MODULE 5
SOME SPECIAL ABSOLUTELY CONTINUOUS DISTRIBUTIONS
LECTURE 21

Topics

5.1 UNIFORM OR RECTANGULAR DISTRIBUTION
5.1.1 Quantile function and uniform distribution

5.2 GAMMA AND RELATED DISTRIBUTIONS

Lemma 1.1

Let \( X \) be a random variable having distribution function \( F_X(\cdot) \) and quantile function \( Q_X(\cdot) \). Let \( x \in \mathbb{R} \), \( p \in (0, 1) \) and \( 0 < p_1 < p_2 < 1 \). Then

(i) \( Q_X(F_X(x)) \leq x \), provided \( 0 < F_X(x) < 1 \);
(ii) \( F_X(Q_X(p)) \geq p \);
(iii) \( F_X(Q_X(p)) = p \), provided there exists an \( x_0 \in \mathbb{R} \) such that \( F_X(x_0) = p \). In particular if \( F_X(\cdot) \) is continuous then \( F_X(Q_X(p)) = p \);
(iv) \( Q_X(p) \leq x \iff F_X(x) \geq p \);
(v) \( Q_X(p) = F_X^{-1}(p) \), provided \( F_X^{-1}(p) \) exists;
(vi) \( Q_X(p_1) \leq Q_X(p_2) \).

Proof. For \( p \in (0, 1) \), define

\[
S_p = \{ s \in \mathbb{R} : F_X(s) \geq p \},
\]

so that \( Q_X(p) = \inf S_p \), \( p \in (0, 1) \).

(i) Let \( x \in \mathbb{R} \) be such that \( 0 < F_X(x) < 1 \). Then \( x \in S_{F_X(x)} = \{ s \in \mathbb{R} : F_X(s) \geq F_X(x) \} \) and, therefore, \( x \geq \inf S_{F_X(x)} = Q(F_X(x)) \), i.e., \( Q_X(F_X(x)) \leq x \).

(ii) Let \( p \in (0, 1) \). Then \( Q_X(p) = \inf S_p \). Thus there exists a sequence \( \{ t_n : n = 1, 2, \ldots \} \) in \( S_p \) such that \( \lim_{n \to \infty} t_n = Q_X(p) \). Consequently \( t_n \geq Q_X(p), n = 1, 2, \ldots \) and \( F_X(t_n) \geq p, n = 1, 2, \ldots \). This implies that \( \lim_{n \to \infty} F_X(t_n) \geq p \). Since \( F_X(\cdot) \) is right continuous, \( t_n \geq Q_X(p), n = 1, 2, \ldots \) and \( \lim_{n \to \infty} t_n = Q_X(p) \), we get
\[ F_X(Q_X(p)) = \lim_{n \to \infty} F_X(t_n) \geq p. \]

(iii) Let \( x_0 \in \mathbb{R} \) be such that \( F_X(x_0) = p \). Then
\[
F_X(x_0) = \{ s \in \mathbb{R} : F_X(s) \geq p \} \Rightarrow x_0 \geq \inf S_p = Q_X(p).
\]
Now using (ii) and the fact that \( F_X(\cdot) \) is non-decreasing, we get
\[
p = F_X(x_0) \geq F_X(Q_X(p)) \geq p \Rightarrow F_X(Q_X(p)) = p.
\]
Note that \( \lim_{x \to -\infty} F_X(x) = 0 \) and \( \lim_{x \to \infty} F_X(x) = 1 \). Thus if \( F_X(\cdot) \) is continuous then the intermediate value property of continuous functions implies that there exists an \( x_0 \in \mathbb{R} \) such that \( F_X(x_0) = p \in (0, 1) \) and therefore \( F_X(Q_X(p)) = p \).

(iv) First suppose that \( Q_X(p) = \inf S_p \leq x \). Then, since \( F_X(\cdot) \) is non-decreasing,
we have
\[
F_X(Q_X(p)) \leq F_X(x) \Rightarrow p \leq F_X(x). \quad \text{(using (ii))}
\]
Now suppose that \( F_X(x) \geq p \). Then \( x \in S_p = \{ s \in \mathbb{R} : F_X(s) \geq p \} \) and, therefore,
\[
x \geq \inf S_p = Q_X(p).
\]
(v) Since \( p_1 < p_2 \), we have
\[
S_{p_2} = \{ s \in \mathbb{R} : F_X(s) \geq p_2 \} \subseteq \{ s \in \mathbb{R} : F_X(s) \geq p_1 \} = S_{p_1}
\]
\[
\Rightarrow S_{p_2} \subseteq S_{p_1}
\]
\[
\Rightarrow Q_X(p_1) = \inf S_{p_1} \leq \inf S_{p_2} = Q_X(p_2). \quad \blacksquare
\]

**Theorem 1.3**

Let \( X \) be a random variable with distribution function \( F_X(\cdot) \) and quantile function \( Q_X(\cdot) \).

(i) **(Probability Integral Transformation)** If the random variable \( X \) is of continuous type then \( Y \equiv F_X(X) \sim U(0, 1) \);

(ii) Let \( U \sim U(0,1) \). Then \( Z \equiv Q_X(U) \equiv X \).

**Proof.**

(i) Let \( G(\cdot) \) be the d.f. of \( Y \equiv F_X(X) \), i.e.,
\[
G(y) = P(\{ F_X(X) \leq y \}), \quad y \in \mathbb{R}.
\]
Clearly, for \( y < 0 \), \( G(y) = 0 \) and, for \( y \geq 1 \), \( G(y) = 1 \). Now suppose that \( y \in (0, 1) \). By Lemma 1.1 (iv) we have
\[
\{ s \in \mathbb{R} : F_X(s) \geq y \} = \{ s \in \mathbb{R} : s \geq Q_X(y) \}
\]
\[ P(\{F_X(X) \geq y\}) = P(\{X \geq Q_X(y)\}) \]

\[ P(\{F_X(X) < y\}) = P(\{X < Q_X(y)\}) \]

\[ P(\{F_X(X) < y\}) = P(\{X \leq Q_X(y)\}). \quad \text{(since } F_X(\cdot) \text{ is continuous)} \quad (1.4) \]

Since \( F_X(\cdot) \) is continuous \( \{x \in \mathbb{R}: F_X(x) = y\} = [x_1, x_2] \), for some real numbers \( x_1 \) and \( x_2 \) such that \( -\infty < x_1 \leq x_2 < \infty \) (see Figures 1.5 (a) & (b)).

Thus, for \( y \in (0, 1) \),

\[
P(\{F_X(X) = y\}) = P(\{x_1 \leq X \leq x_2\})
\]

\[
= F_X(x_2) - F_X(x_1)
\]

\[
= y - y = 0. \quad (1.5)
\]

Using (1.4), (1.5) and Lemma 1.1 (iii) we get, for \( y \in (0, 1) \),

\[
G(y) = P(\{F_X(X) \leq y\}) = P(\{F_X(X) < y\}) = P(\{X \leq Q_X(y)\}) = y.
\]

Also right continuity of d.f. \( G(\cdot) \) implies that

\[
G(0) = \lim_{x \rightarrow 0} G(x) = \lim_{x \rightarrow 0} x = 0.
\]

Therefore we have

\[
G(y) = \begin{cases} 
0, & \text{if } y < 0 \\
y, & \text{if } 0 \leq y < 1 \\
1, & \text{if } y \geq 1
\end{cases}
\]

i.e., \( Y \overset{d}{=} F_X(X) \sim U(0, 1) \).
(ii) Let $U \sim U(0, 1)$, so that $P\{\{U \leq u\}\} = u, \forall u \in [0, 1]$ and $P\{\{0 < U < 1\}\} = 1$. Then the d.f. of $Z \equiv Q_X(U)$ is

$$H(z) = P\{\{Z \leq z\}\}$$
$$= P\{\{Q_X(U) \leq z\}\}$$
$$= P\{\{Q_X(U) \leq z, 0 < U < 1\}\} \text{ (since } P\{\{0 < U < 1\}\} = 1\}$$
$$= P\{\{F_X(z) \geq U, 0 < U < 1\}\} \text{ (using Lemma 1.1 (iv))}$$
$$= P\{\{U \leq F_X(z)\}\}$$
$$= F_X(z), \ z \in \mathbb{R}$$

$\Rightarrow Z \overset{d}{=} X$. ■

**Remark 1.3**

The above theorem provides a method to generate observations from any arbitrary distribution using observations from $U(0, 1)$ distribution. Suppose that we require an observation $X$ from a distribution having known d.f. $F(\cdot)$ and quantile function $Q(\cdot)$. To do so, the above theorem suggests that, generate an observation $U$ from the $U(0,1)$ distribution and take $X = Q(U)$. ■

**Example 1.2**

Using a random observation $U \sim U(0, 1)$, describe a method to generate a random observation $X$ from the distribution having

(i) probability density function

$$f(x) = \frac{e^{-|x|}}{Z}, -\infty < x < \infty;$$

(ii) probability mass function

$$g(x) = \begin{cases} \binom{n}{x} \theta^x (1 - \theta)^{n-x}, & \text{if } x \in \{0, 1, \ldots, n\}, \\ 0, & \text{otherwise} \end{cases}$$

where $n \in \mathbb{N}$ and $\theta \in (0, 1)$ are real constants.

**Solution.**

(i) For $x < 0$, we have

$$F(x) = P\{\{X \leq x\}\}$$

$$= \int_{-\infty}^{x} f_X(t) dt$$
\[ F(x) = P([X \leq x]) = \int_{-\infty}^{x} f_X(t) dt = \int_{-\infty}^{0} f_X(t) dt + \int_{0}^{x} f_X(t) dt = \int_{-\infty}^{0} e^{t} \frac{1}{2} dt + \int_{0}^{x} e^{-t} \frac{1}{2} dt = 1 - e^{-x} \]

Thus the d.f. of \( X \) is given by

\[
F(x) = \begin{cases} 
\frac{e^x}{2}, & \text{if } x < 0 \\
1 - \frac{e^{-x}}{2}, & \text{if } x \geq 0
\end{cases}
\]

and the q.f. of \( X \) is given by

\[
Q(p) = F^{-1}(p) = \begin{cases} 
\ln(2p), & \text{if } 0 < p < \frac{1}{2} \\
-\ln(2(1 - p)), & \text{if } \frac{1}{2} \leq p < 1
\end{cases}
\]

Using Theorem 1.3 (ii) the desired random observation is given by

\[
X = Q(U) = \begin{cases} 
\ln(2U), & \text{if } 0 < U < \frac{1}{2} \\
-\ln(2(1 - U)), & \text{if } \frac{1}{2} \leq U < 1
\end{cases}
\]

(ii) The distribution function of \( X \) is given by

\[
G(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\sum_{j=0}^{n} \binom{n}{j} \theta^j (1 - \theta)^{n-j}, & \text{if } k \leq x < k + 1; \; k = 0, 1, ..., n - 1, \\
1, & \text{if } x \geq n
\end{cases}
\]

and the quantile function of \( X \) is given by

\[ Q(p) = \inf\{ s \in \mathbb{R} : G(s) \geq p \} \]
\[
Y = \begin{cases}
1, & \text{if } 0 < p \leq (1 - \theta)^n \\
k, & \text{if } \sum_{j=0}^{k-1} \binom{n}{j} \theta^j (1 - \theta)^{n-j} < p \leq \sum_{j=0}^{k} \binom{n}{j} \theta^j (1 - \theta)^{n-j}; \\
\quad k = 0, 1, \ldots, n-1 \\
n, & \text{if } \sum_{j=0}^{n-1} \binom{n}{j} \theta^j (1 - \theta)^{n-j} < p < 1
\end{cases}
\]

Now, using Theorem 1.3 (ii), the desired random observation is given by

\[
X = \begin{cases}
1, & \text{if } 0 < U \leq (1 - \theta)^n \\
k, & \text{if } \sum_{j=0}^{k-1} \binom{n}{j} \theta^j (1 - \theta)^{n-j} < U \leq \sum_{j=0}^{k} \binom{n}{j} \theta^j (1 - \theta)^{n-j}; \\
\quad k = 0, 1, \ldots, n-1 \\
n, & \text{if } \sum_{j=0}^{n-1} \binom{n}{j} \theta^j (1 - \theta)^{n-j} < U < 1
\end{cases}
\]

### 5.2 GAMMA AND RELATED DISTRIBUTIONS

We begin this section with the definition of gamma function.

**Definition 2.1**

The function \( \Gamma: (0, \infty) \to (0, \infty) \), defined by,

\[
\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt, \quad \alpha > 0
\]

is called the *gamma function*.

To examine convergence of the integral

\[
\int_0^\infty e^{-t} t^{\alpha-1} dt, \quad \alpha \in \mathbb{R},
\]

consider the following cases.

**Case I** \( \alpha \leq 0 \)
In this case the integral
\[ \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt \]
will converge if, and only if, both the integrals
\[ \int_{0}^{1} e^{-t} t^{\alpha-1} dt \quad \text{and} \quad \int_{1}^{\infty} e^{-t} t^{\alpha-1} dt \]
converge. Note that, for \( \alpha \leq 0 \),
\[ e^{-t} t^{\alpha-1} \geq \frac{t^{\alpha-1}}{e}, \quad \forall t \in (0,1) \]
and the integral
\[ \int_{0}^{1} t^{\alpha-1} dt \]
diverges. This implies that, for \( \alpha \leq 0 \), the integral
\[ \int_{0}^{1} e^{-t} t^{\alpha-1} dt \]
diverges. Consequently the integral
\[ \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt \]
diverges for \( \alpha \leq 0 \).

**Case II** \( 0 < \alpha < 1 \)

In this case again the integral
\[ \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt \]
will converge if, and only if, both the integrals
\[ \int_{0}^{1} e^{-t} t^{a-1} dt \quad \text{and} \quad \int_{1}^{\infty} e^{-t} t^{a-1} dt \]

converge. Note that, for \( \alpha > 0 \),

\[ 0 \leq e^{-t} t^{a-1} \leq t^{a-1}, \quad \forall t \in (0,1) \]

and the integral

\[ \int_{0}^{1} t^{a-1} dt \]

is convergent. Therefore the integral

\[ \int_{0}^{1} e^{-t} t^{a-1} dt \]

is convergent for any \( \alpha > 0 \).

Now let us examine the convergence of the integral

\[ \int_{1}^{\infty} e^{-t} t^{a-1} dt. \]

Fix \( \alpha \in \mathbb{R} \) and choose \( k_0 \in \mathbb{N} \) such that \( k_0 > \alpha \). Then we know that

\[ e^{t} \geq \frac{t^{k_0}}{k_0!}, \quad \forall t > 0 \]

\[ \Rightarrow 0 \leq e^{-t} t^{a-1} \leq \frac{k_0!}{t^{k_0-\alpha+1}}, \quad \forall t > 0. \]

Also \( k_0 - \alpha + 1 > 1 \) and, therefore, the integral

\[ \int_{1}^{\infty} \frac{1}{t^{k_0-\alpha+1}} dt \]

converges. Consequently

\[ \int_{1}^{\infty} e^{-t} t^{a-1} dt \]
converges for any $\alpha \in \mathbb{R}$. From the above discussion it follows that the integral
\[ \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt \]
converges for $0 < \alpha < 1$.

**Case III $\alpha \geq 1$**

In this case the integral
\[ \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt \]
will converge if, and only if, the integral
\[ \int_{1}^{\infty} e^{-t} t^{\alpha-1} dt \]
converges. We have seen in the Case II above that the integral
\[ \int_{1}^{\infty} e^{-t} t^{\alpha-1} dt \]
converges for any $\alpha \in \mathbb{R}$.

On combining cases I–III we conclude that the integral
\[ \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt \]
converges if, and only if, $\alpha > 0$.

Using integration by parts, for $\alpha > 0$, we have
\[ \Gamma(\alpha + 1) = \int_{0}^{\infty} e^{-t} t^{\alpha} dt \]
\[ = [e^{-t} t^{\alpha}]_0^\infty - \alpha \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt \]
\[ \Gamma(\alpha + 1) = \alpha \Gamma(\alpha), \quad \alpha > 0. \]  
(2.1)

Note that
\[ \Gamma(1) = \int_{0}^{\infty} e^{-t} \, dt = 1. \]  
(2.2)

For \( n \in \mathbb{N} \), using (2.1) and (2.2), we have
\[ \Gamma(n + 1) = n \Gamma(n) = n(n - 1) \Gamma(n - 1) = \cdots = n(n - 1) \cdots 3 \cdot 2 \cdot 1 \Gamma(1) = n!. \]  
(2.3)

On combining (2.1), (2.2) and (2.3) we get
\[ \Gamma(n) = (n - 1)!, \quad n \in \mathbb{N}, \]  
(2.4)

with the convention that \( 0! = 1 \).

We have
\[ \Gamma\left(\frac{1}{2}\right) = \int_{0}^{\infty} e^{-t} t^{-1/2} \, dt \]
\[ = 2 \int_{0}^{\infty} e^{-x^2} \, dx \]
\[ \Rightarrow \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4 \left[ \int_{0}^{\infty} e^{-x^2} \, dx \right] \left[ \int_{0}^{\infty} e^{-y^2} \, dy \right] \]
\[ = 4 \int_{0}^{\infty} \int_{0}^{\infty} e^{-(x^2+y^2)} \, dxdy. \]

On making the transformation \( x = r \cos \theta \) and \( y = r \sin \theta \) in the above integral (so that the Jacobian of the transformation is \( r \)), we have
\[ \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4 \int_{0}^{\infty} \int_{0}^{\pi/2} r e^{-r^2} \, d\theta \, dr \]
\[ = 2\pi \int_0^\infty re^{-r^2} \, dr \]
\[ = \pi \int_0^\infty e^{-t} \, dt \]
\[ = \pi. \]

Since
\[ \Gamma \left( \frac{1}{2} \right) = \int_0^\infty e^{-t} t^{1/2-1} \, dt \geq 0, \]
we get
\[ \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \quad (2.5) \]

Also, using (2.1),
\[ \Gamma \left( \frac{3}{2} \right) = \frac{1}{2} \Gamma \left( \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2}, \]
and
\[ \Gamma \left( \frac{5}{2} \right) = \frac{3 \cdot 1}{2 \cdot 2} \Gamma \left( \frac{1}{2} \right) = \frac{1 \cdot 3}{2^2} \sqrt{\pi}, \]
In general
\[ \Gamma \left( \frac{2n + 1}{2} \right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n} \sqrt{\pi}, \quad n \in \mathbb{N}, \quad (2.6) \]
i.e., for \( n \in \mathbb{N}, \)
\[ \Gamma \left( \frac{2n + 1}{2} \right) = \frac{(2n)!}{n! 4^n} \sqrt{\pi}, \quad n \in \mathbb{N}. \quad (2.7) \]