MODULE 2
RANDOM VARIABLE AND ITS DISTRIBUTION
LECTURE 11

Topics

2.4 TYPES OF RANDOM VARIABLES: DISCRETE, CONTINUOUS AND ABSOLUTELY CONTINUOUS

The following theorem establishes that any function \( g: \mathbb{R} \rightarrow [0, \infty) \) satisfying the above two properties is a p.d.f. of some r.v. of absolutely continuous type.

**Theorem 4.3**

Suppose that there exists a non-negative function \( g: \mathbb{R} \rightarrow \mathbb{R} \) satisfying:

(i) \( g(x) \geq 0, \forall x \in \mathbb{R}; \)
(ii) \( \int_{-\infty}^{\infty} g(t) \, dt = 1. \)

Then there exists an absolutely continuous type random variable \( X \) on some probability space \((\Omega, \mathcal{B}_1, P)\) such that the p.d.f. \( X \) is \( g. \)

**Proof.** Define the set function \( P: \mathcal{B}_1 \rightarrow \mathbb{R} \) by

\[
P(B) = \int_{-\infty}^{\infty} g(t) I_B(t) \, dt, \quad B \in \mathcal{B}_1.
\]

It is easy to verify that \( P \) is a probability measure on \( \mathcal{B}_1 \), i.e., \((\mathbb{R}, \mathcal{B}_1, P)\) is a probability space. Define \( X: \mathbb{R} \rightarrow \mathbb{R} \) by \( X(\omega) = \omega, \omega \in \mathbb{R}. \) Clearly \( X \) is a random variable on the probability space \((\mathbb{R}, \mathcal{B}_1, P)\). The space \((\mathbb{R}, \mathcal{B}_1, P)\) is also the probability space induced by \( X. \) Clearly, for \( x \in \mathbb{R}, \)

\[
F_X(x) = P_X((-\infty, x])
= P((-\infty, x])
= \int_{-\infty}^{\infty} g(t) I_{(-\infty,x]}(t) \, dt
\]
\[ x = \int_{-\infty}^{\infty} g(t) dt. \]

It follows that \( X \) is of absolutely continuous type and \( g \) is the p.d.f. of \( X \). □

**Example 4.7**

Let \( X \) be r.v. with the d.f.

\[
F_X(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{x^2}{2}, & \text{if } 0 \leq x < 1 \\
\frac{x}{2}, & \text{if } 1 \leq x < 2 \\
1, & \text{if } x \geq 2
\end{cases}
\]

Show that the r.v. \( X \) is of absolutely continuous type and find the p.d.f. of \( X \).

**Solution.** Clearly \( F_X \) is differentiable everywhere except at points 1 and 2. Let \( D = \{1, 2\} \), so that

\[
\int_{-\infty}^{\infty} F_X'(t) I_D(t) dt = \int_0^1 t dt + \int_1^2 \frac{1}{2} dt = 1.
\]

Using Remark 4.2 (vii) it follows that the r.v. \( X \) is of absolutely continuous type with a p.d.f.

\[
f_X(x) = \begin{cases} 
x, & \text{if } 0 \leq x < 1 \\
a, & \text{if } x = 1 \\
\frac{1}{2}, & \text{if } 1 < x < 2 \\
b, & \text{if } x = 2 \\
0, & \text{otherwise}
\end{cases}
\]

where \( a \) and \( b \) are arbitrary nonnegative constants. In particular a p.d.f. of \( X \) is

\[
f_X(x) = \begin{cases} 
x, & \text{if } 0 < x < 1 \\
\frac{1}{2}, & \text{if } 1 < x < 2. \nonumber \end{cases}
\]

**Example 4.8**

Let \( X \) be an absolutely continuous type r.v. with p.d.f.

\[
f_X(x) = \begin{cases} 
k - |x|, & \text{if } |x| < \frac{1}{2} \\
0, & \text{otherwise}
\end{cases}
\]
where \( k \in \mathbb{R} \).

(i) Find the value of constant \( k \);

(ii) Evaluate: \( P(\{X < 0\}) \), \( P(\{X \leq 0\}) \), \( P(\{0 < X \leq \frac{1}{4}\}) \), \( P(\{0 \leq X < \frac{1}{4}\}) \) and \( P(\{-\frac{1}{8} \leq X \leq \frac{1}{4}\}) \);

(iii) Find the d.f. of \( X \).

Solution.

(i) Since \( f_X \) is a p.d.f.

\[
\int_{-\infty}^{\infty} f_X(x) \, dx = 1
\]

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} (k - |x|) \, dx = 1
\]

\[
\Rightarrow k = \frac{5}{4}.
\]

Also, for \( k = \frac{5}{4}, f_X(x) \geq 0, \forall x \in \mathbb{R} \).

(ii) Since the r.v. \( X \) is of absolutely continuous type, \( P(\{X = x\}) = 0, \forall x \in \mathbb{R} \) (see Remark 4.2 (iv)). Therefore

\[
P(\{X < 0\}) = P(\{X \leq 0\}) = \int_{-\infty}^{0} f_X(x) \, dx = \int_{-\frac{1}{2}}^{0} \left( \frac{5}{4} + x \right) \, dx = \frac{1}{2},
\]

\[
P(\{0 < X \leq \frac{1}{4}\}) = P(\{0 \leq X < \frac{1}{4}\}) = \int_{0}^{\frac{1}{4}} f_X(x) \, dx = \int_{0}^{\frac{1}{4}} \left( \frac{5}{4} - x \right) \, dx = \frac{9}{32},
\]

and

\[
P\left(-\frac{1}{8} \leq X \leq \frac{1}{4}\right) = \int_{-\frac{1}{8}}^{\frac{1}{4}} f_X(x) \, dx
\]

\[
= \int_{-\frac{1}{8}}^{0} \left( \frac{5}{4} + x \right) \, dx + \int_{0}^{\frac{1}{4}} \left( \frac{5}{4} - x \right) \, dx
\]
Clearly, for $x < -\frac{1}{2}$, $F_X(x) = 0$ and, for $x \geq \frac{1}{2}$, $F_X(x) = 1$. For $-\frac{1}{2} \leq x < 0$,
\[
F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt
\]
\[
= \int_{-\frac{1}{2}}^{x} \left( \frac{5}{4} + t \right) \, dt
\]
\[
= \frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}
\]
and, for $0 \leq x < \frac{1}{2}$,
\[
F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt
\]
\[
= \int_{-\frac{1}{2}}^{0} \left( \frac{5}{4} + t \right) \, dt + \int_{0}^{x} \left( \frac{5}{4} - t \right) \, dt
\]
\[
= -\frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}.
\]
Therefore the d.f. of $X$ is
\[
F_X(x) = \begin{cases} 
0 & \text{if } x < -\frac{1}{2} \\
-\frac{x|x|}{2} + \frac{5}{4}x + \frac{1}{2}, & \text{if } -\frac{1}{2} \leq x < \frac{1}{2} \\
1, & \text{if } x \geq \frac{1}{2}.
\end{cases}
\]

Theorem 4.4

Let $F_X$ be the distribution function of a random variable $X$. Then $F_X$ can be decomposed as $F_X(x) = \alpha F_d(x) + (1 - \alpha) F_c(x)$, $x \in \mathbb{R}$, where $\alpha \in [0,1]$ , $F_d$ is a distribution function of some random variable of discrete type and $F_c$ is a distribution function of some random variables of continuous type.

Proof. Let $D_X$ denote the set of discontinuity points of $F_X$. We will prove the result for the case when $D_X$ is finite. The idea of the proof for the case when $D_X$ is countably infinite is similar but slightly involved. First suppose that $D_X = \phi$. In this case the result
follows trivially by taking $\alpha = 0$ and $F_c \equiv F_X$. Now suppose that $D_X = \{a_1, a_2, ..., a_n\}$ for some $n \in \mathbb{N}$. Without loss of generality let $-\infty < a_1 < a_2 < \cdots < a_n < \infty$.

Define

$$p_i = P(\{X = a_i\}) = F_X(a_i) - F_X(a_i -), \quad i = 1, 2, ..., n,$$

so that $p_i > 0$, $i = 1, ..., n$.

Let $\alpha = \sum_{i=1}^{n} p_i$ so that $\alpha \in (0, 1]$. Define $F_d : \mathbb{R} \to \mathbb{R}$ by

$$F_d(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{\sum_{i=1}^{i} p_j}{\alpha}, & \text{if } a_i \leq x < a_{i+1}, \quad i = 1, ..., n - 1. \\ 1, & \text{if } x \geq a_n \end{cases}$$

Clearly $F_d$ is non-decreasing, right continuous $F_d(-\infty) = 0$ and $F_d(\infty) = 1$. The set of discontinuity points of $F_d$ is $\{a_1, ..., a_n\}$ and

$$\sum_{i=1}^{n} [F_d(a_i) - F_d(a_i -)] = \sum_{i=1}^{n} \left\{ \frac{\sum_{j=1}^{i} p_j}{\alpha} - \frac{\sum_{j=1}^{i-1} p_j}{\alpha} \right\} = \frac{1}{\alpha} \sum_{i=1}^{n} p_i = 1.$$

It follows that $F_d$ is a d.f. of some r.v. of discrete type. If $\alpha = 1$ then the result follows on taking $F_d \equiv F_X$. Now suppose that $\alpha \in (0, 1)$.

Define $F_c : \mathbb{R} \to \mathbb{R}$ by

$$F_c(x) = \frac{F_X(x) - \alpha F_d(x)}{1 - \alpha}, \quad x \in \mathbb{R}.\]$$

For $A \subseteq \mathbb{R}$, let $S(A) = \{i \in \{1, ..., n\}: a_i \in A\}$. Then, for $-\infty < x < y < \infty$,

$$F_d(y) - F_d(x) = \sum_{i \in S((-\infty, y])} \frac{p_i}{\alpha} - \sum_{i \in S((-\infty, x])} \frac{p_i}{\alpha},$$

$$F_X(y) - F_X(x) = P(\{x < X \leq y\})$$
\[
\sum_{i \in S(x,y)} p_i \geq \alpha(F_d(y) - F_d(x)),
\]

where, for \( A \subseteq \mathbb{R} \), \( \sum_{i \in S(A)} p_i = 0 \), if \( S(A) = \emptyset \).

Therefore, for \(-\infty < x < y < \infty\),
\[
F_c(y) - F_c(x) = \frac{F_X(y) - F_X(x) - \alpha(F_d(y) - F_d(x))}{1 - \alpha} \geq 0,
\]
i.e., \( F_c \) is non-decreasing. Note that \( F_X(a_i) - F_X(a_i -) = \alpha(F_d(a_i) - F_d(a_i -)) = p_i, i = 1, ..., n \) and \( F_X(x) - F_X(x -) = 0 \), if \( x \notin \{a_1, ..., a_n\} \). It follows that
\[
F_c(x) - F_c(x -) = \frac{F_X(x) - F_X(x -) - \alpha(F_d(x) - F_d(x -))}{1 - \alpha} = 0, \quad \forall x \in \mathbb{R},
\]
i.e., \( F_c \) is continuous everywhere. Since \( F_X(-\infty) = F_d(-\infty) = 0 \) and \( F_X(\infty) = F_d(\infty) = 1 \) we also have \( F_c(-\infty) = 0 \) and \( F_c(\infty) = 1 \). Therefore \( F_c \) is a d.f. of some r.v. of continuous type. Hence the result follows.

**Example 4.9**

Let \( X \) be a r.v. having the d.f. \( F_X \) (see Example 3.2 (iii)) given by

\[
F_X(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{x}{4}, & \text{if } 0 \leq x < 1 \\
\frac{x}{3}, & \text{if } 1 \leq x < 2 \\
\frac{3x}{8}, & \text{if } 2 \leq x < \frac{5}{2} \\
1, & \text{if } x \geq \frac{5}{2}
\end{cases}
\]

Decompose \( F_X \) as \( F_X(x) = \alpha H_d(x) + (1 - \alpha) H_c(x), x \in \mathbb{R} \), where \( \alpha \in [0,1] \), \( H_d \) is a d.f. of some r.v. \( X_d \) of discrete type and \( H_c \) is a d.f. of some r.v. \( X_c \) of continuous type.

**Solution**. The set of discontinuity points \( F_X \) is \( D_X = \{1, 2, 5/2\} \) with
\[
p_1 = P\{X = 1\} = F_X(1) - F_X(1-) = \frac{1}{12}.
\]
\[ p_2 = P\{X = 2\} = F_X(2) - F_X(2-) = \frac{1}{12} \]

and

\[ p_3 = P\left(\left\{X = \frac{5}{2}\right\}\right) = F_X\left(\frac{5}{2}\right) - F_X\left(\frac{5}{2}-\right) = \frac{1}{16}. \]

Thus,

\[ \alpha = p_1 + p_2 + p_3 = \frac{11}{48}. \]

\[ P\left(\left\{X_d = 1\right\}\right) = \frac{p_1}{\alpha} = \frac{4}{11}, \quad P\left(\left\{X_d = 2\right\}\right) = \frac{p_2}{\alpha} = \frac{4}{11}, \quad P\left(\left\{X_d = \frac{5}{2}\right\}\right) = \frac{p_3}{\alpha} = \frac{3}{11}, \]

\[ H_d(x) = \begin{cases} 0, & \text{if } x < 1 \\ \frac{4}{11}, & \text{if } 1 \leq x < 2 \\ \frac{8}{11}, & \text{if } 2 \leq x < \frac{5}{2} \\ 1, & \text{if } x \geq \frac{5}{2} \end{cases} \]

and

\[ H_c(x) = \frac{H(x) - \alpha H_d(x)}{1 - \alpha} \]

\[ = \begin{cases} 0, & \text{if } x < 0 \\ \frac{12}{37} x, & \text{if } 0 \leq x < 1 \\ \frac{4(4x-1)}{37}, & \text{if } 1 \leq x < 2 \\ \frac{2(9x-4)}{37}, & \text{if } 2 \leq x < \frac{5}{2} \\ 1, & \text{if } x \geq \frac{5}{2} \end{cases}. \]
Figure 4.5. Plot of distribution function $H_d(x)$

Figure 4.6. Plot of distribution function $H_c(x)$