Analysis of Variance and Design of Experiments-I

MODULE - I

LECTURE - 1

SOME RESULTS ON LINEAR ALGEBRA, MATRIX THEORY AND DISTRIBUTIONS

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We need some basic knowledge to understand the topics in analysis of variance.

**Vectors**

A vector \( Y \) is an ordered \( n \)-tuple of real numbers. A vector can be expressed as row vector or a column vector as

\[
Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}
\]

is a column vector of order \( n \times 1 \).

and

\[
Y' = (y_1, y_2, \ldots, y_n)
\]

is a row vector of order \( 1 \times n \).

If all \( y_i = 0 \) for all \( i = 1, 2, \ldots, n \) then \( Y' = (0, 0, \ldots, 0) \) is called the **null vector**.

If

\[
X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}
\]

then

\[
X + Y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad kY = \begin{pmatrix} ky_1 \\ ky_2 \\ \vdots \\ ky_n \end{pmatrix}
\]
\[ X + (Y + Z) = (X + Y) + Z \]
\[ X' (Y + Z) = X' Y + X' Z \]
\[ k(X' Y) = (kX)' Y = X'(kY) \]
\[ k(X + Y) = kX + kY \]
\[ X' Y = x_1y_1 + x_2y_2 + \ldots + x ny_n \]

where \( k \) is a scalar.

**Orthogonal vectors**

Two vectors \( X \) and \( Y \) are said to be orthogonal if \( X'Y = Y'X = 0 \).

The null vector is orthogonal to every vector \( X \) and is the only such vector.

**Linear combination**

if \( x_1, x_2, \ldots, x_m \) are \( m \) vectors of same order and \( k_1, k_2, \ldots, k_m \) are scalars, Then

\[ t = \sum_{i=1}^{m} k_i x_i \]

is called the linear combination of \( x_1, x_2, \ldots, x_m \).
**Linear independence**

If $X_1, X_2, \ldots, X_m$ are $m$ vectors then they are said to be linearly independent if there exist scalars $k_1, k_2, \ldots, k_m$ such that

$$
\sum_{i=1}^{m} k_i X_i = 0 \Rightarrow k_i = 0 \text{ for all } i = 1, 2, \ldots, m.
$$

If there exist $k_1, k_2, \ldots, k_m$ with at least one $k_i$ to be nonzero, such that $\sum_{i=1}^{m} k_i x_i = 0$ then $x_1, x_2, \ldots, x_m$ are said to be linearly dependent.

- Any set of vectors containing the null vector is linearly dependent.
- Any set of non-null pair-wise orthogonal vectors is linearly independent.
- If $m > 1$ vectors are linearly dependent, it is always possible to express at least one of them as a linear combination of the others.
**Linear function**

Let \( K = (k_1, k_2, \ldots, k_m)' \) be a \( m \times 1 \) vector of scalars and \( X = (x_1, x_2, \ldots, x_m) \) be a \( m \times 1 \) vector of variables, then

\[
K'Y = \sum_{i=1}^{m} k_i y_i
\]

is called a linear function or linear form. The vector \( K \) is called the **coefficient vector**.

For example, mean of \( x_1, x_2, \ldots, x_m \) can be expressed as

\[
\bar{x} = \frac{1}{m} \sum_{i=1}^{m} x_i = \frac{1}{m} \begin{pmatrix} 1, & 1, & \ldots, & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \frac{1}{m} \mathbf{1}_m'X
\]

where \( \mathbf{1}_m' \) is a \( m \times 1 \) vector of all elements unity.

**Contrast**

The linear function \( K'X = \sum_{i=1}^{m} k_i x_i \) is called a contrast in \( x_1, x_2, \ldots, x_m \) if \( \sum_{i=1}^{m} k_i = 0 \).

For example, the linear functions

\[
x_1 - x_2, 2x_1 - 3x_2 + x_3, \frac{x_1}{2} - x_2 + \frac{x_3}{3}
\]

are contrasts.

- A linear function \( K'X \) is a contrast if and only if it is orthogonal to a linear function \( \sum_{i=1}^{m} x_i \) or to the linear function \( \bar{x} = \frac{1}{m} \sum_{i=1}^{m} x_i \).
- Contrasts \( x_1 - x_2, x_1 - x_3, \ldots, x_i - x_j \) are linearly independent for all \( j = 2, 3, \ldots, m \).
- Every contrast in \( x_1, x_2, \ldots, x_n \) can be written as a linear combination of \((m - 1)\) contrasts \( x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_m \).
A matrix is a rectangular array of real numbers. For example,

\[
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

is a matrix of order \( m \times n \) with \( m \) rows and \( n \) columns.

- If \( m = n \), then \( A \) is called a square matrix.
- If \( a_{ij} = 0, \ i \neq j, m = n \), then \( A \) is a diagonal matrix and is denoted as \( A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn}) \).
- If \( m = n \), (square matrix) and \( a_{ij} = 0 \) for \( i > j \), then \( A \) is called an upper triangular matrix. On the other hand if \( m = n \), and \( a_{ij} = 0 \) for \( i < j \) then \( A \) is called a lower triangular matrix.
- If \( A \) is a \( m \times n \) matrix, then the matrix obtained by writing the rows of \( A \) and columns of \( A \) as columns of \( A \) and rows of \( A \) respectively, is called the transpose of a matrix \( A \) and is denoted as \( A' \).
- If \( A = A' \) then \( A \) is a symmetric matrix.
- If \( A = -A' \) then \( A \) is skew symmetric matrix.
- A matrix whose all elements are equal to zero is called as null matrix.
- An identity matrix is a square matrix of order \( p \) whose diagonal elements are unity (ones) and all the off diagonal elements are zero. It is denotes as \( I_p \).
• If $A$ and $B$ are matrices of order $m \times n$ then
  
  
  
  $(A + B)' = A' + B'$.

• If $A$ and $B$ are the matrices of order $m \times n$ and $n \times p$ respectively and $k$ is any scalar, then

  
  
  
  $(AB)' = B'A'$

  
  
  
  
  $(kA)B = A(kB) = k(AB) = kAB$.

• If the orders of matrices $A$ is $m \times n$, $B$ is $n \times p$ and $C$ is $n \times p$ then

  
  
  
  
  $A(B + C) = AB + AC$.

• If the orders of matrices $A$ is $m \times n$, $B$ is $n \times p$ and $C$ is $p \times q$ then

  
  
  $(AB)C = A(BC)$.

• If $A$ is the matrix of order $m \times n$ then

  
  
  
  $I_mA = AI_n = A$. 
Trace of a matrix

The trace of $n \times n$ matrix $A$, denoted as $tr(A)$ or $\text{trace}(A)$ is defined to be the sum of all the diagonal elements of $A$, i.e., $tr(A) = \sum_{i=1}^{n} a_{ii}$.

- If $A$ is of order $m \times n$ and $B$ is of order $n \times m$, then
  $$tr(AB) = tr(BA).$$

- If $A$ is $n \times n$ matrix and $P$ is any nonsingular $n \times n$ matrix then
  $$tr(A) = tr(P^{-1}AP).$$
  If $P$ is an orthogonal matrix then $tr(A) = tr(P'AP)$.

- If $A$ and $B$ are $n \times n$ matrices, $a$ and $b$ are scalars then
  $$tr(aA + bB) = a tr(A) + b tr(B).$$

- If $A$ is a $m \times n$ matrix, then
  $$tr(A' A) = tr(AA') = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}^2$$
  and
  $$tr(A' A) = tr(AA') = 0 \text{ if and only if } A = 0.$$ 

- If $A$ is $n \times n$ matrix then
  $$tr(A') = trA.$$
**Rank of a matrix**

The rank of a matrix $A$ of $m \times n$ is the number of linearly independent rows in $A$.

Let $B$ be another matrix of order $n \times q$.

- A square matrix of order $m$ is called **non-singular** if it has a full rank.
- $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.
- $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.
- Rank of $A$ is equal to the maximum order of all nonsingular square sub-matrices of $A$.
- $\text{rank}(AA') = \text{rank}(A'A) = \text{rank}(A) = \text{rank}(A')$.
- $A$ is of full row rank if $\text{rank}(A) = m < n$.
- $A$ is of full column rank if $\text{rank}(A) = n < m$. 
Inverse of matrix

The inverse of a square matrix $A$ of order $m$, is a square matrix of order $m$, denoted as $A^{-1}$, such that $A^{-1}A = AA^{-1} = I_m$.

The inverse of $A$ exists if and only if $A$ is non singular.

- $(A^{-1})^{-1} = A$.
- If $A$ is non singular, then $(A')^{-1} = (A^{-1})'$.
- If $A$ and $B$ are non-singular matrices of same order, then their product, if defined, is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

Idempotent matrix

A square matrix $A$ is called idempotent if $A^2 = AA = A$.

If $A$ is an $n \times n$ idempotent matrix with $\text{rank}(A) = r \leq n$. Then

- the eigenvalues of $A$ are 1 or 0.
- $\text{trace}(A) = \text{rank}(A) = r$.
- If $A$ is of full rank $n$, then $A = I_n$.
- If $A$ and $B$ are idempotent and $AB = BA$, then $AB$ is also idempotent.
- If $A$ is idempotent then $(I - A)$ is also idempotent and $A(I - A) = (I - A)A = 0$. 