \(\mathcal{A} = \{(-\infty, f(a')) : a \in X\}\) is an open cover for the compact subspace \(f(X)\) of \(\mathbb{R}\) (continuous image of a compact space is compact). Hence there exist \(a_1, a_2, \ldots, a_n \in X\) such that \(f(X) \subseteq \bigcup_{i=1}^{n} (-\infty, f(a'_i))\). This implies \(f(X) \subseteq (-\infty, f(a_0))\) for some \(a_0 \in \{a'_1, a'_2, \ldots, a'_n\}\). Hence for this \(a_0 \in X\) by our assumption
there exists a $a' \in X$ such that $f(a_0) < f(a'_0)$. But $f(X) \subseteq (-\infty, f(a_0))$ implies $f(a'_0) < f(a_0)$. Hence we have got a contradiction. This means $f(a) < f(a')$ cannot be true for all $a \in X$ and hence there should exist at least one $x_0 \in X$ such that $f(x) \leq f(x_0)$ for all $x \in X$. ■

**Remark 4.1.15.** In a similar way we can prove that continuous image of a compact set attains its minimum at a point $y_0 \in X$. ♦

**Theorem 4.1.16. Tychonoff.** Let $X$ and $Y$ be compact topological spaces. Then the product space $X \times Y$ is compact.

**Proof.** For each $x_0 \in X$, $y \rightarrow (x_0, y)$ is a surjective continuous function and $Y$ is a compact space implies $x_0 \times Y$ is a compact subset of $X \times Y$. Let $\mathcal{A}$ be a collection of basic open sets such that $X \times Y = \bigcup_{U \times V \in \mathcal{A}} U \times V$.

This implies that $x_0 \times Y \subseteq \bigcup_{U \times V \in \mathcal{A}} U \times V$ implies there exist $U_1 \times V_1, U_2 \times V_2, \cdots, U_n \times V_n$ such that

$$x_0 \times Y \subseteq (U_1 \times V_1) \cup (U_2 \times V_2) \cup \cdots \cup (U_n \times V_n). \quad (4.4)$$
Also if for some \( i \), \((U_i \times V_i) \cap (x_0 \times Y) = \phi\), then we do not require to include such an
\( U_i \times V_i \) in our finite subcover \( \{U_i \times V_i\}_{i=1}^n \). So assume that each \((U_i \times V_i) \cap (x_0 \times Y) \neq \phi\).

This in turn implies that \( x_0 \in U_i, \forall i = 1, 2, \ldots, n \) and hence \( x_0 \in W_{x_0} = \bigcap_{i=1}^n U_i \). Now it is clear that
\( W_{x_0} \times Y \subseteq (U_1 \times V_1) \cup (U_2 \times V_2) \cup \cdots \cup (U_n \times V_n) \). Consider \((x, y) \in W_{x_0} \times Y\). Then \( x \in U_i \) for all \( i \) and \( y \in Y \). Hence from Eq. (4.4),
\( (x_0, y) \in U_j \times V_j \) for some \( j \). This implies \((x, y) \in U_j \times V_j \) for the same \( j \). That is for
each \( x_0 \in X \), the tube \( W_{x_0} \times Y \) is covered by finitely many members of \( \mathcal{A} \).

Now let us prove that \( X \times Y \) is covered by finitely many such tubes \( W_x \times Y \).
Now \( \{W_x : x \in X\} \) is an open cover for \( X \). Hence \( X \) is a compact space implies there
exist \( x_1, x_2, \ldots, x_k \in X \) such that \( X = \bigcup_{i=1}^k W_{x_i} \). Now \((x, y) \in X \times Y \Rightarrow x \in W_{x_i} \) for
some \( i, 1 \leq i \leq k \) and hence \((x, y) \in W_{x_i} \times Y \). This implies that \( X \times Y \subseteq \bigcup_{i=1}^k W_{x_i} \times Y \)
and hence \( X \times Y \) is covered by finitely many members of \( \mathcal{A} \). This proves that \( X \times Y \)
is a compact topological space.

4.2 Local Compactness

A Hausdorff topological space \((X, \mathcal{J})\) is said to be locally compact if and only
if for each \( x \in X \) and for each open set \( U \) containing \( x \) there exists an open set \( V \)
containing \( x \) such that \( V \) is compact and \( V \subseteq U \). Now it is easy to prove that a
Hausdorff topological space \((X, \mathcal{J})\) is locally compact if and only if for each \( x \in X \)
there exists an open set \( V \) such that \( x \in V \) and \( V \) is a compact set in \( X \).

Examples 4.2.1. (i) If \( X \) is a compact Hausdorff space then \( X \) is locally compact.
(ii) \( \mathbb{R}^n \) with Euclidean topology is locally compact but not compact. Here \( n \in \mathbb{N} \)
and \( \mathcal{J} \) is the topology induced by the metric \( d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \left( \sum_{k=1}^n |x_k - y_k|^2 \right)^{\frac{1}{2}} \).
Also it is easy to prove that if, for $1 \leq p \leq \infty$, 
\[d_p((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \left(\sum_{k=1}^{n} |x_k - y_k|^p\right)^{\frac{1}{p}}\] and 
\[d_\infty((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max\{|x_k - y_k| : k = 1, 2, \ldots, n\}\] then $d_p$ is a metric on $\mathbb{R}^n$. (Note. Proof of $d_p(x, y) \leq d_p(x, z) + d(z, y)$ for all $x, y, z \in \mathbb{R}^n$ is not that easy.)

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, 
\[\|x + y\| \leq \|x\| + \|y\|\] is known as Minkowski’s inequality, $1 \leq p \leq \infty$.

If we use this inequality then 
\[d_p(x, y) = \|x - y\|_p = \|(x - z) + (y - z)\|_p \leq \|x - z\|_p + \|z - y\|_p = d_p(x, z) + d_p(z, y).\]

Also it is to be noted that the topology $\mathcal{J}_p$ on $\mathbb{R}^n$ induced by the metric $d_p$ is same as $\mathcal{J}_2 = \mathcal{J}$.

**Definition 4.2.2.** A topological space $(X, \mathcal{J})$ is said to be **limit point compact** if every infinite subset of $X$ has a limit point.

**Theorem 4.2.3.** Every compact topological space $(X, \mathcal{J})$ is limit point compact.

**Proof.** Let $A$ be an infinite subset of $X$. Suppose $A' = \phi$. That is $A$ does not have any limit point. Note that $\overline{A} = A \cup A' = A$ implies $A$ is a closed set, then $x \in A$ implies $x \notin A'$ implies there exists an open set $U_x$ such that $x \in U_x$, $U_x \cap A \backslash \{x\} = \phi$. (That is $U_x \cap A \cap \{x\} = \phi \Rightarrow U_x \cap A = \{x\}$. Now $\{U_x : x \in A\}$ is an open cover for the closed subset $A$ of the given compact topological space. Hence there exists a natural number $n$ and $x_1, x_2, \ldots, x_n \in A$ such that $A \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. This gives that 
\[A = (U_{x_1} \cup \cdots \cup U_{x_n}) \cap A = (U_{x_1} \cap A) \cup (U_{x_2} \cap A) \cup \cdots \cup (U_{x_n} \cap A) = \{x_1, x_2, \ldots, x_n\}.\] Hence we have arrived at a contradiction by assuming $A' = \phi$. Therefore $A' \neq \phi$. That is we have proved that every infinite subset $A$ of the given compact topological space has at least one limit point. This means that $(X, \mathcal{J})$ is a limit point compact topological space. ■
What about the converse of the above theorem? Is every limit point compact topological space compact? Limit point compact does not imply compact.

**Example 4.2.4.** Let \( X = \{0, 1\} \), \( J = \{\phi, X\} \) and \( Y = \mathbb{N} = \{1, 2, \ldots\} \), the set of all natural numbers and \( J' = \mathcal{P}(\mathbb{N}) \), that \( J' \) is discrete topology on \( \mathbb{N} \). Let \( X_0 = X \times Y \) be the product space. Here \( \{X \times \{n\}\} \) is an open cover for \( X \times Y \). But for any fixed \( k \in \mathbb{N}, X \times Y = X \times \mathbb{N} \not\subseteq (X \times \{1\}) \cup \cdots \cup (X \times \{k\}) \) (note: \((1, k+1) \not\in \bigcup_{j=1}^{k} X \times \{j\}\)). This gives that \( X \times Y \) is not a compact topological space.

Now let \( A \) be a nonempty subset of \( X \times Y \). Then there exists \( k \in \mathbb{N} \) such that \((0, k) \in A \) or \((1, k) \in A \). Let us say \((0, k) \in A \). In this case we claim that \((1, k) \in A' \). Take a basic open set \( U \) containing \((1, k)\) then \( U = X \times \{k\} \). Now \((0, k) \in U \cap A \backslash \{(1, k)\} \neq \phi \). Hence we have proved that \((1, k)\) is a limit point of \( A \). Note that if \((1, k) \in A \) then we can prove that \((0, k)\) is a limit point of \( A \). So we have proved that every nonempty subset \( A \) of \( X \times Y \) has a limit point. In particular every infinite subset of \( X \times Y \) has a limit point. Therefore \( X \times Y \) is a limit point compact.

### 4.3 One Point Compactification of a Topological Space \((X, J)\)

It is given that \((X, J)\) is a non compact Hausdorff topological space. Our aim is take an element say \( \infty \) (just a notation) which is not in \( X \). For each \( x \in X \), we have open sets containing \( x \). For \( \infty \in X^* = X \cup \{\infty\} \) we aim to define open sets satisfying: If \( U \) is an open set containing \( \infty \) then each such open set is so large that the complement of \( U \) (with respect to \( X^* \)) is rather a small set. That is we want that \( X^* \backslash U = C \), where \( C \) is a compact set in \((X, J)\) and since \( \infty \in U, \infty \notin C \). So, if we start with a collection \( \mathcal{A} \) of open sets in our new topological space (note: we have not yet defined such a topology on \( X^* \)) which covers \( X^* \), then \( \infty \in A_0 \) for some
$A_0 \in A$. Fix one such $A_0$. Now $X^* \setminus A_0 = C$ a compact subset of $(X, \mathcal{J})$.

So, if our proposed topology say $\mathcal{J}^*$ on $X^*$ is such that $\mathcal{J}^*_X = \mathcal{J}$ then $A$ is also an open cover for $C$ and $C$ is a compact subspace of $(X, \mathcal{J})$ implies $C$ is also a compact subspace of $(X^*, \mathcal{J}^*)$. Hence there exists $n \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n \in A$ such that $C \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$. Therefore $X^* = (X^* \setminus A) \cup A = C \cup A_0 \subseteq A_1 \cup \cdots \cup A_n \cup A_0$.

Hence every open cover $A$ of $(X^*, \mathcal{J}^*)$ has a finite subcover. So we see that if we could define such a topology $\mathcal{J}^*$ on $X^*$ such that $\mathcal{J}^*_X = \mathcal{J}$ then $(X^*, \mathcal{J}^*)$ is a compact topological space.

We also want to retain the Hausdorff property. So keeping these points in mind we define $\mathcal{J}^*$ as follows:

- if a subset $A$ of $X^*$ is such that $\infty \notin A$ then $A \subseteq X$. In such a case $A \in \mathcal{J}^*$ if and only if $A \in \mathcal{J}$,

- if $\infty \in A$ then, $A \in \mathcal{J}^*$ if and only if $A = X^* \setminus C$, for some compact subset $C$ of $X$.

Now it is easy to prove that $\mathcal{J}^*$ is a topology on $X^*$.

**Theorem 4.3.1.** Let $(X, \mathcal{J})$ be a locally compact Hausdorff space. Then there exists a topological space $(X^*, \mathcal{J}^*)$ satisfying the following conditions:

(i) $(X, \mathcal{J})$ is a subspace of $(X^*, \mathcal{J}^*)$,

(ii) $X^* \setminus X$ is a set containing exactly one element,

(iii) $(X^*, \mathcal{J}^*)$ is a compact Hausdorff space.

**Proof.** Keeping the above requirements in mind we have defined $\mathcal{J}^*$, we have taken care that $\mathcal{J} \subseteq \mathcal{J}^*$ and $\mathcal{J}^*_X = \mathcal{J}$. Also $X^* \setminus X = \{\infty\}$. 

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Now let us prove that \((X^*, J^*)\) is a compact Hausdorff space. So start with an collection \(A\) of open sets in \((X^*, J^*)\) (that is \(A \subseteq J^*\)) such that \(X^* = \bigcup_{A \in A} A\). Now \(\infty \in X^*\) implies \(\infty \in A_0\) for some \(A_0 \in A\). It is quite possible that \(\infty\) belongs to more than one such \(A \in A\). From such \(A\) just fix one \(A_0 \in A\). By the definition of \(J^*\), \(X^* \backslash A_0 = C\) is a compact subset of \(X\). Note that \(A\) is a collection of open sets in \(X^*\) and \(C\) is a compact subspace of \(X\), and hence of \(X^*\). Now \(C \subseteq \bigcup_{A \in A} A\) implies there exists \(n \in \mathbb{N}\) and \(A_1, A_2, \ldots, A_n \in A\) such that \(C \subseteq A_1 \cup A_2 \cup \cdots \cup A_n\) implies \(X^* = (X^* \backslash C) \cup C \subseteq A_0 \cup A_1 \cup \cdots \cup A_n\) \((X^* \backslash A_0 = C\) implies \(A_0 = X^* \backslash C\)\) (it is possible that \(A_0 = A_j\), for some \(j \in \{1, 2, \ldots, n\}\). So we have proved that \(X^* \subseteq A_0 \cup A_1 \cup \cdots \cup A_n\). This means that the started open cover \(A\) of \(X^*\) has finite subcover \(\{A_0, A_1, A_2, \ldots, A_n\}\). Hence \((X^*, J^*)\) is a compact space.

Now let us prove that \((X^*, J^*)\) is a Hausdorff space. So start with \(x, y \in X^*\) with \(x \neq y\).

Case 1: \(x, y \in X\) (means \(x \neq \infty, y \neq \infty\)).

Now \(x, y \in X\), \((X, J)\) is a Hausdorff topological space implies there exist \(U, V \in J\) such that (i) \(x \in U, y \in V\), (ii) \(U \cap V = \emptyset\). But \(J \subseteq J^*\). Hence we have \(U, V \in J^*\) satisfying (i) and (ii) and this is what we wanted to prove.

Case 2: \(x \in X, y = \infty\).

Here we require the fact that \((X, J)\) is locally compact space. Now \(x \in X\), \((X, J)\) is locally compact Hausdorff space implies there exists an open set \(U\) containing \(x\) such that \(C = \overline{U}\) is a compact subset of \((X, J)\) (here \(\overline{U}\) is the closure of \(U\) in \(X\)). Hence by the definition of \(J^*\), \(V = X^* \backslash C\) is an open set containing \(\infty\) and \(U\) is an open set containing \(x\). Further \(U \cap V = \emptyset\) and this is what we wanted to prove. □
**Remark 4.3.2.** It is easy to prove that $\infty$ is a limit point of $X$. To prove this, start with an open set $U$ containing $\infty$. Then we have to prove that $U \cap X \neq \emptyset$. If $X$ is not a compact space, then $U \cap X \setminus \{\infty\} = U \cap X \neq \emptyset$. Hence if $(X, \mathcal{J})$ is not a compact Hausdorff space then $\infty$ is a limit point of $X$.

**Examples 4.3.3.** (i) Take $X = (a, b]$ and consider $X$ as a subspace of $\mathbb{R}$ here $a, b \in \mathbb{R}, a < b$. Now $X$ is considered as a subspace of $\mathbb{R}$, is a locally compact Hausdorff space of $\mathbb{R}$. Also $X$ is not a compact space.

What is the one point compactification of $X$? Here our $X = (a, b]$ and $a \notin X$. So take $\infty = a$. Note that while defining the one point compactification of $X$ we just took an object or (say an element) which we denoted by $\infty$ and $\infty \notin X$. So what we need is $\infty \notin X$. Now what are the open sets containing our $\infty = a$ in $X^*$. $U \subseteq X^*$ is an open set containing $a$ if and only if $X^* \setminus U = C$ is a compact subset of $X$. Now it is easy to prove that $(X^*, \mathcal{J}^*)$ is homeomorphic to $[a, b]$ as $f(x) = x$ for all $x \in X^*$. Now let us prove that $f$ is a homeomorphism. Here it is enough to prove that $f$ is a continuous map. So start with a nonempty open set $U$ in $[a, b]$ (here $[a, b]$ is considered as a subspace of $\mathbb{R}$).

Case 1: $a \notin U$.

Then $U \subseteq (a, b]$. In this case by our definition $U$ is open in $X^*$. That is $f^{-1}(U) = U$ is an open set in $X^*$.

Case 2: $a \in U$.

It is enough to consider a basic open set containing $a$. Hence $U = [a, a + \epsilon)$ for some $0 < \epsilon < b - a$. Is $f^{-1}(U) = f^{-1}([a, a + \epsilon))$ is an open set in $(X^*, \mathcal{J}^*)$. Note that $U$ is an open set containing $a$ if and only if $X^* \setminus U = [a, b] \setminus [a, a + \epsilon)$ is a compact subset of $X (X = (a, b])$. In our case $X^* \setminus U = [a + \epsilon, b]$ which is a compact subset of $(a, b)$. 

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Hence from our definition of $\mathcal{J}^*$, $f^{-1}(U)$ is an open set in $X^*$. So from cases 1 and 2 we see that $f$ is a continuous map. Now $f : (X^* \mathcal{J}^*) \to [a,b]$ such that

- $f$ is bijective and continuous.
- $(X^*, \mathcal{J}^*)$ is a compact space.
- $[a,b]$ is Hausdorff space implies $f$ is a homeomorphism.

Hence we have proved that there is a homeomorphism between the one-point compactification of $(X^* \mathcal{J}^*)$ and the compact Hausdorff space $[a,b]$ and our $X = (a,b]$ is such that $X$ is proper subspace of $[a,b]$ whose closure equals $[a,b]$. In such a case we say that $[a,b]$ is a compactification of $X$.) So we define compactification of a topological space $(X, \mathcal{J})$ as follows:

**Definition 4.3.4.** A compact Hausdorff topological space $(Y, \mathcal{J}^*)$ is said to be a **compactification** of a topological space $(X, \mathcal{J})$ if and only if

(i) $(X, \mathcal{J})$ is a proper subspace of $(Y, \mathcal{J}^*)$,

(ii) $\overline{X} = Y$.

![Figure 4.3](image.png)
Note. If $Y \setminus X$ is a single point then we say that $(Y, J^*)$ is the one point compactification of $(X, J)$. Now it is easy to prove the following statements:

(i) the one point compactification of $\mathbb{R}$ is homeomorphic to the unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$,

(ii) the one point compactification of $\mathbb{R}^2$ is homeomorphic to the sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^2 : x_1^2 + x_2^2 + x_3^2 = 1\}$. So if we identify $\mathbb{R}^2$ with the complex plane (there is a homeomorphism between $\mathbb{R}^2$ and $\mathbb{C}$) then the one point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ is known as the Riemann sphere or the extended complex plane.

### 4.4 Tychonoff Theorem for Product Spaces

Now let us prove that if $(X_\alpha, J_\alpha), \alpha \in J$ is an arbitrary collection of compact topological spaces then the product space $\prod_{\alpha \in J} X_\alpha$ is also a compact topological space. This theorem is due to Tychonoff and different proofs are available in the literature. To prove Tychonoff theorem we will use Zorn’s lemma. Let us recall the following:

**Definition 4.4.1.** Let $X$ be a nonempty set and $R \subseteq X \times X$, that is $R$ is a relation on $X$. If $(x, y) \in R$ then we say that $x$ is related to $y$ and write $x \leq y$. The pair $(X, R)$ is said to be a partially ordered set if and only if

(i) $x \leq x$ ($\leq$ is a reflexive),

(ii) for $x, y \in X$, $x \leq y$ and $y \leq x \Rightarrow x = y$. (That is $\leq$ is against symmetry in the sense that $x \leq y$ and $y \leq x$ can happen only when $x = y$.) In this case we say that $\leq$ is antisymmetry,

(iii) for $x, y, z \in X x \leq y$ and $y \leq z \Rightarrow x \leq z$. ($\leq$ is transitive.)

In this case we say that $(X, \leq)$ is a partially ordered set (PO set).
**Definition 4.4.2.** Let \((X, \leq)\) be a partially ordered set and \(A\) be a nonempty subset of \(X\). Then an element \(x \in X\) (note: \(x\) need not be in \(A\)) is called an **upper bound** of \(A\) if and only if \(a \leq x\) for all \(a \in A\). An element \(y \in Y\) is called a **lower bound** of \(A\) if and only if \(y \leq a\) for all \(a \in A\). If there exists an \(x_0 \in X\) such that (i) \(x_0\) is an upper bound of \(A\), (ii) \(x \in X\) is an upper bound of \(A\) implies \(x_0 \leq x\) then such an upper bound \(x_0\) is called the **least upper bound** (lub) of \(A\) and we can easily show that l.u.b of \(A\) is unique, when it exists. An element \(x_0 \in X\) is called the **greatest lower bound** (glb) of \(A\) if it satisfies the following: (i) \(x_0\) is a lower bound of \(A\), (ii) if \(y_0 \in X\) is a lower bound of \(A\) implies \(y_0 \leq x_0\).

**Definition 4.4.3.** An element \(x_0 \in X\) of a partially ordered set is called a **maximal element** of \(X\) if \(x \in X\) is such that \(x_0 \leq x\) then \(x = x_0\). An element \(y_0 \in X\) is called a **minimal element** of \(X\) if \(y \in X\) is such that \(y \leq y_0\) then \(y = y_0\).

**Example 4.4.4.** Let \(X = \{1, 2, 3, 4, 5\}\), \(R = \{(1, 2), (3, 4), (n, n) : n \in \{1, 2, 3, 4, 5\}\}\). If \((x, y) \in R\) then we say that \(x \leq y\). Here \(2, 4, 5 \in X\) and they are maximal elements of \(X\). Note that \((2, 3) \notin R\) and hence \(2\) is not related to \(3\). That is \(2 \leq 3\) is not true. Similarly \(2\) is not related to \(4\) and \(2\) is not related to \(5\). So \(2\) is not smaller than other elements of \(X\) and hence \(2\) is a maximal element of \(X\). Since \(3 \leq 4\) and \(3 \neq 4\), \(3\) is not maximal element of \(X\). If \(y_0 \in X\) is such that \(y_0\) is not larger than any other element of \(X\) then we say that \(y_0\) is a minimal element of \(X\). That is if there exists \(y \in X\) such that \(y \leq y_0\) then \(y = y_0\).

A nonempty subset \(A\) of \(X\) is said to be a chain (also known as totally ordered set) if for \(x, y \in A\), \(x \leq y\) or \(y \leq x\). That is any pair of elements \(x, y\) in \(A\) are comparable.

Now we are in a position to state Zorn’s lemma:
Lemma 4.4.5. Zorn’s Lemma. Let \((X, \leq)\) be a partially ordered set. Further suppose every chain \(C \subseteq X\) has an upper bound in \(X\). Then \(X\) will have at least one maximal element.

We observe the following: A topological space \((X, J)\) is compact if and only if whenever \(\mathcal{A}\) is a collection of subsets of \(X\) which has finite intersection property (f.i.p) then \(\bigcap_{A \in \mathcal{A}} A \neq \emptyset\).

Theorem 4.4.6. Tychonoff theorem. Let \((X_\alpha, J_\alpha), \alpha \in J\) be a collection of compact topological spaces. Then the product space \(\prod_{\alpha \in J} X_\alpha\) is also a compact space.

Proof. Start with a collection \(\mathcal{A}\) of subsets of \(X = \prod_{\alpha \in J} X_\alpha\) which has f.i.p. Then we aim to prove that \(\bigcap_{A \in \mathcal{A}} A \neq \emptyset\).

Step 1:
Let \(\mathcal{F} = \{\mathcal{D} : \mathcal{D}\) is a collection of subsets of \(X\) containing \(\mathcal{A}\) and \(\mathcal{D}\) has f.i.p }\).

For \(\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{F}\), define \(\mathcal{D}_1 \leq \mathcal{D}_2\) if \(\mathcal{D}_1 \subseteq \mathcal{D}_2\). Then \((\mathcal{F}, \leq)\) is a partially ordered set. Now let \(\mathcal{C}\) be a chain in \(\mathcal{F}\) and \(\mathcal{A}_0 = \bigcup_{\mathcal{D} \in \mathcal{C}} \mathcal{D}\) (here \(\mathcal{C} \subseteq \mathcal{F}\) and \(\mathcal{D} \in \mathcal{F}\)). It is easy to prove that \(\mathcal{A}_0\) is an upper bound for \(\mathcal{C}\). For this, we will have to prove that \(\mathcal{A}_0 \in \mathcal{F}\) and \(\mathcal{D} \leq \mathcal{A}_0\) for all \(\mathcal{D} \in \mathcal{C}\). First let us prove that \(\mathcal{A}_0\) has f.i.p. Let \(A_j \in \mathcal{A}_0\) for \(j = 1, 2, \ldots, n\). Then \(A_j \in \mathcal{D}_j\), for some \(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n \in \mathcal{C}\). As \(\mathcal{C}\) is a chain for \(j \in \{1, 2, \ldots, n\}\) either \(\mathcal{D}_i \subseteq \mathcal{D}_j\) or \(\mathcal{D}_j \subseteq \mathcal{D}_i\). Hence there exists \(k, 1 \leq k \leq n\) such that \(\mathcal{D}_j \subseteq \mathcal{D}_k\) for all \(j \in \{1, 2, \ldots, n\}\). Then \(A_j \in \mathcal{D}_k\) for all \(j\) and \(\mathcal{D}_k\) has f.i.p implies \(\bigcap_{j=1}^{n} A_j \neq \emptyset\). Also \(\mathcal{A} \subseteq \mathcal{A}_0\). Hence \(\mathcal{A}_0 \subseteq \mathcal{F}\). By the definition of \(\mathcal{A}_0\), \(\mathcal{D} \subseteq \mathcal{A}_0\) for all \(\mathcal{D} \subseteq \mathcal{C}\). This proves that \(\mathcal{A}_0 \in \mathcal{F}\) is an upper bound for \(\mathcal{C}\).

Now we have proved that every chain \(\mathcal{C}\) in \(\mathcal{F}\) has an upper bound in \(\mathcal{F}\). Therefore by Zorn’s lemma \(\mathcal{F}\) will have a maximal element say \(B \in \mathcal{F}\). This \(B \in \mathcal{F}\) is
such that (i) $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{B}$ has f.i.p, (ii) whenever $\mathcal{A}'$ is a collection of subsets of $X$ such that $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{A}'$ has f.i.p then $\mathcal{A}' \subseteq \mathcal{B}$.

**Step 2:** Now let us prove that $\mathcal{B}$ has the following properties:

(i) For $n \in \mathbb{N}, A_1, A_2, \ldots, A_n \in \mathcal{B}$ implies $A_1 \cap A_2 \cap \cdots \cap A_n \in \mathcal{B}$.

(ii) If $A$ is subset of $X$ such that $A \cap B \neq \phi$, for all $B \in \mathcal{B}$ then $A \in \mathcal{B}$.

To prove (i), let $A_0 = A_1 \cap A_2 \cap \cdots \cap A_n$ and $\mathcal{B}_0 = \mathcal{B} \cup \{A_0\}$. Then $\mathcal{B}_0 \in \mathcal{F}$ and $B \subseteq \mathcal{B}_0$. Since $\mathcal{B}$ is maximal, $B = \mathcal{B}_0$. This proves that $A_0 \in \mathcal{B}$.

To prove (ii), take $\mathcal{B}_0 = \mathcal{B} \cup \{A\}$. Then $\mathcal{B}_0 \in \mathcal{F}$ and hence by step 1, $A \in \mathcal{B}$.

**Step 3:** Let us prove that $\bigcap_{A \in \mathcal{B}} \overline{A} \neq \phi$.

For each $\alpha \in J$, $\{P_{\alpha}(A) : A \in \mathcal{B}\}$ is a collection of subsets of $(X_{\alpha}, \mathcal{J}_{\alpha})$. If $A_1, A_2, \ldots, A_n \in \mathcal{B}$, then $\mathcal{B}$ has f.i.p and $\bigcap_{j=1}^{n} A_j \neq \phi$. Let $x \in \bigcap_{j=1}^{n} A_j$. Now $P_{\alpha}(x) \in P_{\alpha}(A_j)$ for all $j = 1, 2, \ldots, n$. Hence $\{P_{\alpha}(A) : A \in \mathcal{B}\}$ is a collection of subsets of the compact topological space $(X_{\alpha}, \mathcal{J}_{\alpha})$. Further this collection has f.i.p.

This gives that $\bigcap_{A \in \mathcal{B}} P_{\alpha}(A) \neq \phi$. Let $x_{\alpha} \in \bigcap_{A \in \mathcal{B}} P_{\alpha}(A)$ and $x = (x_{\alpha})_{\alpha \in J}$. (That is, we define $f : J \to \bigcup_{\alpha \in J} X_{\alpha}$ as $f(\alpha) = x_{\alpha} \in X_{\alpha}$ and we identify $f$ with $x$.) Now we aim to prove that $x \in \overline{A}$, for each $A \in \mathcal{B}$. So fix $A \in \mathcal{B}$ and let $P_{\beta}^{-1}(V_{\beta})$ be a subbasic open set containing $x$. Now $x = (x_{\alpha}) \in P_{\beta}^{-1}(V_{\beta})$ implies $x_{\beta} \in V_{\beta}$. We have $x_{\beta} \in \overline{P_{\beta}(A)}$ and hence $V_{\beta}$ is an open set in $(X_{\alpha}, \mathcal{J}_{\alpha})$ containing $x_{\beta}$ implies $V_{\beta} \cap P_{\beta}(A) \neq \phi$ implies there exists $y \in A$ such that $P_{\beta}(y) \in V_{\beta}$. This gives that $y \in P_{\beta}^{-1}(V_{\beta}) \cap A$. Hence $P_{\beta}^{-1}(V_{\beta}) \cap A \neq \phi$ for all $A \in \mathcal{B}$ implies $P_{\beta}^{-1}(V_{\beta}) \in \mathcal{B}$. Again if $B$ is a basic open set containing $x$ in the product space $(X, \mathcal{J})$ then $B = P_{\beta_1}^{-1}(V_{\beta_1}) \cap P_{\beta_2}^{-1}(V_{\beta_2}) \cap \cdots \cap P_{\beta_n}^{-1}(V_{\beta_n})$ for some $V_{\beta_i} \in \mathcal{J}_{\beta_i}, i = 1, 2, 3, \ldots, n$. We have proved that each $P_{\beta_i}^{-1}(V_{\beta_i}) \in \mathcal{B}$ and hence $B \in \mathcal{B}$. Hence whenever $B$ is a basic open set containing $x$, then $B \cap A \neq \phi (A \in \mathcal{B})$ implies $x \in \overline{A}$, for all $A \in \mathcal{B}$ implies $x \in \bigcap_{A \in \mathcal{B}} \overline{A} \neq \phi$. Now
A \subseteq B implies \bigcap_{A \in A} \overline{A} \neq \phi. That is, whenever A is a collection of closed subsets of the product space \((X, \mathcal{J})\) and further A has f.i.p then \bigcap_{A \in A} \overline{A} \neq \phi. This proves that \((X, \mathcal{J})\) is a compact topological space. ■

Now let us introduce the notion of a generalized sequence, known as net and convergence of a net in a topological space.

Let \((X, \leq)\) be a partially ordered set. Further suppose for \(\alpha, \beta \in X\) there exist \(\gamma \in X\) such that \(\alpha \leq \gamma\) and \(\beta \leq \gamma\). Then we say that \((X, \leq)\) is a directed set.

(In the above case if \(\alpha \leq \gamma\) then we also say \(\gamma \geq \alpha\).)

**Definition 4.4.7.** Let \(X\) be a nonempty set and \((D, \leq)\) be a directed set. Then any function \(f : D \to X\) is called a net in \(X\). For each \(\alpha \in D\), \(f(\alpha) = x_\alpha \in X\) and we say that \(\{x_\alpha\}_{\alpha \in D}\) is a net in \(X\).

**Example 4.4.8.** Let \(D = \mathbb{N}\) and \(\leq\) be the usual relation on \(\mathbb{N}\). Then \((\mathbb{N}, \leq)\) is a directed set. If \(X\) is a nonempty set and \(f : \mathbb{N} \to X\) then for each \(n \in \mathbb{N}\), \(f(n) = x_n \in X\). Hence our net \(\{x_n\}_{n \in \mathbb{N}}\) is the well known concept namely sequence in \(X\). In this sense we say that every sequence is a net. Now take \(D = [0, 1]\). Then \((D, \leq)\) is also a directed set. Define \(f : [0, 1] \to \mathbb{R}\) as \(f(\alpha) = \alpha + 3\), \(\forall \alpha \in [0, 1]\). Here \(f = \{f(\alpha)\}_{\alpha \in D} = \{\alpha + 3\}_{\alpha \in [0, 1]}\) is a net (generalized sequence) in \(\mathbb{R}\).

It is intuitively clear that the net \(\{\alpha + 3\}_{\alpha \in [0, 1]}\) approaches to 4. What do we mean by saying that the net \(\{x_\alpha\}_{\alpha \in D}\) approaches 4? Can we also say that the net \(\{x_\alpha\}_{\alpha \in D}\) approaches 3? Well, in \(\mathbb{R}\) consider a sequence \(\{x_n\}_{n \in \mathbb{N}} = \{x_n\}_{n=1}^\infty\). We know that \(\lim_{n \to \infty} x_n = x\) (i.e. \(x_n \to x\) as \(n \to \infty\)) if and only if for each \(\epsilon > 0\) there exists \(n_0 \in \mathbb{N}\) such that \(x_n \in (x - \epsilon, x + \epsilon)\) for all \(n \geq n_0\). Note that \(x_n \to x\) as \(n \to \infty\) if and only if for each open set \(U\) containing \(x\) there exists \(n_0 \in \mathbb{N}\) such that \(x_n \in U\) for all \(n \geq n_0\).
Keeping this in mind, we define:

Let \((X, \mathcal{J})\) be a topological space and \(\{x_\alpha\}_{\alpha \in D}\) be a net in \(X\). Then we say that the net \(\{x_\alpha\}_{\alpha \in D}\) converges to an element \(x \in X\) if and only if for each open set \(U\) containing \(x\) there exists \(\alpha_0 \in D\) such that \(x_\alpha \in U, \forall \alpha \geq \alpha_0\) (that is \(\alpha \in D\) with \(\alpha_0 \leq \alpha\)). If \(\{x_\alpha\}_{\alpha \in D}\) converges to \(x\) then we write \(x_\alpha \to x\).

In a metric space \((X, d)\) we know that a sequence \(\{x_n\}_{n=1}^\infty\) in \(X\) converges to at most one element \(x\) in \(X\).

What about in a topological space? Whether a net \(\{x_\alpha\}_{\alpha \in D}\) in a topological space converges to at most one element in \(X\). Obviously the answer is no. For example, let \(X\) be any set containing at least two elements and \(\mathcal{J} = \{\phi, X\}\). Take \(x_1, x_2 \in X, x_1 \neq x_2\). Now with usual \(\leq\), \((\mathbb{N}, \leq)\) is a directed set.

Define \(f : \mathbb{N} \to X\) as

\[
 f(n) = \begin{cases} 
 x_1 & \text{when } n \text{ is odd} \\
 x_2 & \text{when } n \text{ is even}
\end{cases}
\]

Here our net is \(\{x_1, x_2, x_1, x_2, \ldots\}\) that is our net is a sequence in \(X\). Let \(x = x_1\). Then the only open set \(U\) containing \(x_1\) is \(X\) and hence \(n_0 = 1 \in \mathbb{N}\). Then for all \(n \geq n_0, x_n \in X = U\). Hence \(x_n \to x\) for any \(x \in X\). Also nothing special about the net \(\{x_1, x_2, x_1, x_2, \ldots\}\). In fact if \(D\) is a directed set and \(\{x_\alpha\}_{\alpha \in D}\) is an arbitrary net in \(X\) then for each \(x \in X, x_\alpha \to x\).

**Example 4.4.9.** Now consider \(X = \mathbb{R}\) and \(\mathcal{J}_f\), the cofinite topology on \(\mathbb{R}\). \(D = \mathbb{R}\) and \(\leq\) is our usual relation. Then \((D, \leq)\) is a directed set. Define \(f : D \to \mathbb{R}\) as \(f(\alpha) = \alpha\) for \(\alpha \in D = \mathbb{R}\). Then \(\{\alpha\}_{\alpha \in \mathbb{R}}\) is a net in \(\mathbb{R}\). Fix an element say \(x \in \mathbb{R}\). Whether \(x_\alpha \to x\)? How to start? Start with an open set \(U\) containing \(x\) in our...
topological space \((\mathbb{R}, J)\). Now \(U \in J\), \(x \in U\) (that is \(U \neq \emptyset\)) implies \(U^c\) is a finite subset of \(\mathbb{R}\).

**Case (i).** \(U^c = \emptyset (\Rightarrow U = X)\).

**Case (ii).** \(U^c \neq \emptyset\).

That is \(U^c\) is a nonempty finite subset of \(\mathbb{R}\). Hence there exists \(n_0 \in \mathbb{N}\) and \(x_1, x_2, \ldots, x_{n_0} \in \mathbb{R} = D\) such that \(U^c = \{\alpha_1, \alpha_2, \ldots, \alpha_{n_0}\}\). Now take a real number say \(\alpha_0\) such that \(\alpha_0 > \alpha_i\) for all \(i = 1, 2, \ldots, n_0\). This \(\alpha_0 \in D\) is such that \(x_\alpha = \alpha \in U\) \(\forall \alpha \geq \alpha_0, (\alpha \geq \alpha_0, \alpha_0 > \alpha_i \Rightarrow \alpha > \alpha_i \Rightarrow \alpha \not\in U^c \Rightarrow \alpha \in U)\).

Conclusion: We started with an open set \(U\) containing \(x\) and we could get an \(\alpha_0 \in D\) (\(\alpha_0\) depends on \(U\)) such that \(x_\alpha \in U, \forall \alpha \geq \alpha_0\). Hence by our definition \(x_\alpha \to x\).

That this net \(\{x_\alpha\} = \{\alpha\}_{\alpha \in D}\) converges to every element \(x\) of the given topological space \((\mathbb{R}, J)\).

(iii) \(D = \{1, 2, \ldots, p\}\) and \(\leq\) is our usual relation. \((D, \leq)\) is a directed set (check).

What about \(\{x_\alpha\}_{\alpha \in D}\)? Here \(D = \{1, 2, \ldots, 10\}\) implies \(\{x_\alpha\}_{\alpha \in D} = \{1, 2, \ldots, 10\}\). Now for any open set \(U\) containing 10 there exists \(\alpha_0 = 10 \in D\) is such that \(\alpha \in D, \alpha \geq \alpha_0 = 10 \Rightarrow \alpha = 10\) and \(x_\alpha = \alpha = 10 \in U\). Hence \(\{x_\alpha\}_{\alpha \in D} \to 10\).

**Theorem 4.4.10.** In a Hausdorff topological space \((X, J)\) a net \(\{x_\alpha\}_{\alpha \in D}\) in \(X\) cannot converge to more than one element.

**Proof.** Suppose a net \(\{x_\alpha\}_{\alpha \in D}\) converge to say \(x, y \in X\), where \(x \neq y\). Now \(x \neq y\), \((X, J)\) is a Hausdorff topological space implies there exist open sets \(U, V\) in \(X\) such that \((i)\) \(x \in U, y \in U\), \((ii)\) \(U \cap V = \emptyset\). Now \(x_\alpha \to x\), \(U\) is an open set containing \(x\) implies

\[
\text{there exists } \alpha_1 \in D \text{ such that } x_\alpha \in U \text{ for all } \alpha \geq \alpha_1. \tag{4.5}
\]
Also \( y_\alpha \to y \), \( V \) is an open set containing \( y \) implies
\[
\text{there exist } \alpha_2 \in D \text{ such that } y_\alpha \in V \text{ for all } \alpha \geq \alpha_2. \tag{4.6}
\]

Note that \( D \) with a relation \( \leq \) is a directed set and hence for \( \alpha_1, \alpha_2 \in D \) there exists \( \alpha_0 \in D \) such that \( \alpha_0 \geq \alpha_1 \) and \( \alpha_0 \geq \alpha_2 \) (that is \( \alpha_1 \leq \alpha_0 \) and \( \alpha_2 \leq \alpha_0 \)). Now \( \alpha_0 \geq \alpha_1 \) implies \( x_{\alpha_0} \in U \) from Eq. (4.5) and \( \alpha_0 \geq \alpha_2 \) implies \( x_{\alpha_0} \in V \) from Eq. (4.6). Hence \( x_{\alpha_0} \in U \cap V \), a contradiction to \( U \cap V = \emptyset \). We arrived at this contradiction by assuming \( x_\alpha \to x \), \( x_\alpha \to y \) and \( x \neq y \). This means \( \{x_\alpha\}_{\alpha \in D} \) cannot converge to more than one element. 

\[\blacksquare\]

**Note.** In a Hausdorff topological space a net \( \{x_\alpha\}_{\alpha \in D} \) may not converge. If a net converges then it converges to a unique limit.

**Theorem 4.4.11.** Let \((X, J)\) be a topological space and \( A \subseteq X \). Then an element \( x \) of \( X \) is in \( \overline{A} \) if and only if there exists a net \( \{x_\alpha\}_{\alpha \in D} \) in \( A \) such that \( x_\alpha \to x \).

**Proof.** Let us assume that \( x \in \overline{A} \). Our tasks are the following: (i) using the fact that \( x \in \overline{A} \) construct a suitable directed set, \((D, \leq)\), (ii) and then define a net \( \{x_\alpha\}_{\alpha \in D} \) that converges to \( x \). Now \( x \in \overline{A} \) implies for each open set \( U \) containing \( x \), \( U \cap A \neq \emptyset \). (If our topology \( J \) is induced by a metric \( d \) on \( X \) then \( J = J_d \). In this case \( B(x, \frac{1}{n}) \cap A \neq \emptyset \) for each \( n \in \mathbb{N} \). So take \( x_n \in B(x, \frac{1}{n}) \cap A \). Then \( d(x_n, x) < \frac{1}{n} \) and \( \frac{1}{n} \to 0 \) as \( n \to \infty \). Hence \( x_n \to x \).) Take \( D = \mathcal{N}_x = \{U \in J : x \in U\} \) that is \( \mathcal{N}_x \) is the collection of all open sets containing \( x \). For \( U, V \in \mathcal{N}_x \), define \( U \leq V \) if and only if \( V \subseteq U \) (reverse set inclusion is our relation \( \leq \)). Now define \( f : \mathcal{N}_x \to X \) as \( f(U) = x_U \in U \cap A \) (\( U \cap A \neq \emptyset \) for each \( U \in \mathcal{N}_x \) implies by axiom of choice such a function exists). Now we have a net \( \{x_U\}_{U \in \mathcal{N}_x} \).
Claim: $x_U \to x$.

Take an open set $U_0$ containing $x$, then such an $U_0 \in \mathbb{N}_x$ implies $f(U_0) \in U_0 \cap A$.

Now $U \in \mathbb{N}_x$ (our directed set) and $U \geq U_0$ implies $U \subseteq U_0$ implies $x_U \in U \subseteq U_0$.

Now $U \geq U_0$ implies $x_U \in U_0$. Hence by definition of convergence of a net, $x_U \to x$.

Conversely, assume that there is a net say $\{x_\alpha\}_{\alpha \in D}$ in $A$ such that $x_\alpha \to x$.

Now we will have to prove that $x \in \overline{A}$. So start with an open set $U$ containing $x$.

Hence $x_\alpha \to x$ implies

$$\text{there exists } \alpha_0 \in D \text{ such that } x_\alpha \in U \text{ for all } \alpha \geq \alpha_0. \quad (4.7)$$

($D$ is a directed set means $(D, \leq)$ is a directed set). In particular when $\alpha = \alpha_0$, $\alpha \geq \alpha_0$ and therefore from Eq. (4.7), $x_{\alpha_0} \in U$. Also $x_{\alpha_0} \in A$. Hence $x_{\alpha_0} \in U \cap A$.

That is for each open set $U$ containing $x$, $U \cap A \neq \emptyset$. This implies $x \in \overline{A}$. ■

**Theorem 4.4.12.** Let $X$, $Y$ be topological spaces and $f: X \to Y$. Then $f$ is continuous if and only if for every net $\{x_\alpha\}_{\alpha \in J}$ converging to an element $x \in X$ the net $\{f(x_\alpha)\}_{\alpha \in J}$ converges to $f(x)$.

**Proof.** Assume that $f : X \to Y$ is a continuous function. Now let $\{x_\alpha\}_{\alpha \in J}$ be a net in $X$ such that $x_\alpha \to x$ for some $x \in X$. We will have to prove that $f(x_\alpha) \to f(x)$.

Let $V$ be an open set containing $f(x)$ in $Y$. Now $V$ is an open set containing $f(x)$ and $f : X \to Y$ is a continuous function implies

$$\text{there exists an open set } U \text{ containing } x \text{ such that } f(U) \subseteq V. \quad (4.8)$$

Now $U$ is an open set containing $x$ and $x_\alpha \to x$ implies there exists an $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. Hence from Eq. (4.8), $f(x_\alpha) \in V$ for all $\alpha \geq \alpha_0$. That is,
for each open set $V$ containing $f(x)$ there exists $\alpha_0 \in J$ such that $f(x_{\alpha}) \in V$ for all $\alpha \geq \alpha_0$. This in turn implies $f(x_{\alpha}) \to f(x)$.

Now let us assume that whenever a net $\{x_{\alpha}\}_{\alpha \in J}$ converges to an element $x$ in $X$ then $f(x_{\alpha}) \to f(x)$ in $Y$. In this case we will have to prove that $f : X \to Y$ is continuous. We know that $f$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. (An element $z$ of $X$ is closer to $A$, that is if $z \in \overline{A}$ then the image $f(z)$ is closer to $f(A)$.) So start with $A \subseteq X$ and an element $y \in f(\overline{A})$ ($f(\overline{A}) = \phi \Rightarrow f(\overline{A}) \subseteq \overline{f(A)}$).

Now $y \in f(\overline{A})$ implies there exists $x \in \overline{A}$ such that $y = f(x)$. Hence $x \in \overline{A}$ implies there exists a net $\{x_{\alpha}\}_{\alpha \in J}$ in $A$ such that $x_{\alpha} \to x$ (refer the previous theorem) this implies by our assumption, $f(x_{\alpha}) \to f(x)$. Now $f(x_{\alpha}) \in f(A)$ and $f(x_{\alpha}) \to f(x)$ implies $f(x) \in \overline{f(A)}$ (again refer the previous theorem). So we have proved that $f(\overline{A}) \subseteq \overline{f(A)}$ whenever $A \subseteq X$. This implies $f : X \to Y$ is a continuous function. ■

**Alternate proof of theorem 4.4.12.**

**Proof.** Assume that whenever a net $\{x_{\alpha}\}_{\alpha \in J}$ converges to an element $x \in X$ then $f(x_{\alpha}) \to f(x)$ in $Y$. Now suppose $f$ is not continuous at $x$. Then there exists an open set $V$ containing $x$ such that $f(U) \notin V$ for every $U \in \mathcal{N}_x$. Then for each $U \in \mathcal{N}_x$ there exists $x_U \in U$ such that $f(x_U) \notin V$. Now observe (refer the proof of the theorem 4.4.11) that the net $\{x_U\}_{U \in \mathcal{N}_x}$ such that $x_U \to x$ in $X$ but $f(x_U) \not\to f(x)$ in $Y$. ■

Now let us define a concept which generalize the concept of a subsequence. Recall that if $X$ is a nonempty set $\{x_n\}_{n=1}^{\infty} = (x_n)_{n \in \mathbb{N}}$ is a sequence in $X$ if and only if there exists a function $f : \mathbb{N} \to X$ satisfying the condition that $f(x_n) = x_n$. Here we have a net $\{x_{\alpha}\}_{\alpha \in J}$ in $X$. Hence in place of $\mathbb{N}$ we have a directed set $(J, \leq)$ and a function $f : J \to X$ satisfying the condition that $f(\alpha) = x_{\alpha}$ for all $\alpha \in J$. 

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What do we mean by saying that \( \{x_{n_k}\}_{k=1}^\infty \) is a subsequence of \( \{x_n\} \)? We have a subset \( \{n_k\}_{k=1}^\infty \) of natural numbers satisfying \( n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \). Let \( D = \{n_k : k \in \mathbb{N}\} \) and we have \( f : \mathbb{N} \to X \) such that \( f(n) = x_n \) for all \( n \in \mathbb{N} \). That is essentially \( f : \mathbb{N} \to X \) is sequence in \( X \). Now we have another function say \( g : D \to \mathbb{N} \) satisfying (i) \( g(k) = n_k \), (ii) \( n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \) that is \( k < l \Rightarrow n_k < n_l \) that is \( k < l \Rightarrow g(k) < g(l) \). Also note that for each \( n_0 \in \mathbb{N} \) there exists \( k_0 \in \mathbb{N} \) such that \( f(k_0) = n_{k_0} > n_0 \). So keeping this motivation in mind we define the concept of subnet of a given net in \( X \). It is given that \( f : J \to X \) is a net in \( X \) (so it is understood that \((J, \leq)\) is a directed set). Suppose \( D \) with a relation \( \leq \) is a directed set (need not be the same relation as given in \( J \). But for the sake of simplicity we use same notation \( \leq \) for both sets). Suppose \( g : D \to J \) such that \( i, j \in D, i \leq j \) implies \( g(i) \leq g(j) \) and for each \( \alpha \in J \) there exists \( \gamma \in D \) such that \( g(\gamma) \geq \alpha \) (this is like saying that, when \( J = \mathbb{N} = D \), for each \( n_0 \in \mathbb{N} \) there exists \( k \in \mathbb{N} \) such that \( g(k) = n_k \geq n_0 \)). In such a case \( f \circ g : D \to X \) is called subnet of \( X \). \((f \circ g)(k) = f(g(k)) = f(n_k) = x_{n_k} \).

**Definition 4.4.13.** Let \((X, \mathcal{J})\) be a topological space and \( \{x_\alpha\}_{\alpha \in J} = (x_\alpha)_{\alpha \in J} \) be a net in \( X \). An element \( x \in X \) is said to be an **accumulation point** of the given net \((x_\alpha)_{\alpha \in J}\) if and only if for each open set \( U \) containing \( x \), the set \( K_U = \{\alpha \in J : x_\alpha \in U\} \) is cofinal in \( J \). Now \( K_U \) is cofinal in \( J \) means for each \( \alpha \in J \) there exists \( \beta \in K_U \) such that \( \beta \geq \alpha \) (it is like saying that \( k \to \infty \) implies \( n_k \to \infty \)).

Now let us prove:

**Theorem 4.4.14.** Let \((x_\alpha)_{\alpha \in J}\) be a net in a topological space. Then a point \( x \) in \( X \) is an accumulation point of the given net \((x_\alpha)_{\alpha \in J}\) if and only if \((x_\alpha)_{\alpha \in J}\) has a subnet and that subnet converges to \( x \).
Proof. ⇒ Assume that \( x \) is an accumulation point of \((x_\alpha)_{\alpha \in J}\). By the definition of accumulation point of a net we have for each open set \( U \) containing \( x \)

\[
K_U = \{ \alpha \in J : x_\alpha \in U \} \text{ is cofinal in } J. \tag{4.9}
\]

Let \( K = \{ (\alpha, U) \in J \times \mathcal{N}_x : x_\alpha \in U \} \), where \( \mathcal{N}_x \) is the collection of all open sets containing \( x \). From Eq. (4.9), \( K_U \neq \emptyset \). (Fix \( \alpha \in J \). Now \( K_U \) is cofinal in \( J \) implies there exists \( \beta \in K_U \) such that \( \beta \geq \alpha \).) For \((\alpha, U), (\beta, V) \in K \) define \((\alpha, U) \leq (\beta, V)\) if and only if \( \alpha \leq \beta \) and \( V \subseteq U \) (reverse set inclusion). It is easy to see that \((K, \leq)\) is a directed set. It is given that \((x_\alpha)_{\alpha \in J}\) is a net in \( X \). Hence \((J, \leq)\) is a directed set and \( f : J \to X \) is such that \( f(\alpha) = x_\alpha \). Now define \( g : K \to J \) as \( g(\alpha, U) = \alpha \) (refer Eq. (4.9)).

Claim: \( g(K) \) is cofinal in \( J \).

So take \( \alpha \in J \). Now \( K_U \) is cofinal in \( J \) (refer Eq. (4.9)) there exists \( \beta \in K_U \) such that \( \beta \geq \alpha \). Now \( \beta \in K_U \) implies \( x_\beta \in U \) that is \((\beta, U) \in K \) is such that \( g(\beta, U) = \beta \geq \alpha \) implies \( g(K) \) is cofinal in \( J \). Also \((\alpha, U), (\beta, V) \in K \), \((\alpha, U) \leq (\beta, V)\) implies \( g(\alpha, U) = \alpha \leq \beta = g(\beta, V) \). Hence \( f \circ g : K \to X \) is a subnet of \( f \) (or say \( f(\alpha) = (x_\alpha) \)). Now let us prove that this subnet converges to \( x \). So take an open set \( U \) containing \( x \). This implies \( K_U \) is cofinal in \( J \). Fix \((\alpha_0, U) \in K \). Now \( \alpha_0 \in J \), \( K_U \) is cofinal in \( J \) implies \( \beta_0 \in K_U \) such that \( \beta_0 \geq \alpha_0 \). Hence \((\alpha, V) \in K \), \((\alpha, V) \geq (\alpha_0, U)\) implies \((f \circ g)(\alpha, V) = f(\alpha) = x_\alpha \in V \subseteq U \). That is for each open set \( U \) containing \( x \) there exists \((\alpha_0, U) \in K \) such that \((\alpha, V) \in K \), \((\alpha, V) \geq (\alpha_0, U)\) implies \((f \circ g)(\alpha, V) \in U \). This proves that \( f \circ g \to x \).

Conversely, suppose there is a subnet of \((f(\alpha))_{\alpha \in J} = (x_\alpha)_{\alpha \in J}\) which converge to an element \( x \in X \). A subnet of \( f \) converges to \( x \) means there exists a directed set
say \((K, \leq)\) and a function say \(g : K \to J\) such that \(i, j \in K, i \leq j\) implies \(g(i) \leq g(j)\), \(g(K)\) is cofinal in \(J\), and \((f \circ g)(i) = f(g(i)) \to x\). Now let us prove that \(x\) is an accumulation point of the net \(f\). So take an open set \(U\) containing \(x\).

**Claim:** \(\{\alpha \in J : f(\alpha) = x_\alpha \in U\}\) is cofinal in \(J\).

Let \(\alpha_0 \in J\). Now \(f \circ g : K \to X\) is a subnet such that \(f \circ g \to x\). Hence for this given \(\alpha_0 \in J\) there exists \(\beta \in K\) such that \(g(\beta) \geq \alpha_0\) (note \(g(K)\) is cofinal in \(J\)).

Now \(f \circ g \to x, U\) is an open set containing \(x\) implies there exists \(\beta_0 \in K\) such that \(\alpha \in K, \alpha \geq \beta_0 \Rightarrow f(g(\alpha)) \in U, \beta \in J\) is such that \(g(\beta) \geq \alpha_0\). Take \(\gamma_0 \in K\) such that \(\alpha_0 \geq \beta, \beta_0\). Then \((f \circ g)(\gamma_0) \in U\) and \(g(\gamma_0) \geq g(\beta) \geq \alpha_0\). That is for \(\alpha_0 \in J\), there exists \(g(\gamma_0) \in J\) such that \(f(g(\gamma_0)) \in U\) implies \(\{\alpha \in J : f(\alpha) = x_\alpha \in U\}\) is cofinal in \(J\). Hence \(x\) is an accumulation point.

Recall that a metric space \((X, d)\) is a compact metric space if and only if every sequence \(\{x_n\}_{n=1}^\infty\) in \(X\) has a subsequence \(\{x_{n_k}\}_{k=1}^\infty\) that converges to an element in \(X\). It is to be noted that this result is not true for an arbitrary topological space. For a topological space we have the following theorem.

**Theorem 4.4.15.** A topological space \((X, J)\) is compact if and only if every net in \(X\) has a subnet that converges to an element in \(X\).

**Proof.** Assume that \((X, J)\) is a compact topological space and \(f : J \to X\) is a net in \(X\). We will have to prove that \(f\) has a subnet that converges to an element in \(X\). So it is enough to prove that \(f\) has an accumulation point.

For each \(\alpha \in J\), let \(A_\alpha = \{x_\beta : \alpha \leq \beta\}\) (note: \(f : J \to X\) is a net means with respect to a relation \(\leq\), \((J, \leq)\) is directed set). Now \(\{A_\alpha\}_{\alpha \in J}\) is a collection of sets which has finite intersection property. For \(A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_k}\) if we take \(\alpha \in J\) such
that \( \alpha \geq \alpha_j \) for all \( j = 1, 2, \ldots, k \), that is \( \alpha_j \leq \alpha \), then \( x_\alpha \in A_{\alpha_j} \), \( \forall j = 1, 2, \ldots, k \) and hence \( x \in \bigcap_{j=1}^{k} A_{\alpha_j} \). Now \((X, J)\) is a compact topological space \( \{\overline{A_\alpha}\}_{\alpha \in J} \) is a collection of closed subsets of \( X \) which has finite intersection property implies \( \bigcap_{\alpha \in J} \overline{A_\alpha} \neq \emptyset \). Let \( x \in \bigcap_{\alpha \in J} \overline{A_\alpha} \).

Now we aim to prove that \( x \) is an accumulation point of \( f \). So, start with an open set \( U \) containing \( x \), and we will have to prove that \( \{\alpha \in J : x_\alpha \in U\} \) is cofinal in \( J \). Take \( \alpha_0 \in J \). Now \( U \) is an open set containing \( x \), \( x \in \overline{A_{\alpha_0}} \) implies \( U \cap A_{\alpha_0} \neq \emptyset \). Hence there exists \( \alpha \geq \alpha_0 \) such that \( x_\alpha \in U \). This proves that \( \{\alpha \in J : x_\alpha \in U\} \) is cofinal in \( J \). Hence we have proved that \( x \) is an accumulation point of the stated net \( f \). This implies there exists a subnet of \( f \) which converges to \( f \).

To prove the converse part let us assume that every net in \( X \) has convergent subnet in \( X \). By assuming this, we aim to prove that \((X, J)\) is a compact topological space.

To prove that \((X, J)\) is a compact topological space, let us prove: if \( A \) is a collection of closed subsets of \( X \) which has finite intersection property then \( \bigcap_{A \in A} A \neq \emptyset \). So, we have a collection \( A \) of closed subsets of \( X \) which has finite intersection property.

Let \( B = \{A \subseteq X : A = A_1 \cap A_2 \cap \cdots \cap A_k, k \in \mathbb{N}, A_1, \ldots, A_k \in A\} \). That is \( B \) is the collection of finite intersection of members of \( A \). (Note. \( \bigcap_{A \in \phi} A = X \) and hence we do not require to consider this case.) For \( A, B \in B \) define \( A \preceq B \), whenever \( B \subseteq A \). Then \((B, \preceq)\) is a directed set. Now define \( f : B \rightarrow X \) as \( f(A) = f(A_1 \cap A_2 \cap \cdots \cap A_k) = x_A \), where \( x_A \in A_1 \cap A_2 \cap \cdots \cap A_k \) is fixed \( (A_1 \cap A_2 \cap \cdots \cap A_k) \) may contains more than one element and in that case first take any one element form \( A_1 \cap A_2 \cap \cdots \cap A_k \). Hence \( f = (f(A))_{A \in B} \) is a net in \( X \). By our assumption this net \( f \) will have a subnet
that will converge to an element say \textit{x} in \textit{X}. So there will exists a directed set \textit{K} and a function \textit{g}: \textit{K} \to \mathcal{B} satisfying \textit{f} \circ \textit{g} is a subnet of \textit{f} and \textit{f} \circ \textit{g} converges to \textit{x}.

Now we claim that \textit{x} \in \textit{A} for each \textit{A} \in \mathcal{A}. Suppose for some \textit{A} \in \mathcal{A}, \textit{x} \notin \textit{A}. Then \textit{x} \in \textit{A}^c = \textit{U}, an open set. Since \textit{f} \circ \textit{g} \to \textit{x} and \textit{U} is an open set containing \textit{x} there exists \alpha_0 \in \textit{K} such that (\textit{f} \circ \textit{g})(\alpha) \in \textit{U} for all \textit{\alpha} \geq \alpha_0. Now \alpha_0 \in \textit{K} implies \textit{g}(\alpha_0) \in \mathcal{B} implies there exists \textit{k} \in \mathbb{N} and \textit{A}_1, \textit{A}_2, \ldots, \textit{A}_\textit{k} \in \mathcal{A} such that \textit{g}(\alpha_0) = \textit{A}_1 \cap \textit{A}_2 \cap \cdots \cap \textit{A}_\textit{k}. \textit{A}_1 \cap \textit{A}_2 \cap \cdots \cap \textit{A}_\textit{k} \in \mathcal{B} is such that \textit{A}_1 \cap \textit{A}_2 \cap \cdots \cap \textit{A}_\textit{k} \geq \textit{g}(\alpha_0). We have \textit{f} \circ \textit{g}(\alpha_0) = \textit{f}(\textit{g}(\alpha_0)) \in \textit{U} = \textit{A}^c. Now \textit{K} is a directed set and \textit{g}(\textit{K}) is cofinal in \mathcal{B} implies there exists \textit{\alpha} \in \textit{K} such that (i) \textit{\alpha} \geq \alpha_0, (ii) \textit{g}(\textit{\alpha}) \geq \textit{A}_1 \cap \textit{A}_2 \cap \cdots \cap \textit{A}_\textit{k}. Now \textit{\alpha} \geq \alpha_0 implies (\textit{f} \circ \textit{g})(\textit{\alpha}) \in \textit{U} = \textit{A}^c but by the definition of \textit{f}, \textit{f}(\textit{g}(\textit{\alpha})) \in \textit{g}(\textit{\alpha}) \subseteq \textit{A}_1 \cap \textit{A}_2 \cap \cdots \cap \textit{A}_\textit{k} \subseteq \textit{A}. So we get a contradiction. Therefore \textit{x} \notin \textit{A} for some \textit{A} \in \mathcal{A} cannot happen. We have proved that if \mathcal{A} is a collection of closed subsets of \textit{X} which has finite intersection property then \bigcap_{\textit{A} \in \mathcal{A}} \textit{A} \neq \phi. Hence (\textit{X}, \mathcal{J}) is a compact topological space. \hfill \blacksquare

**Definition 4.4.16.** A topological property is any property so that if (\textit{X}, \mathcal{J}), (\textit{Y}, \mathcal{J}') are topological spaces and \textit{f}: (\textit{X}, \mathcal{J}) \to (\textit{Y}, \mathcal{J}') is a homeomorphism (that is (\textit{X}, \mathcal{J}) is homeomorphic to (\textit{Y}, \mathcal{J}')) then (\textit{X}, \mathcal{J}) has the property if and only if (\textit{Y}, \mathcal{J}') has the same property.

**Example 4.4.17.** Compactness, connectedness, local compactness are all topological properties.
Exercises.

1. Let $a, b \in \mathbb{R}, a < b$ and $\mathcal{A}$ be a collection of open sets in $\mathbb{R}$ such that $[a, b] \subseteq \bigcup_{A \in \mathcal{A}} A$. Let $C = \{x \in [a, b] : [a, x] \text{ is covered by finitely many members of } \mathcal{A}\}$. Now $C$ is a nonempty bounded subset of $\mathbb{R}$ implies by LUB axiom $z = \sup C = \text{lub } C$ exists. Prove that $z = b$ and $b \in C$.

2. Let $(X, \mathcal{J})$ be a locally compact Hausdorff space. Further assume that $(X, \mathcal{J})$ is not a compact space. Suppose there exists a compact Hausdorff space $(Y, \mathcal{J}')$ such that (i) $Y \setminus X$ is a single point, and (ii) $\overline{X} = Y$. Then there exists a homeomorphism $f : (X^*, \mathcal{J}^*) \rightarrow (Y, \mathcal{J})$ such that $f(x) = x$ for all $x \in X$, where $(X^*, \mathcal{J}^*)$ is the one point compactification of $(X, \mathcal{J})$.

3. Let $(X, \mathcal{J})$ be a Hausdorff topological space. Further suppose every subset $A$ of $X$ is compact. Then prove that every subset of $X$ is open.

4. Say true or false (justify your answer)

   • $A = \{(n, m) : 1 \leq n, m \leq 100\}$ is a compact subset of $\mathbb{R}^2$.
   • Let $X = \mathbb{R}$ and $\mathcal{J}_c$ be the co-countable topology on $\mathbb{R}$. Then every countable subset of $(\mathbb{R}, \mathcal{J}_c)$ is compact.
   • $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is a compact subset of $\mathbb{R}$.

5. Is it true that continuous image of a locally compact space is locally compact? Justify your answer.

6. Let $A_1, A_2, \ldots, A_n (n \in \mathbb{N})$ be compact subsets of a topological space $(X, \mathcal{J})$ then prove that $\bigcup_{i=1}^{n} A_i$ is compact.

7. Is it true that $\{A_n\}_{n=1}^{\infty}$ are compact subsets of a topological space then $\bigcup_{n=1}^{\infty} A_n$ is also compact set? Justify your answer.
8. Prove or disprove: Let \( \{A_n\}_{n=1}^{\infty} \) be a collection of compact sets such that \( A_n \subseteq A_{n+1} \) for all \( n \in \mathbb{N} \). Then \( \bigcup_{n=1}^{\infty} A_n \) is also a compact set.

9. Prove that \( A = [-1, 0] \cup \{1/2, 1/3, \cdots\} \) is a compact subset of \( \mathbb{R} \).

10. Is \( A = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |x_k| \leq \frac{1}{k}, k = 1, 2, \ldots, n\} \) a compact subset of \( \mathbb{R}^n \)? Justify your answer (for \( x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \), let \( d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n| \) then \((\mathbb{R}^n, d_1)\) is a metric space).

11. Prove that a subset \( A \) of \( \mathbb{R} \) is compact if and only if every countable collection of open sets which covers \( A \) has a finite subcollection which also covers \( A \).

12. Can there exist a continuous one-one function from \([a, b] \) onto \([a, b] \times [c, d] \), where \( a, b, c, d \in \mathbb{R}, a < b, c < d \).

13. Is the intersection of finite number of compact subsets of a topological space compact? Justify your answer.

14. Can there exist a homeomorphism between \((0, 1)\) and \([0,1]\)? Justify your answer.

15. Let \( \mathcal{J}_f \) be the cofinite topology on \( \mathbb{R} \).
   
   (i) For each \( x \in \mathbb{R} \), prove that \( \{\frac{1}{n}\}_{n \in \mathbb{N}} \) converges to \( x \) in \((\mathbb{R}, \mathcal{J}_f)\).
   
   (ii) Is \( \{1, 0, 2, 0, 3, 0, 4, 0, 5, 0, \ldots\} \) converges to 0 in \((\mathbb{R}, \mathcal{J}_f)\)? Justify your answer.

16. For each \( n \in \mathbb{N} \), let \( X_n = \{0, 1\}, \mathcal{J}_n = \{\emptyset, \{0\}, \{1\}, X_n\}, \mathcal{J} \) - product topology on \( X = \prod_{n \in \mathbb{N}} X_n = \bigcap_{n=1}^{\infty} X_n \) and \( \mathcal{J}_b \) - box topology on \( X \). Is \((X, \mathcal{J})\) a compact space? Is \((X, \mathcal{J}_b)\) a compact space? Justify your answer.
Chapter 5

Countability and Separation Axioms

5.1 First and Second Countable Topological Spaces

Definition 5.1.1. A topological space $(X, T)$ is said to have a countable local basis (or countable basis) at a point $x \in X$ if there exists a countable collection say $\mathcal{B}_x$ of open sets containing $x$ such that for each open set $U$ containing $x$ there exists $V \in \mathcal{B}_x$ with $V \subseteq U$.

Definition 5.1.2. A topological space $(X, T)$ is said to be first countable or said to satisfy the first countability axiom if for each $x \in X$ there exists a countable local base at $x$.

Examples 5.1.3. (i) Let $(X, d)$ be a metric space then for each $x \in X$, $\mathcal{B}_x = \{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable local basis at $x$. Hence $(X, T_d)$ is a first countable space. So, we say that every metric space $(X, d)$ is a first countable space.

(ii) Let $X = \mathbb{N}$ and $T = \{\emptyset, X, \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}, \ldots\}$ then obviously $(X, T)$ is a first countable topological space.

Note that this is not an interesting example of a first countable topological space. Once the topology $T$ is a countable collection then $(X, T)$ is a first countable space.

Example 5.1.4. Let $X = \mathbb{R}$ and $T_l$ be the lower limit topology on $\mathbb{R}$ generated by $\{[a, b) : a, b \in \mathbb{R}, a < b\}$. For each $x \in X$, $\mathcal{B}_x = \{[x, x + \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable
local base at \( x \). Hence \((\mathbb{R}, \mathcal{J}_l) = \mathbb{R}_l\) is a first countable topological space. Now let us see a stronger version of first countable topological space.

**Definition 5.1.5.** If a topological space \((X, \mathcal{J})\) has a countable basis \(\mathcal{B}\) then we say that \((X, \mathcal{J})\) is a **second countable topological space** or it satisfies the second countability axiom.

**Exercise 5.1.6.** Though it is trivial from the definition, prove that every second countable topological space \((X, \mathcal{J})\) is a first countable topological space.

What about the converse?

Let \( X \) be any uncountable set and \( \mathcal{J}_D \) be the discrete topology on \( X \). Then \((X, \mathcal{J}_D)\) is first countable, but it is not second countable. In fact, for each \( x \in X \), \( \mathcal{B}_x = \{\{x\}\} \) is a countable local base at \( x \). Take any open set \( U \) containing \( x \) then there exists \( V = \{x\} \in \mathcal{B}_x \) such that \( x \in V \subseteq U \). Hence \( \mathcal{B}_x \) is a local base at \( x \).

How to prove that \((X, \mathcal{J})\) is not a second countable topological space? Well we use the method of proof by contradiction.

Suppose there exists a countable basis say \( \mathcal{B} = \{B_1, B_2, \ldots\} \) for \((X, \mathcal{J})\). Let us assume that each \( B_k \neq \emptyset \). For each \( k \in \mathbb{N} \), let \( x_k \in B_k \). Since \( X \) is an uncountable set we can select an \( x \in X \) such that \( x \neq x_k \) for all \( k \in \mathbb{N} \). Now \( \{x\} \), the singleton set containing \( x \), is an open set and \( \mathcal{B} \) is a basis for \((X, \mathcal{J})\) implies there exists \( k \in \mathbb{N} \) such that \( x \in B_k \subseteq \{x\} \) this implies \( B_k = \{x\} \). But \( x_k \in B_k \) implies \( x = x_k \), a contradiction to our assumption that \( x \in X \) such that \( x \neq x_k \) for all \( k \in \mathbb{N} \). Hence if \( X \) is an uncountable set then the discrete topological space \((X, \mathcal{J}_D)\) is first countable but not second countable.

Also we have seen that the lower limit topological space \( \mathbb{R}_l \) is first countable. Now let us prove that \( \mathbb{R}_l = (\mathbb{R}, \mathcal{J}_l) \) is not a second countable topological space. That
is we will have to prove that if \( \mathcal{B} \) is a basis for \((\mathbb{R}, \mathcal{J})\) then \( \mathcal{B} \) is not a countable collection. So, fix a basis say \( \mathcal{B} \) for \((\mathbb{R}, \mathcal{J})\). For each \( x \in \mathbb{R}, [x, x + 1) \in \mathcal{J} \). Hence \( \mathcal{B} \) is a basis for \((\mathbb{R}, \mathcal{J})\) implies there exists \( B_x \in \mathcal{B} \) such that \( x \in B_x \subseteq [x, x + 1) \).

\[
\begin{array}{cccc}
  x & y & x + 1 & y + 1 \\
\end{array}
\]

Figure 5.1

For \( x, y \in \mathbb{R}, x \neq y \) we have \([x, x + 1) \neq [y, y + 1)\). Also \( B_x \subseteq [x, x + 1) \) implies \( \inf B_x \geq \inf [x, x + 1) = x \). Also \( x \in B_x \) implies \( x \geq \inf B_x \). Hence \( x = \inf B_x \). Now define \( f : \mathbb{R} \to \mathcal{B} \) as \( f(x) = B_x \). Then \( x \neq y \) implies \( B_x \neq B_y \). \( B_x = B_y \) implies \( \inf B_x = \inf B_y \) That is \( f(x) \neq f(y) \). Hence \( f \) is an one-one function. This implies that \( f : \mathbb{R} \to f(\mathbb{R}) \subseteq \mathcal{B} \) is a bijective function. Therefore \( f(\mathbb{R}) \) is an uncountable set and hence \( \mathcal{B} \) is an uncountable set. We have proved that if \( \mathcal{B} \) is a basis for \((\mathbb{R}, \mathcal{J})\) then \( \mathcal{B} \) is an uncountable set. Hence \((\mathbb{R}, \mathcal{J})\) cannot have a countable basis and therefore \((\mathbb{R}, \mathcal{J})\) is not a second countable topological space.

It is a simple exercise to check \( \mathcal{Q} = \mathbb{R} \) in \((\mathbb{R}, \mathcal{J})\). That is \( \mathcal{Q} \) is a countable dense subset of \( \mathbb{R} \) with respect to \((\mathbb{R}, \mathcal{J})\). Such a topological space is known as a separable topological space.

**Definition 5.1.7.** A topological space \((X, \mathcal{J})\) is said to be a **separable topological space** if there exists a countable subset say \( A \) of \( X \) such that \( \overline{A} = X \).

**Definition 5.1.8.** A topological space \((X, \mathcal{J})\) is said to be a **Lindelöf** space if for any collection \( \mathcal{A} \) of open sets such that \( X = \bigcup_{A \in \mathcal{A}} A \), there exists a countable subcollection...
say $\mathcal{B} \subseteq \mathcal{A}$ such that $X = \bigcup_{B \in \mathcal{B}} B$. That is, a topological space $(X, \mathcal{J})$ is said to be a Lindelöf space if and only if every open cover of $X$ has a countable subcover for $X$.

By definition every compact topological space $(X, \mathcal{J})$ is a Lindelöf space. But the converse need not be true. It is easy to prove that $\mathbb{R}$ (with usual topology) is a Lindelöf space. But $\mathbb{R}$ is not compact space.

Now let us prove that every second countable topological space is a Lindelöf space.

**Theorem 5.1.9.** If $(X, \mathcal{J})$ is a second countable topological space then $(X, \mathcal{J})$ is a Lindelöf space.

**Proof.** Let $\mathcal{B} = \{B_1, B_2, B_3, \ldots\}$ be a countable basis for $(X, \mathcal{J})$ and $\mathcal{A}$ be an open cover for $X$. Let us assume that, $X \neq \phi$, $A \neq \phi$ for each $A \in \mathcal{A}$ and $B \neq \phi$, for each $B \in \mathcal{B}$. Fix $A \in \mathcal{A}$ and $x \in A$. Now $x \in A$, $A$ is an open set implies there exists $B \in \mathcal{B}$ such that

$$x \in B \subseteq A.$$  \hfill (5.1)

For each $n \in \mathbb{N}$, let $\mathcal{F}_n = \{A \in \mathcal{A} : B_n \subseteq A\}$. Here it is possible that $\mathcal{F}_n = \phi$, for some $n \in \mathbb{N}$. At the same time note that, from Eq. (5.1), $\{n \in \mathbb{N} : \mathcal{F}_n \neq \phi\}$ is a nonempty set. Let $\{n \in \mathbb{N} : \mathcal{F}_n \neq \phi\} = \{n_1, n_2, \ldots, n_k, \ldots\}$ (it may be a finite set) and for each such $k$ take $A_{n_k} \in \mathcal{F}_{n_k}$. This will give us $B_{n_k} \subseteq A_{n_k} \in \mathcal{A}$.

Let us prove that $\bigcup_{k=1}^{\infty} A_{n_k} = X$. So, let $x \in X$. Now $A$ is an open cover for $X$ implies $x \in A$ for some $A$. Now $x \in A$, $\mathcal{B}$ is a basis for $(X, \mathcal{J})$ implies there exists $k \in \mathbb{N}$ such that $x \in B_{n_k} \subseteq A$. This implies that $A \in \mathcal{F}_{n_k}$. Also $A_{n_k} \in \mathcal{F}_{n_k}$. Hence by our definition of $\mathcal{F}_{n_k}$, $B_{n_k} \subseteq A_{n_k}$. Hence $x \in X$ implies $x \in A_{n_k}$, for some $k \in \mathbb{N}$.
This implies that \( X \subseteq \bigcup_{k=1}^{\infty} A_{nk} \). That is \( \{A_{nk}\}_{k=1}^{\infty} \) is a countable subcover for \( \mathcal{A} \). Therefore every open cover \( \mathcal{A} \) of \( X \) has a countable subcover. Hence \((X, J)\) is a Lindelöf space. \( \blacksquare \)

**Note.** Recall that, for \( 1 \leq p < \infty \), \( l_p = \{x = (x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^p < \infty \} \) is a second countable metric space, where for \( x = (x_n) \in l_p \), \( y = (y_n) \in l_p \),

\[
d_p(x_n, y_n) = d_p(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}.
\]

Also note that \( \mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\} \) is a second countable metric space with respect to any of the metric given by \( d_p(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \) for \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \), \( 1 \leq p < \infty \) or \( d_\infty(x, y) = \max\{|x_k - y_k| : k = 1, 2, \ldots, n\} \). So, all the above mentioned metric spaces are all Lindelöf spaces. But none of these metric spaces is a compact space. \( \star \)

Now let us prove that a second countable topological space is a separable space.

**Theorem 5.1.10.** Every second countable topological space \((X, J)\) is a separable space.

**Proof.** Given that \((X, J)\) is a second countable topological space. Hence there exists a countable basis say \( \mathcal{B} = \{B_1, B_2, \ldots\} \) for \((X, J)\). When we write \( \mathcal{B} = \{B_1, B_2, \ldots\} \), it does not mean that \( \mathcal{B} \) is a countably infinite set. It means that either for some \( n \in \mathbb{N} \), \( \mathcal{B} = \{B_1, B_2, \ldots, B_n\} \) or \( \mathcal{B} = \emptyset \) or \( \mathcal{B} \) is a countably infinite set. If \( X \neq \emptyset \) then \( \mathcal{B} \neq \emptyset \). If for some \( k \in \mathbb{N} \), \( B_k = \emptyset \), then \( \mathcal{B}' = \{B_1, B_2, \ldots, B_{k-1}, B_{k+1}, \ldots\} \) is also a basis for \((X, J)\).

So, let us assume that each \( B_n \neq \emptyset \) for all \( n \). Since \( B_n \neq \emptyset \), for each \( n \in \mathbb{N} \), let \( x_n \in B_n \) (note that by axiom of choice there exists a function \( f : \mathbb{N} \to \bigcup_{n=1}^{\infty} B_n \) such
that \( x_n = f(n) \in B_n \) and \( A = \{x_1, x_2, x_3, \ldots \} \). Here also it is quite possible that \( A \) is a finite set. Now let us prove that \( \overline{A} = X \). So, take an \( x \in X \) and an open set \( U \) containing \( x \). Now \( \mathcal{B} \) is a basis for \((X, \mathcal{J})\), \( U \) is an open set containing \( x \) implies there exists \( B_n \in \mathcal{B} \) such that \( x \in B_n \) and \( B_n \subseteq U \). Also \( x_n \in B_n \). Hence \( x_n \in U \cap A \). This gives that \( U \cap A \neq \phi \). That is we have proved that \( U \cap A \neq \phi \) for each open set \( U \) containing \( x \). Hence \( x \in \overline{A} \). That is \( x \in X \) and hence \( x \in \overline{A} \) and hence \( \overline{A} = X \). Therefore \((X, \mathcal{J})\) has a countable dense subset and therefore \((X, \mathcal{J})\) is a separable space.

Now let us prove that subspace of a separable metric space is separable.

**Theorem 5.1.11.** Let \((X, d)\) be a separable metric space and \( Y \) be a subspace of \( X \) (that is \( Y \subseteq X \), and for \( x, y \in Y \), \( d_Y(x, y) = d(x, y) \)). Then \((Y, d_Y)\) is a separable space.

**Proof.** \((X, d)\) is a separable metric space implies there exists a countable subset say \( A = \{x_1, x_2, x_3, \ldots \} \) of \( X \) such that \( \overline{A} = X \) (here \( \overline{A} \) denotes the closure of \( A \) with respect to \((X, d)) \). We will have to find a countable subset say \( B \) of \( Y \) such that \( \overline{B_Y} = Y \) (here \( \overline{B_Y} = \overline{B} \cap Y \), the closure of \( B \) with respect to the subspace \((Y, d_Y)) \). For \( n \in \mathbb{N} \), let \( A_{n,k} = B(x_n, \frac{1}{k}) \cap Y \). Here we do not know whether \( A_{n,k} = \phi \) or \( A_{n,k} \neq \phi \). If \( A_{n,k} \neq \phi \) \((n, k \in \mathbb{N})\) let \( a_{n,k} \in A_{n,k} \) be a fixed element. Then \( B = \{a_{n,k} : a_{n,k} \in A_{n,k} \text{ whenever } A_{n,k} \neq \phi \} \) is a countable subset of \( Y \). Now let us prove that \( \overline{B_Y} = \overline{B} \cap Y = Y \). So let \( x \in Y \) and \( U \) be an open set in \( X \) containing \( x \). Hence there exists \( k \in \mathbb{N} \) such that \( B(x, \frac{1}{k}) \subseteq U \). Again \( x \in \overline{A} = X \) implies \( B(x, \frac{1}{2k}) \cap A \neq \phi \). Then there exists \( x_n \in A \) such that \( x_n \in B(x, \frac{1}{2k}) \). Therefore \( x \in B(x_n, \frac{1}{2k}) \cap Y = A_{n,2k} \). Hence \( A_{n,2k} \neq \phi \). Now \( A_{n,2k} \neq \phi \) implies \( a_{n,2k} \in B \). Further \( a_{n,2k} \in B(x_n, \frac{1}{2k}) \). Now \( d(x, a_{n,2k}) \leq d(x, x_n) + d(x_n, a_{n,2k}) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \).
Hence \( a_{n,2k} \in B(x, \frac{1}{k}) \cap B \subseteq U \cap B \). That is \( U \cap B \neq \emptyset \) for each open set \( U \) containing \( x \). This implies \( x \in B \). Also \( x \in Y \). Therefore \( x \in B \cap Y = B_Y \). This implies \( Y \subseteq B_Y \subseteq Y \) and hence \( B_Y = Y \). That is \( B \) is a countable dense subset of \( Y \). This proves that the subspace \( (Y, d_Y) \) is a separable metric space. ■

**Subspace of a separable topological space need not be separable.**

We give an example to show that subspace of a separable topological space need not be separable.

Let \( X = \{(x, y) : x \in \mathbb{R}, y \geq 0\} \). Basic open sets are of the type:

(i) for \((x, y) \in \mathbb{R}^2, x \in \mathbb{R}, y > 0\) basic open sets containing \((x, y)\) are of the form \( B((x, y), r), 0 < r < y \), and (ii) for \((x, 0) \in \mathbb{R}^2, x \in \mathbb{R}\), basic open sets are of the form \((B(x, 0), r) \cap X) \setminus \{(y, 0) : 0 < |y - x| < r\}, r > 0\). Here \( B((x, y), r) = \{(a, b) \in \mathbb{R}^2 : d((x, y), (a, b)) = \sqrt{(x-a)^2 + (y-b)^2} < r\} \), the open ball centered at \((x, y)\) and radius \( r \) with respect to the Euclidean metric \( d \) on \( \mathbb{R}^2 \).

It is easy to see that the above collection of sets will form a basis for a topology on \( X \).

Let \( \mathcal{J} \) be the topology on \( X \) induced by the collection of basic open sets described as above and if \( A = \{(x, y) : x, y \in \mathbb{Q}, y \geq 0\} \) then \( A \) is a countable subcollection of \( X \) such that \( \overline{A} = X \). That is \( A \) is a countable dense subset of \( X \), and hence \( (X, \mathcal{J}) \) is a separable topological space.

Now for \( Y = \{(x, 0) : x \in \mathbb{R}\} \) (an uncountable set), \( \mathcal{J}_Y = \mathcal{P}(Y) \). That is the subspace \((Y, \mathcal{J}_Y)\) of \((X, \mathcal{J})\) is the discrete topological space. That is every subset \( U \) of \( Y \) is both open and closed in \((Y, \mathcal{J}_Y)\). Therefore if \( B \) is a countable subset of \( Y \) then \( \overline{B} = B \neq Y \) here \( \overline{B} = \overline{B}_{\mathcal{J}_Y} \), the is closure of \( B \) in \((Y, \mathcal{J}_Y)\). This proves that \((Y, \mathcal{J}_Y)\) is not a separable subspace, though \((X, \mathcal{J})\) is a separable topological space.
Note that we have already proved that every second countable topological space is separable.

**Exercise 5.1.12.** Prove that every separable metric space is second countable. □

**Exercise 5.1.13.** Prove that \( \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\} \) is a separable metric space for \( 1 \leq p < \infty \), \( d_p(x, y) = \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}} \), \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \) and \( d_\infty(x, y) = \max \{|x_i - y_i| : 1 \leq i \leq n\} \). □

It is easy to prove that if \( J_2 \) is the topology induced by \( d_2 \) then \( J_p = J_2 \), \( \forall p \geq 1 \). That is all these metrics \( d_p, 1 \leq p \leq \infty \) will induce the same topology on \( \mathbb{R}^n \). So if we want to prove that \( (\mathbb{R}^n, d_p) \) is a separable metric space, it is enough to prove that \((\mathbb{R}^n, J_1)\) (or say \((\mathbb{R}^n, J_2)\)) is separable.

For \( 1 \leq p < \infty \), let \( l_p = \{x = (x_n) : x_n \in \mathbb{R} \text{ for all } n \text{ and } \sum_{n=1}^{\infty} |x_n|^p < \infty\} \). If we define, \( d_p(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \), then \( d_p \) is a metric on \( l_p \). Now let us see how to prove that \((l_p, d_p)\) is a separable metric space.

**Step 1:** For each \( n \in \mathbb{N} \), let \( A_n = \{(r_1, r_2, \ldots, r_n, 0, 0, \ldots) : r_i \in \mathbb{Q}, i = 1, 2, \ldots, n\} \).

If we define \( f : \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q} \) (\( n \) times) = \( \mathbb{Q}^n \rightarrow A_n \) as \( f(r_1, r_2, \ldots, r_n) = (r_1, r_2, \ldots, r_n, 0, 0, \ldots) \). Then \( f \) is a bijective function. Now \( \mathbb{Q} \) is a countable set implies \( \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q} \) (finite product of countable sets is countable) is countable. Hence there is a bijection between \( \mathbb{Q}^n \) and \( A_n \) implies \( A_n \) is a countable set. Now each \( A_n \) is a countable set implies \( \bigcup_{n=1}^{\infty} A_n \) is also countable.

We leave it as an exercise to prove that \( \bigcup_{n=1}^{\infty} A_n = l_p \). That is \( \bigcup_{n=1}^{\infty} A_n \) is a countable dense subset of \( l_p \). Hence \((l_p, d_p)\) is a separable metric space. These spaces
are important examples of Banach spaces. If \( l_{\infty} = \{ x = (x_n) : (x_n) \) is a bounded real sequence \} and \( d_{\infty}(x, y) = \sup\{|x_n - y_n| : n \geq 1\} \), then \( (l_{\infty}, d_{\infty}) \) is also a metric space. Let \( X = \{ x = (x_n) : x_n = 0 \text{ or } x_n = 1 \} \). For \( x, y \in X \), \( x \neq y \), \( d(x, y) = 1 \). Hence \( (X, d_{\infty}) \) (that is \( d_{\infty} \) is restricted to the subspace \( X \) of \( l_{\infty} \)) is a metric space.

Now the topology \( J \) on \( X \) induced by the metric \( d_{\infty} \) is the discrete topology on \( X \). In this topological space \( (X, J) \) every subset \( A \) of \( X \) is both open and closed. Therefore, if \( A \) is a countable subset of \( X \), then \( \overline{A} = A \neq X \), (note that \( X \) is an uncountable subset of \( X \)). Now the subspace \( X \) of \( l_{\infty} \) is not a separable space implies \( l_{\infty} \) is not a separable space.

5.2 Properties of First Countable Topological Spaces

**Theorem 5.2.1.** If \( (X, J) \) is a first countable topological space then for each \( x \in X \) there exists a countable local base say \( \{ V_n(x) \}_{n=1}^{\infty} \) such that \( V_{n+1}(x) \subseteq V_n(x) \).

**Proof.** Fix \( x \in X \). Now \( (X, J) \) is a first countable topological space implies there exists a countable local base say \( \{ U_n \}_{n=1}^{\infty} \) at \( x \). Let \( V_n(x) = U_1 \cap U_2 \cap \cdots \cap U_n \), then \( \{ V_n(x) \}_{n=1}^{\infty} \) is a collection of open sets such that \( V_{n+1}(x) \subseteq V_n(x) \) for all \( n \in \mathbb{N} \). So, it is enough to prove that \( \{ V_n(x) \}_{n=1}^{\infty} \) is a local base at \( x \). So start with an open set \( V \) containing \( x \). Now \( \{ U_n \}_{n=1}^{\infty} \) is a local base at \( x \) and \( V \) is an open set containing \( x \) implies there exists \( n_0 \in \mathbb{N} \) such that \( U_{n_0} \subseteq V \). By the definition of \( V_n(x) \)'s we have \( V_{n_0}(x) \subseteq U_{n_0} \). Hence we have the following: for each open set \( V \) containing \( x \) there exists \( n_0 \in \mathbb{N} \) such that \( V_{n_0}(x) \subseteq V \). This implies that \( \{ V_n(x) \} \) is a local base at \( x \) satisfying \( V_{n+1}(x) \subseteq V_n(x) \) for all \( n \in \mathbb{N} \). ■

Let us use the above characterization of a first countable base to show that, in some sense, first countable topological spaces behave like metric spaces.
Theorem 5.2.2. Let \((X, \mathcal{J})\) be a first countable topological space and \(A\) be a nonempty subset of \(X\). Then for each \(x \in X\), \(x \in \overline{A}\) if and only if there exists a sequence \(\{x_n\}_{n=1}^{\infty}\) in \(A\) such that \(x_n \to x\) as \(n \to \infty\).

Proof. First let us assume that \(x \in \overline{A}\). Now \((X, \mathcal{J})\) is a first countable topological space implies there exists a countable local base say \(\mathcal{B} = \{V_n\}_{n=1}^{\infty}\) such that \(V_{n+1} \subseteq V_n\), for all \(n \in \mathbb{N}\). Hence \(x \in \overline{A}\) implies \(A \cap V_n \neq \emptyset\), for each \(n \in \mathbb{N}\). Let \(x_n \in A \cap V_n\).

Claim: \(x_n \to x\) as \(n \to \infty\).

So start with an open set \(U\) containing \(x\) (enough to start with \(V_n\)) then there exists \(n_0 \in \mathbb{N}\) such that \(x \in V_{n_0} \subseteq U\). Hence \(x_n \in V_n \subseteq V_{n_0} \subseteq U\) for all \(n \geq n_0\). That is \(x_n \in U\) for all \(n \geq n_0\). This means \(x_n \to x\) as \(n \to \infty\).

Conversely, suppose there exists a sequence \(\{x_n\}_{n=1}^{\infty}\) in \(A\) such that \(x_n \to x\). Then for each open set \(U\) containing \(x\) there exists a positive integer \(n_0\) such that \(x_n \in U\) for all \(n \geq n_0\). In particular \(x_{n_0} \in U \cap A\). Hence for each open set \(U\) containing \(x\), \(U \cap A \neq \emptyset\) and this implies \(x \in \overline{A}\).

Theorem 5.2.3. Let \(X\) and \(Y\) be topological spaces and further suppose \(X\) is a first countable topological space. Then a function \(f : X \to Y\) is continuous at a point \(x \in X\) if and only if for every sequence \(\{x_n\}_{n=1}^{\infty}\) in \(X\), \(x_n \to x\) as \(n \to \infty\), then the sequence \(\{f(x_n)\}_{n=1}^{\infty}\) converges to \(f(x)\) in \(Y\).

Proof. Assume that \(f : X \to Y\) is continuous at a point \(x \in X\). Also assume that \(\{x_n\}_{n=1}^{\infty}\) is a sequence in \(X\) such that \(x_n \to x\) as \(n \to \infty\).

To prove: \(f(x_n) \to f(x)\) in \(Y\).

So start with an open set \(V\) in \(Y\) containing \(f(x)\). Since \(f\) is continuous at \(x\), \(U = f^{-1}(V)\) is an open set in \(X\). Now \(f(x) \in V\) implies \(x \in f^{-1}(V) = U\). That is
$U$ is an open set containing $x$. Hence $x_n \to x$ implies there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. This implies $f(x_n) \in V$ for all $n \geq n_0$. That is, whenever $x_n \to x$ as $n \to \infty$ then $f(x_n) \to f(x)$ as $n \to \infty$.

Conversely, suppose that $\{x_n\}$ is a sequence in $X$, $x_n \to x$ as $n \to \infty$ implies $f(x_n) \to f(x)$. Now we will have to prove that $f$ is continuous at $x$. It is to be noted that to prove this converse part we will make use of the fact that $X$ is a first countable space. Now $X$ is a first countable space implies there exists a local base $\{V_n\}_{n=1}^{\infty}$ at $x$ such that $V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}$. We will use the method of proof by contradiction. If $f$ is not continuous at $x$ then there should exist an open set $W$ containing $f(x)$ such that $f(U) \not\subseteq W$ for any open set $U$ containing $x$. In particular for such an open set $W$, $f(V_n) \not\subseteq W$ for all $n = 1, 2, 3, \ldots$. Hence there exists $x_n \in V_n$ such that $f(x_n) \not\in W$.

Claim: $x_n \to x$ as $n \to \infty$. So start with an open set $V$ in $X$ containing $x$.

Now $\{V_n\}_{n=1}^{\infty}$ is a local base at $x$ implies there exists $n_0 \in \mathbb{N}$ such that $V_{n_0} \subseteq V$. Hence $x_n \in V_n \subseteq V_{n_0} \subseteq V$ for all $n \geq n_0$. That is for each open set $V$ containing $x$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq n_0$. Hence $x_n \to x$ as $n \to \infty$. But this sequence $\{x_n\}$ in $X$ is such that $f(x_n) \not\in W$, where $W$ is an open set containing $f(x)$. So we have arrived at a contradiction to our assumption namely $x_n \in X$, $x_n \to x \in X$ implies $f(x_n) \to f(x)$. We arrived at this contradiction by assuming $f$ is not continuous at $x$. Therefore our assumption is wrong and hence $f$ is continuous at $x$.

\[\blacksquare\]

**Example 5.2.4.** Let $\mathcal{J}_c = \{A \subseteq \mathbb{R} : A^c$ is countable or $A^c = \mathbb{R}\}$, the co-countable topology on $\mathbb{R}$, and $X = (\mathbb{R}, \mathcal{J}_c), Y = (\mathbb{R}, \mathcal{J}_s)$, where $\mathcal{J}_s$ is the standard topology on $\mathbb{R}$. Define $f : X \to Y$ as $f(x) = x$ for all $x \in X$. Suppose $\{x_n\}$ is a sequence
in $X$ such that $\{x_n\}$ converges to $x \in X = \mathbb{R}$. Then it is easy to prove that there exists $n_0 \in \mathbb{N}$ such that $x_n = x$ for all $n \geq n_0$. (If this statement is not true then there exists a subsequence $\{x_n\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \neq x$ for all $k \in \mathbb{N}$. Then $U = \mathbb{R}\{x_n : k \in \mathbb{N}\}$ is an open set in $X$ containing $x$. Hence $\{x_n\}$ converges to $x$ in $X$ implies there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. In particular for $k \geq n_0, n_k \geq k \geq n_0$ and this implies $x_{n_k} \in U$.) So we have the following: $x_n \to x$ in $X$ implies $f(x_n) \to f(x)$ in $Y$. But the given function $f : X \to Y$ is not a continuous function (note: $f^{-1}(0,1) = (0,1)$ is not an open set in $(\mathbb{R}, J_c)$). This example does not give any contradiction to theorem 5.2.3. From this example we conclude that $X = (\mathbb{R}, J_c)$ is not a first countable topological space.

5.3 Regular and Normal Topological Spaces

**Definition 5.3.1.** A topological space $(X, J)$ is called a $T_1$ **space** if for each $x \in X$, the singleton set $\{x\}$ is a closed set in $(X, J)$.

**Definition 5.3.2.** A $T_1$-topological space $(X, J)$ is called a **regular space** if for each $x \in X$ and for each closed subset $A$ of $X$ with $x \notin A$, there exist open sets $U$, $V$ in $X$ satisfying the following:

(i) $x \in U, A \subseteq V$, (ii) $U \cap V = \phi$.

**Result 5.3.3.** Every regular topological space $(X, J)$ is a Hausdorff space.

**Proof.** Let $x, y \in X, x \neq y$. By definition every regular space is a $T_1$-space. Hence $\{y\}$ is a closed set. Also $x \neq y$ implies $x \notin A = \{y\}$. Now $\{y\}$ is a closed set which does not contain $x$. Since $(X, J)$ is a regular space, there exist open sets $U, V$ in $X$ satisfying the following:

(i) $x \in U, A = \{y\} \subseteq V$, 
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(ii) $U \cap V = \emptyset$ that is $U, V$ are open sets in $X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Hence $(X, \mathcal{J})$ is a Hausdorff topological space.

Exercise 5.3.4. Prove that every Hausdorff space is a $T_1$-space.

Example 5.3.5. Let $X$ be an infinite set and $\mathcal{J}_f$ be the cofinite topology on $X$. Then $(X, \mathcal{J}_f)$ is a $T_1$-space, but $(X, \mathcal{J}_f)$ is not a Hausdorff space. For each $x \in X$, $U = X \setminus \{x\}$ is an open set. Hence $U^c = X \setminus U = \{x\}$ is a closed set in $X$. That is for each $x \in X$, the singleton set $\{x\}$ is a closed set. Therefore $(X, \mathcal{J})$ is a $T_1$-space. Take any $x, y \in X, x \neq y$. Suppose there exist open sets $U, V$ in $X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Now $U, V$ are nonempty open subsets of the cofinite topological space $(X, \mathcal{J}_f)$ implies $U^c, V^c$ are finite sets. Hence $X = \phi^c = (U \cap V)^c = U^c \cup V^c$ is a finite set. Therefore there cannot exist any open sets $U, V$ in $(X, \mathcal{J}_f)$ satisfying $x \in U, y \in V$ and $U \cap V = \emptyset$. This means $(X, \mathcal{J}_f)$ is not a Hausdorff space.

Now let us give an example of a topological space which is Hausdorff but not regular. Take $X = \mathbb{R}$ and $\mathcal{B}_K = \{(a, b), (a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$, where $K = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Now it is easy to prove that (left as an exercise) $\mathcal{B}_K$ is a basis for a topology on $\mathbb{R}$. Let $\mathcal{J}_K$ be the topology on $\mathbb{R}$ generated by $\mathcal{B}_K$. If $\mathcal{J}$ is the usual topology on $\mathbb{R}$ then we know that $\mathcal{J}$ is generated by $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$. Since we have $\mathcal{B} \subseteq \mathcal{B}_K$ and this implies that $\mathcal{J} = \mathcal{J}_\mathcal{B} \subseteq \mathcal{J}_\mathcal{B}_K = \mathcal{J}_K$.

From this, it is clear that $(\mathbb{R}, \mathcal{J}_K)$ is a Hausdorff space. For $x, y \in \mathbb{R}, x \neq y$, $(\mathbb{R}, \mathcal{J})$ is a Hausdorff space implies there exist open sets $U$ and $V$ in $(\mathbb{R}, \mathcal{J})$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. But $\mathcal{J} \subseteq \mathcal{J}_K$. Hence $U, V \in \mathcal{J}_K$ are such that $x \in U, y \in V$ and $U \cap V = \emptyset$ and this shows that $(X, \mathcal{J}_K)$ is a Hausdorff topological space.
Is \( K = \{1, \frac{1}{2}, \frac{1}{3}, \ldots \} \) a closed set? Here \( K \) is a subset of \( \mathbb{R} \) and \( J, J_K \) are two different topologies on \( \mathbb{R} \), \( 0 \in K \) and \( 0 \notin K \) with respect to \((\mathbb{R}, J)\). Hence \( K \) is not a closed set in \((\mathbb{R}, J)\). But \( \mathbb{R} \setminus K = \bigcup_{n=1}^{\infty} A_n \), where \( A_n = (-n, n) \setminus K \) for each \( n \in \mathbb{N} \). Each \( A_n \) is an open set in \((\mathbb{R}, J_K)\) implies \( \mathbb{R} \setminus K \) is an open set in \((\mathbb{R}, J_K)\). This implies \( K \) is a closed set in \((\mathbb{R}, J_K)\). Also \( 0 \notin K \). What are the open sets containing \( K \)? If \( V \) is an open set containing \( K \), then for each \( n \in \mathbb{N} \), \( \frac{1}{n} \in V \), there exists a basic open set say \((a_n, b_n)\) such that \( \frac{1}{n} \in (a_n, b_n) \subseteq V \) \( (\frac{1}{n} \notin (a_n, b_n) \setminus K) \) and \( 0 < a_n < b_n \) implies \( K \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \subseteq V \).

Suppose \( U, V \) are open sets such that \( 0 \in U \) and \( K \subseteq V \). Since \( 0 \in U \), there exists a basic open set \( B \) such that \( 0 \in B \subseteq U \). If \( B \) is of the form \((a, b)\) then \((a, b) \cap K \neq \phi \). So \( U \cap V \neq \phi \). If \( B \) is of the form \((a, b) \setminus K \), choose \( n_0 \in \mathbb{N} \) such that \( \frac{1}{n_0} < b \). Since \( \frac{1}{n_0} \in V \), there exists an open interval \((c, d)\) such that \( \frac{1}{n_0} \in (c, d) \subseteq V \). Now since \((a, b) \cap (c, d) \) is not empty (it contains \( \frac{1}{n_0} \)), it is an interval and hence uncountable. As \( K \) is countable, \( ((a, b) \cap (c, d)) \setminus K \neq \phi \), i.e., \( ((a, b) \setminus K) \cap (c, d) \neq \phi \). Therefore \( U \cap V \neq \phi \).

So we have proved that there cannot exist open sets \( U, V \) in \((\mathbb{R}, J_K)\) with \( 0 \in U \), \( K \subseteq V \) and \( U \cap V = \phi \). This shows that \((\mathbb{R}, J_K)\) is not a regular space.

**Definition 5.3.6.** A topological space \((X, J)\) is said to be a **normal space** if and only if it satisfies:

(i) \((X, J)\) is a \( T_1 \)-space,

(ii) \( A, B \) closed sets in \( X \), \( A \cap B = \phi \) implies there exist open sets \( U, V \) in \( X \) such that \( A \subseteq U, B \subseteq V \) and \( U \cap V = \phi \).

**Remark 5.3.7.** It is to be noted that every normal space is a regular space.
Theorem 5.3.8. Every metric space \((X, d)\) is a normal space, That is if \(J_d\) is the topology induced by the metric then the topological space \((X, J_d)\) is a normal space.

**Proof.** Let \(A, B\) be disjoint closed subsets of \(X\). Then for each \(a \in A\), \(a \notin B = \overline{B}\) implies \(d(a, B) = \inf\{d(a, b) : b \in B\} > 0\). If \(r_a = d(a, B) > 0\) then \(B(a, r_a) \cap B = \emptyset\) (if there exists \(b_0 \in B\) such that \(d(b_0, a) < r_a\), then \(r_a = d(a, B) \leq d(a, b_0) < r_a\) a contradiction). Similarly for each \(b \in B\) there exists \(r_b > 0\) such that \(B(b, r_b) \cap A = \emptyset\). Let \(U = \bigcup_{a \in A} B(a, \frac{r_a}{3})\), \(V = \bigcup_{b \in B} B(a, \frac{r_b}{3})\). Now it is easy to prove that \(U \cap V = \emptyset\). Hence if \(A, B\) are disjoint closed subsets of \(X\) then there exist open sets \(U, V\) in \(X\) such that \(A \subseteq U, B \subseteq V\) and \(U \cap V = \emptyset\). This implies \((X, J_d)\) is a normal space. □

Theorem 5.3.9. A \(T_1\)-topological space \((X, J)\) is regular if and only if whenever \(x\) is a point of \(X\) and \(U\) is an open set containing \(x\) then there exists an open set \(V\) containing \(x\) such that \(V \subseteq U\).

**Proof.** Assume that \((X, J)\) is a regular topological space, \(x \in X\) and \(U\) is an open set containing \(x\). Now \(x \in U\) implies \(x \notin A = U^c = X \setminus U\), the complement of the open set \(U\). Now \(A\) is a closed set and \(x \notin A\). Hence \(X\) is a regular space implies there exist open sets \(V\) and \(W\) of \(X\) such that \(x \in V, A = U^c \subseteq W\) and \(V \cap W = \emptyset\). Now \(V \cap W = \emptyset\) implies \(V \subseteq W^c \subseteq U\) (we have \(U^c \subseteq W\)), \(V \subseteq W^c\) implies \(V \subseteq \overline{W^c} = W^c\) (\(W\) is an open set implies \(W^c\) is a closed set) implies \(V \subseteq U\). Hence for \(x \in X\) and for each open set \(U\) containing \(x\), there exists an open set \(V\) containing \(x\) such that \(V \subseteq U\).

Now let us assume that the above statement is satisfied. Our aim here is to prove that \((X, J)\) is a regular space. So take a closed set \(A\) of \(X\) and a point \(x \in X \setminus A\). Now \(A\) is a closed subset of \(X\) implies \(U = X \setminus A\) is an open set containing \(x\). Hence
by our assumption there exists an open set \( V \) containing \( x \) such that \( V \subseteq U = A^c \).
Now \( V \subseteq A^c \) implies \( A \subseteq (V)^c = X \setminus V \). So \( V \) and \( (V)^c = W \) are open sets satisfying \( x \in V, A \subseteq W \) and \( V \cap W = V \cap (V)^c \subseteq V \cap V^c = \phi \). \((V) \subseteq \bar{V} \) implies \((V)^c \subseteq V^c.\)

Hence by definition \((X, J)\) is a regular space. \(\blacksquare\)

In a similar way we prove the following theorem.

**Theorem 5.3.10.** A \( T_1 \)-topological space is a normal space if and only if whenever \( A \) is a closed subset of \( X \) and \( U \) is an open set containing \( A \), then there exists an open set \( V \) containing \( A \) such that \( V \subseteq U \).

**Proof.** Assume that \((X, J)\) is a normal topological space. Now take a closed set \( A \) and an open set \( U \) in \( X \) such that \( A \subseteq U \). Now \( A \subseteq U \) implies \( U^c \subseteq A^c \). Here \( A, U^c = B \) are closed sets such that \( A \cap B = A \cap U^c \subseteq U \cap U^c = \phi \). That is \( A, B \) are disjoint closed subsets of the normal space \((X, J)\). Hence there exist open sets \( U, W \) in \( X \) such that \( A \subseteq V, B = U^c \subseteq W \) and \( V \cap W = \phi \). Further \( \bar{V} \subseteq W^c \) (note: \( V \subseteq W^c \) implies \( \bar{V} \subseteq \bar{W}^c = W^c \)). Now \( \bar{V} \subseteq W^c \subseteq U \). Hence whenever \( A \) is a closed set and \( U \) is an open set containing \( A \) then there exists an open set \( V \) such that \( A \subseteq V, \bar{V} \subseteq U \).
Now let us assume that the above statement is satisfied. So our aim is to prove that \((X, J)\) is a normal space. So start with disjoint closed subsets say \( A, B \) of \( X \). Now \( A \cap B = \phi \) implies \( A \subseteq B^c = U \). That is \( U \) is an open set containing the closed set \( A \). Hence by our assumption there exists an open set \( V \) such that \( A \subseteq V, \bar{V} \subseteq U \). Now \( \bar{V} \subseteq U \) implies \( U^c \subseteq (\bar{V})^c \) implies \( B \subseteq (\bar{V})^c \). Further \( V \cap (\bar{V})^c \subseteq V \cap V^c = \phi \). That is whenever \( A, B \) are closed subsets of \( X \), then there exist open sets \( V \) and \((\bar{V})^c = W \) such that \( A \subseteq V, B \subseteq W \) and \( V \cap W = \phi \). Therefore by definition \((X, J)\) is a normal space. \(\blacksquare\)
Example of a topological space which is regular but not normal.

Let $\mathcal{J}_l$ be a lower limit topology on $\mathbb{R}$. That is $\mathbb{R}_l = (\mathbb{R}, \mathcal{J}_l)$. Now let us prove that the product space $\mathbb{R}_l \times \mathbb{R}_l$ is a regular space. (If $X, Y$ are regular topological spaces then the product space $X \times Y$ is a regular space. Hence it is enough to prove that $\mathbb{R}_l$ is a regular space.) For $(x, y) \in \mathbb{R}^2$, each basic open set $U$ of the form $U = [x, a) \times [y, b)$ is both open and closed. Hence for each basic neighbourhood $U$ of $(x, y)$ in $\mathbb{R}_l \times \mathbb{R}_l$ there exists a neighbourhood $V = U$ of $(x, y)$ such that $\overline{V} = \overline{U} \subseteq U$. Now if $U'$ is any open set containing $(x, y)$ then there exists a basic open set $U$ as given above such that $(x, y) \in U = [x, a) \times [y, b) \subseteq U'$. Therefore $V = U$ is an open set containing $(x, y)$ and $\overline{V} = \overline{U} = U \subseteq U'$. Also $\mathbb{R}_l \times \mathbb{R}_l$ is a Hausdorff space. Hence $\mathbb{R}_l \times \mathbb{R}_l$ is a regular space. Now let us take $Y = \{(x, y) \in \mathbb{R}^2 : y = -x \}$ then for each $(x, y) \in Y$ there exists $a, b \in \mathbb{R}, x < a, y < b$ such that $([x, a) \times [y, b)) \cap Y = \{(x, y)\}$. Hence each singleton $\{(x, y)\}$ is open in the subspace $Y$ of $\mathbb{R}_l \times \mathbb{R}_l$. This proves that the subspace $Y$ of $\mathbb{R}_l \times \mathbb{R}_l$ is discrete. Also $Y$ is a closed subset of $\mathbb{R}_l \times \mathbb{R}_l$. Let $A = \{(x, y) \in \mathbb{R}^2 : y = -x \in \mathbb{Q}\}, B = \{(x, y) \in \mathbb{R}^2 : y = -x \in \mathbb{Q}^c\}$. Now $A, B$ are closed sets in $Y$ and $Y$ is a closed set in $\mathbb{R}_l \times \mathbb{R}_l$ implies $A, B$ are closed in $\mathbb{R}_l \times \mathbb{R}_l$. Also $A \cap B = \emptyset$. Suppose there exist open sets $U, V$ in $\mathbb{R}_l \times \mathbb{R}_l$ satisfying $A \subseteq U, B \subseteq V$. Then we can observe that $U \cap V \neq \emptyset$. Therefore the product space $\mathbb{R}_l \times \mathbb{R}_l$ is not a normal space.

**Remark 5.3.11.** We can prove that $(\mathbb{R}, \mathcal{J}_l) = \mathbb{R}_l$ is a normal space. So, $\mathbb{R}_l \times \mathbb{R}_l$ is a regular space but it is not a normal space.

We have already proved that every compact subset of Hausdorff topological space is closed. Essentially we use the same proof technique used there to prove the following theorem:
Theorem 5.3.12. Every compact Hausdorff topological space \((X, \mathcal{J})\) is a regular space.

**Proof.** Let \(A\) be a closed subset of \(X\) and \(x \in X \setminus A\), then for each \(y \in A, x \neq y\). Hence \(X\) is a Hausdorff space implies that there exist open sets \(U_y, V_y\) in \(X\) satisfying \(x \in U_y, y \in V_y\) and \(U_y \cap V_y = \emptyset\). We know that closed subset of a compact space is compact. Here \(A \subseteq \bigcup_{y \in A} V_y\), that is \(\{V_y : y \in A\}\) is an open cover for the compact space \(A\). Therefore there exists \(n \in \mathbb{N}\) and \(y_1, y_2, \ldots, y_n \in A\) such that \(A \subseteq \bigcup_{i=1}^n V_{y_i}\). Let \(U = \bigcap_{i=1}^n U_{y_i}\) and \(V = \bigcup_{i=1}^n V_{y_i}\). Then \(U, V\) are open sets in \(X\) satisfying \(x \in U, A \subseteq V\) and \(U \cap V \subseteq U \cap (V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}) = (U \cap V_{y_1}) \cup (U \cap V_{y_2}) \cup \cdots \cup (U \cap V_{y_n}) \subseteq (U_{y_1} \cap V_{y_1}) \cup (U_{y_2} \cap V_{y_2}) \cup \cdots \cup (U_{y_n} \cap V_{y_n}) = \emptyset\). Hence by definition \((X, \mathcal{J})\) is a regular space. 

Now let us prove that every compact Hausdorff space is a normal space.

Theorem 5.3.13. Every compact Hausdorff space \((X, \mathcal{J})\) is a normal space.

**Proof.** Let \(A, B\) be disjoint closed sets in \(X\). Then for each \(x \in A, x \not\in B\). Now \((X, \mathcal{J})\) is a regular space implies there exist open sets \(U_x, V_x\) satisfying: \(x \in U_x; B \subseteq V_x\) and \(U_x \cap V_x = \emptyset\). Now \(\{U_x : x \in A\}\) is an open cover for \(A\) implies there exists \(n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in A\) such that \(A \subseteq \bigcup_{i=1}^n U_{x_i}\). Let \(U = U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}\) and \(V = V_{x_1} \cap V_{x_2} \cap \cdots \cap V_{x_n}\). Then \(U, V\) are open sets in \(X\) satisfying \(A \subseteq U, B \subseteq V\) and \(U \cap V = \emptyset\). Hence by definition \((X, \mathcal{J})\) is a normal space. 

Theorem 5.3.14. Closed subspace of a normal topological space \((X, \mathcal{J})\) is normal.

**Proof.** Let \(Y\) be a closed subspace of \((X, \mathcal{J})\). That is \(Y\) is a closed subset of \((X, \mathcal{J})\) and \(\mathcal{J}_Y = \{A \cap Y : A \in \mathcal{J}\}\) is a topology on \(Y\). So we will have to prove that \((Y, \mathcal{J}_Y)\) is a normal space. To prove this, take a closed set \(A \subseteq Y\) and an open set \(U\) in \((Y, \mathcal{J}_Y)\). 128
such that $A \subseteq U$. Now $U$ is an open set in $(Y, \mathcal{J}_Y)$ implies there exists $V \in \mathcal{J}$ such that $U = V \cap Y$. Also $A$ is a closed set in the subspace implies $A = \overline{A}_Y = \overline{A} \cap Y$ (here $\overline{A}_Y$ denotes the closure of $A$ in $(Y, \mathcal{J}_Y)$ and $\overline{A}$ denotes the closure of $A$ in $(X, \mathcal{J})$).

Now $\overline{A}, Y$ are closed sets in $X$ implies $\overline{A} \cap Y$ is also a closed set in $X$. Hence $A$ is a closed set in $(X, \mathcal{J})$ and $V$ is an open set in $(X, \mathcal{J})$ containing $A$ and $(X, \mathcal{J})$ is a normal topological space implies there exists an open set $W$ in $(X, \mathcal{J})$ such that $A \subseteq W$ and $\overline{W} \subseteq V$. Now $W \cap Y$ is an open set in $(Y, \mathcal{J}_Y)$ and $A \subseteq W \cap Y$ and $\overline{W \cap Y} \subseteq \overline{W} \subseteq V \cap Y \subseteq U$. We started with a closed set $A$ in $(Y, \mathcal{J}_Y)$ and an open set $U$ in $(Y, \mathcal{J}_Y)$ such that $A \subseteq U$. Now we have proved that there exists an open set $W \cap Y$ in $(Y, \mathcal{J}_Y)$ satisfying $A \subseteq W \cap Y$ and $(\overline{W \cap Y})_Y = \overline{W} \cap Y \cap Y = \overline{W \cap Y} \subseteq U$. That is $W \cap Y$ is an open set in the subspace containing $A$ and closure of this open set with respect to the subspace $(Y, \mathcal{J}_Y)$ is contained in $U$. Hence $(Y, \mathcal{J}_Y)$ is a normal space. $\blacksquare$

### 5.4 Urysohn Lemma

Now let us prove the following important theorem known as Urysohn lemma.

**Theorem 5.4.1.** Let $(X, \mathcal{J})$ be a normal space and $A$, $B$ be disjoint nonempty closed subsets of $X$. Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for every $x$ in $A$, and $f(x) = 1$ for every $x$ in $B$.

**Proof.** $A \cap B = \phi$ implies $A \subseteq B^c = X \setminus B$. Hence $B^c$ is an open set containing the closed set $A$. Now $X$ is a normal space implies there exists an open set $U_0$ such that $A \subseteq U_0$ and $\overline{U}_0 \subseteq B^c = U_1$. Now $[0, 1] \cap \mathbb{Q}$ is a countable set implies there exists a bijective function say $f : \mathbb{N} \rightarrow [0, 1] \cap \mathbb{Q}$ satisfying $f(1) = 1, f(2) = 0$ and $f(\mathbb{N} \setminus \{1, 2\}) = (0, 1) \cap \mathbb{Q}$. That is $[0, 1] \cap \mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ such that $r_1 = 1, r_2 = 0$ and $r_n = f(n)$ for $n \geq 3$. Hence $f$ is a continuous function.
and \( f(k) = r_k \) for \( k \geq 3 \).

**Aim:** To define a collection \( \{U_p\}_{p \in [0,1] \cap \mathbb{Q}} \) of open sets such that for \( p, q \in [0, 1] \cap \mathbb{Q} \), \( p < q \) implies \( \overline{U_p} \subseteq U_q \).

Let \( P_n = \{r_1, r_2, \ldots, r_n\} \). Assume that \( U_p \) is defined for all \( p \in P_n \), where \( n \geq 2 \) and this collection satisfies the property namely \( p, q \in [0, 1] \cap \mathbb{Q}, p < q \) implies \( \overline{U_p} \subseteq U_q \).

Note that this result is true when \( n = 2 \). Now let us prove this result for \( P_{n+1} \). Here \( P_{n+1} = P_n \cup \{r_{n+1}\} \).

Let \( p, q \in P_{n+1} \) be such that \( p = \max\{r \in P_{n+1} : r < r_{n+1}\} \) and \( q = \min\{r \in P_{n+1} : r > r_{n+1}\} \). Now \( p, q \neq r_{n+1} \) implies \( p, q \in P_n \). By our assumption \( U_p, U_q \) are known and \( \overline{U_p} \subseteq U_q \). Now \( U_q \) is an open set containing the closed set \( \overline{U_p} \) and \( X \) is a normal space. Hence there exists an open set say \( U_{r_{n+1}} \) such that \( \overline{U_p} \subseteq U_{r_{n+1}} \) and \( U_{r_{n+1}} \subseteq U_q \).

If \( r, s \in P_n \), then we are through.

Suppose \( r \in P_n \) and \( s = r_{n+1} \) then \( r \leq p \) or \( r \geq q \). If \( r \leq p \), \( \overline{U_r} \subseteq U_p \subseteq \overline{U_p} \subseteq U_s \). If \( r \geq q \), \( U_s \subseteq U_q \subseteq \overline{U_q} \subseteq \overline{U_p} \) and therefore by induction \( U_p \) is defined for all \( p \in [0, 1] \cap \mathbb{Q} \) and \( p, q \in [0, 1] \cap \mathbb{Q}, p < q \) implies \( \overline{U_p} \subseteq U_q \).

Now define \( U_p = \phi \), if \( p \in \mathbb{Q} \), \( p < 0 \) and \( U_p = X \) if \( p \in \mathbb{Q}, p > 1 \). Then \( p, q \in \mathbb{Q}, p < q \) implies \( \overline{U_p} \subseteq U_q \). Define \( f : X \to [0, 1] \) as \( f(x) = \inf\{p \in \mathbb{Q} : x \in U_p\} \). Now \( x \in A \), then \( x \in U_0 \). Hence \( x \in U_p \) for all \( p \geq 0 \). In this case \( \{p \in \mathbb{Q} : x \in U_p\} = [0, \infty) \cap \mathbb{Q} \). Hence \( \inf\{p \in \mathbb{Q} : x \in U_p\} = 0 \).
That is \( x \in A \) implies \( f(x) = 0 \). Now suppose \( x \in B = U_1^c \) then \( x \notin U_p \) for all \( p \leq 1 \). Hence \( \{ p \in \mathbb{Q} : x \in U_p \} = [1, \infty) \cap \mathbb{Q} \) implies \( f(x) = 1 \) for all \( x \in B \).

Now let us prove that \( f \) is a continuous function. \( S = \{ [0, a), (a, 1] : 0 < a < 1 \} \) is a subbase for \([0, 1]\). Hence it is enough to prove that for each \( a \), \( 0 < a < 1 \), \( f^{-1}([0, a)) \) and \( f^{-1}((a, 1]) \) are open sets in \( X \). For \( 0 < a < 1 \), let us prove that \( f^{-1}([0, a)) = \{ x \in X : 0 \leq f(x) < a \} = \bigcup_{p \leq a} U_p \). Now \( x \in f^{-1}([0, a)) \) implies \( f(x) < a \) implies there exists a rational number \( p \) such that \( f(x) < p < a \). By the definition of \( f(x) \), \( x \in U_p \). Hence

\[
 f^{-1}([0, a)) \subseteq \bigcup_{p \leq a} U_p. \tag{5.2}
\]

Now let \( x \in U_p \) for \( p < a \) implies \( f(x) \leq p \) implies \( x \in f^{-1}([0, a)) \). Hence we have

\[
 \bigcup_{p \leq a} U_p \subseteq f^{-1}([0, a)). \tag{5.3}
\]

From Eqs. (5.2) and (5.3) we have \( f^{-1}([0, a)) = \bigcup_{p \leq a} U_p \). Now each \( U_p \) is an open set implies that \( \bigcup_{p \leq a} U_p \) is an open set in \( X \). In a similar way we can prove that \( f^{-1}((a, 1]) \) is also an open subset of \( X \) for each \( 0 < a < 1 \). Now \( f : X \to [0, 1] \) such that inverse image of each subbasic open set is an open set implies that \( f : X \to [0, 1] \) is a continuous function.

\textbf{Theorem 5.4.2.} Let \((X, \mathcal{J})\) be a normal space and \( A, B \) be disjoint nonempty closed subsets of \( X \). Then for \( a, b \in \mathbb{R}, a < b \) there exists a continuous function \( f : X \to [a, b] \) such that \( f(x) = a \) for every \( x \) in \( A \), and \( f(x) = b \) for every \( x \) in \( B \).

\textbf{Proof.} Define \( g : [0, 1] \to [a, b] \) as \( g(t) = a + (b - a)t \) then \( g \) is continuous. Now by theorem 5.4.1 there is a continuous function \( f_1 : X \to [0, 1] \) such that \( f_1(x) = 0 \), for all \( x \in A \) and \( f_1(x) = 1 \) for all \( x \in B \). The function \( f = g \circ f_1 : X \to [a, b] \)
is a continuous function and further \( f(x) = g(f_1(x)) = g(0) = a \) for all \( x \in A \) and \( f(x) = g(f_1(x)) = g(1) = b \) for all \( x \in B \). ■

**Remark 5.4.3.** Let \( A, B \) be nonempty disjoint closed subsets of a metric space \((X, d)\). Define \( f : X \to \mathbb{R} \) as \( f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)} \). Observe that \( f \) is a continuous function satisfying the condition that \( f(x) = 0 \) for all \( x \in A \) and \( f(x) = 1 \) for all \( x \in B \). It shows that the proof of Urysohn lemma is trivial (or say simple) if our topological space is a metrizable topological space. ◆

**5.5 Tietze Extension Theorem**

**Theorem 5.5.1.** Tietze Extension Theorem. *Let \( A \) be a nonempty closed subset of a normal space \( X \) and let \( f : A \to [-1, 1] \) be a continuous function. Then there exists a continuous function \( g : X \to [-1, 1] \) such that \( g(x) = f(x) \) for all \( x \) in \( A \).*

**Proof.** The sets \([-1, \frac{-2}{3}]\), \([\frac{1}{3}, 1]\) are closed subsets of \([-1, 1]\) and \( f : A \to [-1, 1] \) is a continuous function implies \( A_1 = f^{-1}(\left[\frac{1}{3}, 1\right]) \), \( B_1 = f^{-1}(\left[-1, \frac{-1}{3}\right]) \) are closed subsets of the subspace \( A \). (Here consider \( A \) as a subspace of \( X \).) Now \( x \in A_1 \cap B_1 \) implies \( f(x) \in [-1, \frac{-1}{3}] \cap \left[\frac{1}{3}, 1\right] \) a contradiction. Hence \( A_1 \cap B_1 = \emptyset \). Now \( A_1, B_1 \) are closed in \( A \) and \( A \) is closed in \( X \) implies \( A_1, B_1 \) are closed in the normal space \( X \). Hence by Urysohn’s lemma there exists a continuous function \( f_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right] \) such that \( f_1(A_1) = \frac{1}{3} \) and \( f_1(B_1) = -\frac{1}{3} \) then \( |f(x) - f_1(x)| \leq \frac{2}{3} \) for all \( x \in A \). Now consider the function \( f - f_1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right] \) then \( A_2 = (f - f_1)^{-1}(\left[\frac{2}{3}, \frac{2}{3}\right]) \) and \( B_2 = (f - f)^{-1}(\left[\frac{-2}{3}, \frac{-2}{3}\right]) \) are disjoint closed subsets of \( X \). By Urysohn lemma there exists a continuous function \( f_2 : X \to \left[-\frac{2}{3}, \frac{2}{3}\right] \) such that \( f_2(A_2) = \frac{2}{3} \) and \( f_2(B_2) = -\frac{2}{3} \). Also \( |f(x) - (f_1(x) + f_2(x))| \leq \frac{4}{9} \) for all \( x \in A \). By proceeding as above by induction
for each \( n \in \mathbb{N} \) there exists a continuous function \( f_n : X \to \left[ -\frac{2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n} \right] \) such that
\[
|f(x) - \sum_{i=1}^{n} f_i(x)| \leq \left( \frac{2}{3} \right)^n \quad \text{for all } x \in A. \tag{5.4}
\]
That is \( f_n : X \to [-1, 1] \) is a sequence of continuous functions such that \( |f_n(x)| \leq \frac{2^{n-1}}{3^n} = M_n \) and \( \sum_{n=1}^{\infty} M_n < \infty \). By Weierstrass M-test, the series \( \sum_{n=1}^{\infty} f_n(x) \) converges uniformly on \( X \). That is, if \( s_n(x) = \sum_{i=1}^{n} f_i(x), \ x \in X \) then \( s_n(x) \) converges uniformly on \( X \). Also each \( s_n : X \to \mathbb{R} \) is continuous. We know, from analysis, if a sequence \( s_n : X \to \mathbb{R} \) of continuous functions converges uniformly to a function \( g : X \to \mathbb{R} \) then \( g \) is also a continuous function. Hence \( g : X \to \mathbb{R} \) be defined as \( g(x) = \sum_{n=1}^{\infty} f_n(x) \) is continuous. Now for each \( x \in A \),
\[
|g(x) - f(x)| = \left| \lim_{n \to \infty} \sum_{i=1}^{n} f_i(x) - f(x) \right| = \lim_{n \to \infty} \left| \sum_{i=1}^{n} f_i(x) - f(x) \right| \leq \lim_{n \to \infty} \left( \frac{2}{3} \right)^n = 0. \]
This implies \( g(x) = f(x) \) for all \( x \in A \).

**Definition 5.5.2.** A topological space \((X, \mathcal{J})\) is said to be **completely regular** if (i) for each \( x \in X \), singleton \( \{x\} \) is closed in \((X, \mathcal{J})\) (that is \((X, \mathcal{J})\) is a \( T_1 \)-space), (ii) for \( x \in X \) and any nonempty closed set \( A \) with \( x \notin A \) there exists a continuous function \( f : X \to [0, 1] \) such that \( f(x) = 0 \) and \( f(y) = 1 \) for all \( y \in A \).

**Result 5.5.3.** Every normal space \((X, \mathcal{J})\) is completely regular.

**Proof.** Let \( x \in X \) and \( A \) be a nonempty closed set with \( x \notin A \). Now \( \{x\}, A \) are disjoint closed sets. Hence by Urysohn’s lemma there exists a continuous function \( f : X \to [0, 1] \) such that \( f(x) = 0 \) and \( f(y) = 1 \) for all \( y \in A \). 

**Result 5.5.4.** If \( Y \) is a subspace of a completely regular space \((X, \mathcal{J})\) then \((Y, \mathcal{J}_Y)\) is also a completely regular space.
Proof. Let \( y \in Y \) and \( A \) be a closed set in \((Y, \mathcal{J}_Y)\) with \( y \notin A \). Since \( A \) is a closed set in \( Y \) there exists a closed set \( F \) in \((X, \mathcal{J})\) such that \( A = F \cap Y, y \notin F, F \) is a closed set in the completely regular space \((X, \mathcal{J})\) implies there exists a continuous function \( f : X \to [0, 1] \) such that \( f(y) = 0 \) and \( f(a) = 1 \) for all \( a \in F \). Now \( f : X \to [0, 1] \) is a continuous function implies \( f|Y = g : (Y, \mathcal{J}_Y) \to [0, 1] \) (here \( g(x) = (f|Y)(x) = f(x) \) for all \( x \in Y \) is a continuous function. Now \( g : (Y, \mathcal{J}_Y) \to [0, 1] \) is a continuous function such that \( g(y) = f(y) = 0 \) and \( g(a) = f(a) = 1 \) for all \( a \in A = F \cap Y \). Also subspace of a \( T_1 \)-space (do it as an exercise) is \( T_1 \)-space. Hence the subspace \((Y, \mathcal{J}_Y)\) is a completely regular space. \( \blacksquare \)

5.6 Baire Category Theorem

Baire category theorem has many applications in topology and analysis. Our aim here is to state and prove this theorem for locally compact Hausdorff topological spaces.

Definition 5.6.1. A subset \( A \) of a topological space \((X, \mathcal{J})\) is said to be **nowhere dense in** \( X \) if and only if \((\overline{A})^\circ = \text{int}(A) = \phi\).

Example 5.6.2. (i) \( \mathbb{N} \) is nowhere dense in \( \mathbb{R} \) (\( \mathbb{R} \) with standard topology).

(ii) \( \mathbb{Q} \) is dense in \( \mathbb{R} \), that is \( \overline{\mathbb{Q}} = \mathbb{R} \), and hence \( \mathbb{Q} \) is not nowhere dense in \( \mathbb{R} \). Here we have \((\overline{\mathbb{Q}})^\circ = \mathbb{R}^\circ = \mathbb{R} \neq \phi\).

Definition 5.6.3. A topological space \((X, \mathcal{J})\) is said to be of **first category** if and only if there exists a countable collection \( \{E_n\}_{n=1}^\infty \) of subsets of \( X \) satisfying:

(i) for each \( n \in \mathbb{N} \), \((\overline{E_n})^\circ = \overline{E_n}^\circ = \phi\), and

(ii) \( x = \bigcup_{n=1}^\infty E_n \).
Definition 5.6.4. A nonempty subset $Y$ of a topological space $(X, \mathcal{J})$ is said to be of \textbf{first category in} $X$ if and only if there exists a countable collection $\{E_n\}_{n=1}^{\infty}$ of subsets of $X$ satisfying:

(i) for each $n \in \mathbb{N}$, $(\overline{E_n})^\circ = \phi$, and

(ii) $Y = \bigcup_{n=1}^{\infty} E_n$.

Remark 5.6.5. If $Y$ is a nonempty subset of a topological space $(X, \mathcal{J})$ then $(Y, \mathcal{J}_Y)$ ($\mathcal{J}_Y = \{U \cap Y : U \in \mathcal{J}\}$) is also a topological space. It is possible that a subset $Y$ of a topological space $(X, \mathcal{J})$ is of first category in $(X, \mathcal{J})$ but the subspace $(Y, \mathcal{J}_Y)$ is not of first category.

Example 5.6.6. Let $X = \mathbb{R}$ and $\mathcal{J}_s$ be the standard topology on $\mathbb{R}$. Then $Y = \mathbb{N}$, the set of all natural numbers, is of first category in $\mathbb{R}$, but the subspace $(\mathbb{N}, \mathcal{J}_s/\mathbb{N})$ is not of first category. For each $n \in \mathbb{N}$ let $E_n = \{n\}$. As $E_n$ contains only one element namely $n$, $(\overline{E_n})^\circ = \{n\}^\circ = \phi$ in $\mathbb{R}$. Also $\mathbb{N} = \bigcup_{n=1}^{\infty} \{n\} = \{1, 2, \ldots\} = \bigcup_{n=1}^{\infty} E_n$. Hence $\mathbb{N}$ is of first category in $(\mathbb{R}, \mathcal{J}_s)$. But note that the subspace $(\mathbb{N}, \mathcal{J}_s/\mathbb{N})$ is the discrete topological space on $\mathbb{N}$. For $n \in \mathbb{N}$, $(n - 1, n + 1)$ is an open set in $\mathbb{R}$ and hence $(n - 1, n + 1) \cap \mathbb{N} = \{n\}$ is an open set in the subspace $(\mathbb{N}, \mathcal{J}_s/\mathbb{N})$. Now it is easy to see that there cannot exist any countable collection $\{A_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}$ satisfying $(\overline{A_n})^\circ = \phi$ and $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$. Note that for $A_n \subseteq \mathbb{N}$, $(\overline{A_n})^\circ = A_n$ with respect to $(\mathbb{N}, \mathcal{J}_s/\mathbb{N})$ and hence $(\mathbb{N}, \mathcal{J}_s/\mathbb{N})$ is not of first category.

Definition 5.6.7. If a topological space $(X, \mathcal{J})$ is \textbf{not of first category} then we say that the topological space $(X, \mathcal{J})$ is of \textbf{second category}.

Note. We have seen that $\mathbb{N}$ is of first category in $(\mathbb{R}, \mathcal{J}_s)$ but the topological space $(\mathbb{N}, \mathcal{J}_s/\mathbb{N})$ is of second category.
Now our main aim is to prove that every locally compact Hausdorff topological space is of second category.

**Remark 5.6.8.** First let us prove that every locally compact Hausdorff topological space \((X, \mathcal{J})\) is a regular space. So let us take a closed set \(A\) in \((X, \mathcal{J})\) and a point \(x \in X \backslash A\).

We have seen that every compact Hausdorff space is a normal space and hence every compact Hausdorff space is a regular space. We know that the one point of compactification \((X^*, \mathcal{J}^*)\) of \((X, \mathcal{J})\) is a compact Hausdorff space and \(\mathcal{J}^* \upharpoonright x = \mathcal{J}\). That is \((X, \mathcal{J})\) is a subspace of the compact Hausdorff space \((X^*, \mathcal{J}^*)\). Also it is easy to prove that subspace of a regular space is regular (and it is to be noted that subspace of a normal space need not be a normal space) and hence \((X^*, \mathcal{J}^*)\) is a regular space implies the subspace \((X, \mathcal{J})\) of \((X^*, \mathcal{J}^*)\) is also a regular space.

Now we are in a position to state and prove the main theorem.

**Theorem 5.6.9. Baire Category Theorem.** Let \((X, \mathcal{J})\) be a locally compact Hausdorff topological space and \(\{E_n\}_{n=1}^{\infty}\) be a countable collection of open sets in \((X, \mathcal{J})\). Further suppose for each \(n \in \mathbb{N}\), \(\overline{E_n} = X\) (\(E_n\) is dense in \(X\)) then \(\bigcap_{n=1}^{\infty} E_n\) is also dense in \(X\). That is \(\left(\bigcap_{n=1}^{\infty} E_n\right) = X\).

**Proof.** We want to prove that \(\bigcap_{n=1}^{\infty} E_n\) is dense in \(X\).

So take \(x \in X\) and an open set \(U\) containing \(x\). Now \((X, \mathcal{J})\) is a locally compact Hausdorff space implies there exists an open set \(V\) containing \(x\) such that \(V\) is compact. Now let \(U_0 = U \cap V\). Then \(U_0\) is an open set containing \(x\). Also \(\overline{U_0} \subseteq V\) implies \(\overline{U_0}\) is a compact set (since closed subset of compact set is compact). Now our aim is to prove that \(U \cap \left(\bigcap_{n=1}^{\infty} E_n\right) \neq \phi\). For each \(n\), \(E_n\) is open and \(\overline{E_n} = X\), that is each \(E_n\) is open and dense in \(X\). Start with \(n = 1\), now \(x \in X = \overline{E_1}\) and \(U_0\) is an open set containing \(x\)
set containing \( x \) implies \( U_0 \cap E_1 \neq \phi \). So take an element say \( x_1 \in U_0 \cap E_1 \). Now \( U_0 \), \( E_1 \) are open sets implies \( U_0 \cap E_1 \) is also an open set. Now \( U_0 \cap E_1 \) is an open set containing \( x_1 \) and \((X, \mathcal{J})\) is a regular space (every locally compact Hausdorff space is a regular space) implies there exists an open set \( U_1 \) in \( X \) satisfying \( x_1 \in U_1, \overline{U_1} \subseteq U_0 \cap E_1 \). Now \( x_1 \in \overline{E_2} = X \) implies \( U_1 \cap E_2 \neq \phi \). Let \( x_2 \in U_1 \cap E_2 \). Since \( X \) is a regular space implies there exists an open set \( U_2 \) in \( X \) satisfying \( x_2 \in U_2, \overline{U_2} \subseteq U_1 \cap E_2 \). Again \( x_2 \in \overline{E_3} = X \) and \( U_2 \) is an open set containing \( x_2 \) implies \( U_2 \cap E_3 \neq \phi \). Let \( x_3 \in U_2 \cap E_3 \). Choose an open set \( U_3 \) such that \( x_3 \in U_3, U_3 \subseteq U_2 \cap E_3 \). Continuing in this way (that is using induction) we get a sequence \( \{x_n\}_{n=1}^{\infty} \) in \( X \) and a sequence of open sets \( \{U_n\}_{n=1}^{\infty} \) satisfying \( x_n \in U_n, \overline{U_n} \subseteq U_{n-1} \cap E_n \) for all \( n \in \mathbb{N} \). Note that \( \overline{U_n} \subseteq U \cap \bigcap_{k=1}^{n} E_k \) for all \( n \in \mathbb{N} \). Then \( \{\overline{U_k}\}_{k=1}^{\infty} \) is a sequence of nonempty closed subsets of \( X \) and hence of the compact subspace \( \overline{U}_0 \). Further \( \overline{U}_{k+1} \subseteq \overline{U}_k \) for any \( k \in \mathbb{N} \) implies \( \{\overline{U}_k\}_{k=1}^{\infty} \) has finite intersection property. That is \( \{\overline{U}_k\}_{k=1}^{\infty} \) is a family of closed subsets of the compact topological space \( \overline{U}_0 \) and further \( \{\overline{U}_k\}_{k=1}^{\infty} \) has finite intersection property. Therefore \( \bigcap_{k=1}^{\infty} \overline{U}_k \neq \phi \). Let \( a \in \bigcap_{k=1}^{\infty} \overline{U}_k \). Then \( a \in \overline{U}_k \) for all \( k \in \mathbb{N} \) and hence \( a \in U \). Also \( a \in E_n \) for all \( n \in \mathbb{N} \). So \( a \in \bigcap_{n=1}^{\infty} E_n \). Thus \( a \in U \cap \bigcap_{n=1}^{\infty} E_n \). That is for each \( x \in X \) and for each open set \( U \) containing \( x \), \( U \cap \bigcap_{n=1}^{\infty} E_n \neq \phi \). Hence \( x \in \bigcap_{n=1}^{\infty} E_n \). This gives that \( X \subseteq \bigcap_{n=1}^{\infty} E_n \) and hence \( X = \bigcap_{n=1}^{\infty} E_n \), that is \( \bigcap_{n=1}^{\infty} E_n \) is dense in \( X \). \[ \square \]

**Exercise 5.6.10.** Prove that a subset \( E \) of a topological space \((X, \mathcal{J})\) is nowhere dense in \( X \) (that is \((E)^c = \phi \) if and only if \((E)^c \) is dense in \( X \).

**Remark 5.6.11.** It is known that every complete metric space \( X \) is of second category. The notion of completeness cannot be defined in a topological space. So we give the following version of Baire Category theorem for a locally compact Hausdorff topological space.
Theorem 5.6.12. Every nonempty locally compact Hausdorff topological space $(X, \mathcal{J})$ is of second category.

\textbf{Proof.} Proof by contradiction.

Suppose $(X, \mathcal{J})$ is of first category. Then there exists a countable collection $\{E_n\}_{n=1}^\infty$ of subsets of $X$ satisfying $\bigcap_{n=1}^\infty E_n = \emptyset$ and $X = \bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty \overline{E_n}$. Therefore $X^c = \phi = \bigcap_{n=1}^\infty E_n^c$ and hence $\phi = \overline{\phi} = \bigcap_{n=1}^\infty \overline{E_n}^c$. But $\{\overline{E_n}^c\}_{n=1}^\infty$ is a countable collection of dense open sets, implies by theorem 5.6.9 $\bigcap_{n=1}^\infty \overline{E_n}^c = X \neq \phi$. This contradicts $\phi = \overline{\phi} = \bigcap_{n=1}^\infty \overline{E_n}^c$. Hence $(X, \mathcal{J})$ is of second category. \hfill \blacksquare

Now we are in a position to prove Urysohn metrization theorem that gives sufficient conditions under which a topological space is metrizable. Also it is interesting to note that the well known Nagata-Smirnov metrization theorem gives a set of necessary and sufficient conditions for metrizability of a topological space.

\section*{5.7 Urysohn Metrization Theorem}

\textbf{Theorem 5.7.1. Urysohn Metrization Theorem.} Every normal space $(X, \mathcal{J})$ with a countable basis is metrizable.

\textbf{Proof.} Let $\mathcal{B} = \{B_1, B_2, \ldots, \}$ be a countable basis for $(X, \mathcal{J})$. Suppose $n, m \in \mathbb{N}$ are such that $\overline{B_n} \subseteq B_m$ then $\overline{B_n} \cap B_m^c = \emptyset$. Hence by Urysohn's lemma there exists a continuous function say $g_{n,m} : X \to \mathbb{R}$ such that

$$g_{n,m}(x) = 0 \quad \text{for all } x \in B_m^c, \quad (5.5)$$

and

$$g_{n,m}(x) = 1 \quad \text{for all } x \in \overline{B_n}. \quad (5.6)$$

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Now take $x_0 \in X$ and an open set $U$ containing $x_0$. Since $\mathcal{B}$ is a basis for $(X, J)$ there exists $B_m \in \mathcal{B}$ such that $x_0 \in B_m \subseteq U$. Now $B_m$ is an open set containing $x_0$ implies there exists an open set $V$ containing $x_0$ such that $\overline{V} \subseteq B_m$. Hence there exists a basic open set $B_n$ containing $x_0$ such that $\overline{B_n} \subseteq \overline{V} \subseteq B_m$. Hence for such pair $(n, m)$ we have a continuous function $g_{n,m} : X \to \mathbb{R}$ satisfying Eq. (5.5).

So if $x_0 \in X$ and $U$ is an open set containing $x_0$ then there exists a continuous function $g_{n,m} : X \to \mathbb{R}$ such that $g_{n,m}(x_0) = 1$ and $g_{n,m}(x) = 0$ for all $x \in U^c \subseteq B_m^c$.

So we have proved that there exists a countable collection of continuous functions $f_n : X \to [0, 1]$ such that for $x_0 \in X$ and open set $U$ containing $x_0$, there exists $n \in \mathbb{N}$ such that $f_n(x_0) = 1 > 0$ and $f_n(x) = 0$ for all $x \in U^c$. It is to be noted that $\{(n, m) : n, m \in \mathbb{N}\}$ is a countable set. We know that (refer chapter 2, and exercise 9 of chapter 5) $\mathbb{R}^w = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$ with product topology is metrizable. That is there is a metric $d$ on $\mathbb{R}^w$ such that $J_d$, the topology on $\mathbb{R}^w$ induced by $d$, coincides with the product topology on $\mathbb{R}^w$.

Now let us define a map $T : X \to \mathbb{R}^w$ as $T(x) = (f_1(x), f_2(x), \ldots)$ and using this map we define $d_1(x, y) = d(T(x), T(y))$ and conclude that $J_{d_1} = J$. This will prove that $(X, J)$ is a metrizable topological space. Now let us prove that $(X, J)$ is homeomorphic to a subspace of $\mathbb{R}^w$. Each $f_n : X \to \mathbb{R}$ is a continuous function implies $T(x) = (f_1(x), f_2(x), \ldots)$ is a continuous function.

To prove $T$ is injective (one-one).

Let $x, y \in X$ be such that $x \neq y$. Then there exist open sets $U, V \in X$ such that $x \in U, y \in V$ and $U \cap V = \phi$. Now $U$ is an open set containing $x$ implies there exists $n \in \mathbb{N}$ such that $f_n(x) = 1$ and $f_n(y) = 0$ (note that $y \in U^c$). This implies $f_n(x) \neq f_n(y)$ for this particular $n \in \mathbb{N}$ and hence $(f_1(x), f_2(x), \ldots, f_n(x), \ldots) \neq \ldots$
(f_1(y), f_2(y), \ldots, f_n(y), \ldots). This means Tx \neq Ty. That is, x, y \in X, x \neq y implies Tx \neq Ty. This implies T is 1-1.

Now it is enough to prove that T maps open set A in X to an open set T(A) in Y = T(X). Let A be an open set and y_0 \in T(A). Now y_0 \in T(A) implies there exists x_0 \in A such that T(x_0) = y_0. Now x_0 \in A, A is an open set implies there exists n_0 \in \mathbb{N} such that f_{n_0}(x_0) = 1 and f_{n_0}(x) = 0 for all x \in A^c. We know that for each n \in \mathbb{N} the projection map p_n : \mathbb{R}^w \to \mathbb{R} defined as p_n((x_k)_{k=1}^{\infty}) = x_n is a continuous map. Hence (0, \infty) is an open set implies V = p_{n_0}^{-1}((0, \infty)) is an open subset of \mathbb{R}^w. This implies V \cap Y is an open set in Y.

Now let us prove that y_0 \in V \cap Y and V \cap Y \subseteq T(A). p_{n_0}(y_0) = (p_{n_0} \cdot T)(x_0) = f_{n_0}(x_0) = 1 > 0 implies y_0 \in V. Also y_0 \in Y. Hence y_0 \in V \cap Y. That is V \cap Y is an open set in Y containing the point y_0.

Now we claim that V \cap Y \subseteq T(A). So, let y \in V \cap Y. Then there exists x \in X such that y = Tx. This implies p_{n_0}(y) \in (0, \infty) and p_{n_0}(y) = p_{n_0}(T(x)) = f_{n_0}(x) \in (0, \infty). Hence x \in A (f_{n_0}(x) = 0 for x \in A^c). So we have proved that y = Tx \in V \cap Y implies y = Tx \in T(A). Hence V \cap Y is an open set in Y containing Tx and this set is contained in T(A). Therefore T(A) is open in Y. Hence we have proved that T : (X, \mathcal{J}) \rightarrow (Y, d_Y) is a homeomorphism. (Here (Y, d_Y) is a subspace of (\mathbb{R}^w, d).)

Now d_1(x, y) = d(Tx, Ty) for all x, y \in X implies d_1 is a metric on X. Also it is easy to see that a subset A of X is open in (X, \mathcal{J}) if and only if A is open in (X, \mathcal{J}_{d_1}). Therefore \mathcal{J}_{d_1} = \mathcal{J}. \blacksquare
EXERCISES

1. Suppose a Hausdorff topological space \((X, \mathcal{J})\) has the following property: \(A\) and \(B\) are disjoint closed subsets of \(X\) implies there exists a continuous function say \(f_{AB} : X \to [0,1]\) such that \(f(x) = 0\) if \(x \in A\) and \(f(x) = 1\) if \(x \in B\) then prove that \((X, \mathcal{J})\) is a normal space.

2. If \((X, \mathcal{J})\) is a connected normal space containing more than one point, then prove that \(X\) is an uncountable set. (In particular every connected metric space containing more than one point is an uncountable set.)

3. Let \(X = l_\infty = \{x = (x_n) : \{x_n\}_{n=1}^\infty\}\) is a bounded sequence of real numbers. For \(x = (x_n) \in l_\infty\), \(y = (y_n) \in l_\infty\), let \(d_\infty(x,y) = \sup_{n \geq 1} |x_n - y_n|\). Prove (i) \(d_\infty\) is a metric on \(l_\infty\), and (ii) \(Y = \{x = (x_n) : x_n = 0\text{ or } 1\}\) is not a separable subspace of \(l_\infty\). (Hint: For \(x = (x_n) \in Y\), \(y = (y_n) \in Y\), \(x \neq y\), \(d(x,y) = 1\).)

4. Prove that every separable metric space \((X, d)\) is a second countable space. (It is interesting to note that every second countable topological space is separable.)

5. Show that every locally compact Hausdorff space is regular.

6. Prove that a subspace of first countable space is first countable and a subspace of a second countable space is second countable.

7. Is it true that product of first (second) countable spaces is first (second) countable? Justify your answer.

8. Prove that every locally compact Hausdorff space \((X, \mathcal{J})\) is completely regular.
9. For \( x, y \in \mathbb{R} \), let \( \bar{d}(x, y) = \min\{d(x, y), 1\} \), then \( \bar{d} \) is a metric on \( \mathbb{R} \) that induces the usual topology on \( \mathbb{R} \). Let \( \mathbb{R}^w = \{ x = (x_1, x_2, \ldots) : x_n \in \mathbb{R}, n \in \mathbb{N} \} \). For \( x = (x_n) \in \mathbb{R}^w \), \( y = (y_n) \in \mathbb{R}^w \), let \( d(x, y) = \sup \left\{ \frac{\bar{d}(x_n, y_n)}{n} \right\} \) then

(i) prove that \( d \) is a metric on \( \mathbb{R}^w \),

(ii) prove that \( J_d \), the topology on \( \mathbb{R}^w \) induced by the metric \( d \) is same as the product topology on \( \mathbb{R}^w = \prod_{n=1}^{\infty} \mathbb{R}_n \), where \( \mathbb{R}_n = \mathbb{R} \), with usual topology on \( \mathbb{R} \).

10. Let \( X, Y \) be compact normal spaces. Then prove that \( X \times Y \) is also a compact normal space.

11. Find all the topologies on \( X = \{1, 2, 3\} \) which are regular.

12. Prove that a \( T_1 \)-topological space \( (X, J) \) is normal if and only if given any two disjoint closed sets \( A, B \) in \( X \) there exist open sets \( U, V \) in \( X \) such that

(i) \( A \subseteq U, B \subseteq V \) and (ii) \( \overline{U} \cap \overline{V} = \phi \).

13. Prove that homeomorphic image of a normal space is normal.

14. Can there exists a continuous function \( f : \mathbb{R}^2 \to [0, 1] \) such that \( f(x, y) = 0, 0 \leq x, y \leq 1 \) and \( f(x, y) = 1, x = 3, 0 \leq y \leq 1 \). Justify your answer.