MODULE 5

SPECIAL ABSOLUTELY CONTINUOUS DISTRIBUTIONS AND THEIR PROPERTIES

PROBLEMS

1. Let \( X \sim U(0, \theta) \), where \( \theta \) is a positive integer and let \( Y = X - [X] \), where \([x]\) is the largest integer \( \leq x \). Show that \( Y \sim U(0, 1) \).

2. Let \( F(\cdot) \) be the d.f. of a r.v. \( X \), where \( P(X = 1) = p = 1 - P(X = 0) \). Find the distribution of \( Y = F(X) \). Does \( Y \sim U(0, 1) \)? Interpret your findings on light of Theorem 1.3 (i).

3. Let the r.v. \( X \) have the p.d.f

\[
 f(x) = \begin{cases} 
 6x(1-x), & \text{if } 0 < x < 1 \\
 0, & \text{otherwise} 
\end{cases}
\]

Show that \( Y = X^2(3 - 2X) \sim U(0, 1) \).

4. Let \( X \sim U(0, 1) \) and let \( U = \min(X, 1 - X) \). Find the p.d.f. of \( Y = (1 - U)/U \). Does \( Y \) has finite expectation?

5. (i) If \( X \sim U(0, 1) \), find the distribution of \( Y = -\lambda \ln X \), where \( \lambda > 0 \);

(ii) Let the r.v. \( X \) have the Cauchy p.d.f. \( f(x) = \pi^{-1}(1 + x^2)^{-1}, \quad -\infty < x < \infty \). Find the p.d.f. of \( Y = X^{-1} \).

6. If \( X \sim U(0, \theta) \), where \( \theta > 0 \), find the distribution of \( Y = \min(X, \theta/2) \). Calculate \( P \left( \frac{\theta}{4} < Y < \frac{\theta}{2} \right) \).

7. Let \( X \sim N(0, 1) \) and let

\[
 Y = \begin{cases} 
 X, & \text{if } |X| \leq 1 \\
 -X, & \text{if } |X| > 1 
\end{cases}
\]

Find the distribution of \( Y \).

8. Let \( X \sim N(\mu, \sigma^2) \). Find the distribution function and probability density function of \( Y = X^2 \).
9. (i) If $X \sim N(12, 16)$, find $P(X \geq 20)$, (use $\Phi(2) = 0.9772$);
(ii) If $X \sim N(\mu, \sigma^2)$, $P(9.6 \leq X \leq 13.8) = 0.7008$ and $P(X \geq 9.6) = 0.8159$, find $\mu, \sigma^2$ and $P(\{X \geq 13.8 \mid X \geq 9.6\})$(use $\Phi(0.9) = 0.8159$ and $\Phi(1.2) = 0.8849$).

10. For $x > 0$, show that
$$(x^{-1} - x^{-3}) \phi(x) < 1 - \Phi(x) < x^{-1} \phi(x).$$
(Hint: Use integration by parts in $(2\pi)^{1/2} (1 - \Phi(x)) = \int_x^\infty t^{-1} (te^{-t^2/2}) dt$).

11. Let $Z \sim N(0, 1)$. Find $E(Z\Phi(Z))$ and $E(Z^2 \Phi(Z))$. (Hint: Use the fact that $\phi'(z) = -z\phi(z)$ and integrate by parts.)