1.2 Well Ordering Principle and the Principle of Mathematical Induction

Axiom 1.2.1 (Well-Ordering Principle)**.** *Every non-empty subset of natural numbers contains its least element.*

We will use Axiom 1.2.1 to prove the weak form of the principle of mathematical induction. The proof is based on contradiction. That is, suppose that we need to prove that "whenever the statement P holds true, the statement Q holds true as well". A proof by contradiction starts with the assumption that "the statement P holds true and the statement Q does not hold true" and tries to arrive at a contradiction to the validity of the statement P being true.

Theorem 1.2.2 (Principle of Mathematical Induction: Weak Form)**.** *Let $P(n)$ be a statement about a positive integer n such that*

1. *$P(1)$ is true, and*
2. *$P(k+1)$ is true whenever one assumes that $P(k)$ is true.*

Then $P(n)$ is true for all positive integer n.

Proof. On the contrary, assume that there exists $n_0 \in \mathbb{N}$ such that $P(n_0)$ is not true. Now, consider the set

$$S = \{m \in \mathbb{N} \ : \ P(m) \ \text{ is false } \}.$$

As $n_0 \in S$, $S \neq \emptyset$. So, by Well-Ordering Principle, S must have a least element, say N. By assumption, $N \neq 1$ as $P(1)$ is true. Thus, $N \geq 2$ and hence $N - 1 \in \mathbb{N}$.

Therefore, from the assumption that N is the least element in S and S contains all those $m \in \mathbb{N}$ for which $P(m)$ is false, one deduces that $P(N-1)$ holds true as $N - 1 < N \leq 2$. Thus, the implication "$P(N-1)$ is true" and Hypothesis 2 imply that $P(N)$ is true.

This leads to a contradiction and hence our first assumption that there exists $n_0 \in \mathbb{N}$, such that $P(n_0)$ is not true is false. ∎

Example 1.2.3. 1. *Prove that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.*

Solution: *Verify that the result is true for $n = 1$. Hence, let the result be true for n. Let us now prove it for $n + 1$. That is, one needs to show that $1 + 2 + \cdots + n + (n+1) = \dfrac{(n+1)(n+2)}{2}$.*

Using Hypothesis 2,

$$1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n+1}{2}\,(n+2)\,.$$

Thus, by the principle of mathematical induction, the result follows.

2. *Prove that* $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

   Solution: *The result is clearly true for* $n = 1$. *Hence, let the result be true for* n *and one needs to show that* $1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \dfrac{(n+1)(n+2)\left(2(n+1)+1\right)}{6}$.

   Using Hypothesis 2,

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n+1}{6}\left(n(2n+1) + 6(n+1)\right) \\ &= \frac{n+1}{6}\left(2n^2 + 7n + 6\right) = \frac{(n+1)(n+2)\left(2n+3\right)}{6}. \end{aligned}$$

   Thus, by the principle of mathematical induction, the result follows.

3. *Prove that for any positive integer* n, $1 + 3 + \cdots + (2n-1) = n^2$.

   Solution: *The result is clearly true for* $n = 1$. *Let the result be true for* n. *That is,* $1 + 3 + \cdots + (2n-1) = n^2$. *Now, we see that*

$$1 + 3 + \cdots + (2n-1) + (2n+1) = n^2 + (2n+1) = (n+1)^2.$$

   Thus, by the principle of mathematical induction, the result follows.

4. **AM-GM Inequality:** *Let* $n \in \mathbb{N}$ *and suppose we are given real numbers* $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. *Then*

   Arithmetic Mean (AM) $:= \dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n} =:$ *(GM) Geometric Mean.*

   Solution: *The result is clearly true for* $n = 1, 2$. *So, we assume the result holds for any collection of* n *non-negative real numbers. Need to prove AM* $\geq$ *GM, for any collection of non-negative integers* $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq 0$.

   So, let us assume that $A = \dfrac{a_1 + a_2 + \cdots + a_n + a_{n+1}}{n+1}$. *Then, it can be easily verified that* $a_1 \geq A \geq a_{n+1}$ *and hence* $a_1 - A$, $A - a_{n+1} \geq 0$. *Thus,* $(a_1 - A)(A - a_{n+1}) \geq 0$. *Or equivalently,*

$$A(a_1 + a_{n+1} - A) \geq a_1 a_{n+1}. \tag{1.1}$$

   Now, according to our assumption, the AM-GM inequality holds for any collection of n *non-negative numbers. Hence, in particular, for the collection* $a_2, a_3, \ldots, a_n, a_1 + a_{n+1} - A$. *That is,*

$$AM \;=\; \frac{a_2 + \cdots + a_n + (a_1 + a_{n+1} - A)}{n} \geq \sqrt[n]{a_2 \cdots a_n \cdot (a_1 + a_{n+1} - A)} = \; GM. \tag{1.2}$$

   But $\dfrac{a_2 + a_3 + \cdots + a_n + (a_1 + a_{n+1} - A)}{n} = A$. *Thus, by Equation (1.1) and Equation (1.2), one has*

$$A^{n+1} \geq (a_2 \cdot a_3 \cdot \cdots \cdot a_n \cdot (a_1 + a_{n+1} - A)) \cdot A \geq (a_2 \cdot a_3 \cdot \cdots \cdot a_n)\, a_1 a_{n+1}.$$

Therefore, we see that by the principle of mathematical induction, the result follows.

5. *Fix a positive integer n and let A be a set with $|A| = n$. Then prove that $\mathcal{P}(A) = 2^n$.*

   Solution: *Using Example 1.1.7, it follows that the result is true for $n = 1$. Let the result be true for all subset A, for which $|A| = n$. We need to prove the result for a set A that contains $n+1$ distinct elements, say $a_1, a_2, \ldots, a_{n+1}$.*

   Let $B = \{a_1, a_2, \ldots, a_n\}$. Then $B \subseteq A$, $|B| = n$ and by induction hypothesis, $|\mathcal{P}(B)| = 2^n$. Also, $\mathcal{P}(B) = \{S \subseteq \{a_1, a_2, \ldots, a_n, a_{n+1}\} : a_{n+1} \not\in S\}$. Therefore, it can be easily verified that
   $$\mathcal{P}(A) = \mathcal{P}(B) \cup \{S \cup \{a_{n+1}\} : S \in \mathcal{P}(B)\}.$$
   Also, note that $\mathcal{P}(B) \cap \{S \cup \{a_{n+1}\} : S \in \mathcal{P}(B)\} = \emptyset$, as $a_{n+1} \not\in S$, for all $S \in \mathcal{P}(B)$. Hence, using Examples 1.1.5.4 and 1.1.5.6, we see that
   $$|\mathcal{P}(A)| = |\mathcal{P}(B)| + |\{S \cup \{a_{n+1}\} : S \in \mathcal{P}(B)\}| = |\mathcal{P}(B)| + |\mathcal{P}(B)| = 2^n + 2^n = 2^{n+1}.$$

   Thus, the result holds for any set that consists of $n+1$ distinct elements and hence by the principle of mathematical induction, the result holds for every positive integer n.

We state a corollary of the Theorem 1.2.2 without proof. The readers are advised to prove it for the sake of clarity.

Corollary 1.2.4 (Principle of Mathematical Induction). *Let $P(n)$ be a statement about a positive integer n such that for some fixed positive integer n_0,*

1. *$P(n_0)$ is true,*

2. *$P(k+1)$ is true whenever one assumes that $P(k)$ is true.*

Then $P(n)$ is true for all positive integer $n \geq n_0$.