# Module 9: Stationary Processes

## Lecture 1  Stationary Processes

### 1 Introduction

A stationary process is a stochastic process whose joint probability distribution does not change when shifted in time or space. For a stationary process, mean and variance, if they exist, do not change over time or position.

Stationarity is a key concept in time series analysis as it allows powerful techniques for modeling and forecasting to be developed. Time series is a set of data ordered in time usually recorded at regular time intervals of time. In probability theory a time series is a collection of random variable \( \{X(t), t \geq 0\} \) indexed by time.

One of the main features of time series is the interdependency of observation over time. This interdependency needs to be accounted in the time series data modeling to improve temporal behavior and forecast of future moment. So stationarity is used as a tool in time series analysis when raw data is often transformed to become stationary.

### 2 Important Definitions

**Definition 1. Mean function** It is defined as \( m(t) = E(X(t)) \), which may or may not be dependent on \( t \).

**Definition 2. Second Order Process** A stochastic process is called a second order process if it’s second order moment is finite for all \( t \) i.e., \( E(X^2(t)) < \infty \).

**Definition 3. Covariance function** It is denoted by \( C(s, t) \) given by

\[
C(s, t) = \text{cov}(X(s), X(t)) = E(X(s)X(t)) - E(X(s))E(X(t))
\]

A Stochastic process has to be a second order process for covariance function to exist. The Covariance function satisfies the following properties:

1. \( C(s, t) = C(t, s) \forall t, s \in T \)
2. Using Schwarz inequality

\[ C(s, t) \leq \sqrt{C(s,s)C(t,t)} \]

3. It is non negative definite i.e, for a set of real numbers \(a_1, a_2, a_3, \ldots, a_n\) and \(t_i \in T\)

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k C(t_j, t_k) = E[\sum a_j X(t_j)^2] \geq 0
\]

4. Sum and product also covariance functions.

**Definition 4. Auto-Correlation function** It is denoted by \(R(s, t)\), given by:

\[
R(s, t) = \frac{E[X(s)X(t)] - E[X(s)]E[X(t)]}{\sqrt{\text{var}(X(s))\text{var}(X(t))}}
\]

Now assuming, \(R(s, t)\) depends only on \(|t - s|\), \(E(X(t)) = \mu\) and \(\text{Var}(X(t)) = \sigma^2\)
we get,

\[
R(\tau) = \frac{(E(X(t)) - \mu)(E(X(t + \tau) - \mu))}{\sigma^2}
\]

**Definition 5. Independent Increments** If for every \(t_1 < t_2 < \ldots < t_n\)

\[X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})\]

are mutually independent random variables \(\forall n\).

**Definition 6. Ergodic Property** It is the time average of a function along a realization or sample exist almost everywhere and is related to the space average. That means whenever the stochastic process is ergodic the time average is same for all almost initial points i.e., the process evolved for a longer time forgets its initial state.

3 Strict Sense and Wide Sense Stationary Process

**Definition 7. Strict Sense Stationary** If for arbitrary \(t_1 < t_2 < \ldots < t_n\) the joint distribution of the random vectors \(\{X(t_1), X(t_2), \ldots, X(t_n)\}\) and \(\{X(t_1+h), X(t_2+h), \ldots, X(t_n+h)\}\) is same \(\forall h\) then the stochastic process \(\{X(t), t \in T\}\) is said to be strict sense stationary of order \(n\). If the above definition holds for every integer \(n\) then the process is called a strict sense stationary Process.
**Definition 8. Wide Sense Stationary Process** A stochastic process is said to be wide sense stationary or weakly stationary or covariance stationary if the following properties hold-

- \( m(t) = E[X(t)] \) is independent of \( t \).
- \( E[X^2(t)] < \infty \).
- \( c(s, t) \) is a function of the time difference \( |t - s| \) only i.e.,
  \[ c(s, t) = f(t - s) \forall t, s. \]

Also neither a strict sense stationary process imply the wide sense stationary nor wide sense stationary imply the strict sense stationary process.

### 4 Examples of Stationary Process

**Example 9.** Let \( X_i \) be iid \( B(1, p) \) random variables. Then for the stochastic process \( \{X_i, i = 1, 2, \ldots\} \) the following can be seen \( m(i) = E[X_i] = p \)

\[ E[X_i^2] = p \]

\[ c(i, j) = E[X_iX_j] - E[X_i]E[X_j] \]

\[ = \begin{cases} 
0, & i \neq j \\
p(1 - p), & i = j.
\end{cases} \]

Since all three properties of wide sense stationary process are satisfied and hence the above process is a wide sense stationary process.

Now we consider the joint distribution of the vectors \( \{X_{i_1}, X_{i_2}, \ldots, X_{i_n}\} \) and \( \{X_{i_1+h}, X_{i_2+h}, \ldots, X_{i_n+h}\} \).

Since the random variables \( X_i \) are iid random variables, therefore the joint distribution of the two vectors will be the product of the distribution of the random variables and hence the joint distribution for the 2 random vectors will be the same. Hence the process \( \{X_i, i = 1, 2, \ldots\} \) is strict sense stationary process. So in this case the process \( \{X_i, i = 1, 2, \ldots\} \) is strict sense as well as wide sense stationary process.

**Example 10.** Let \( \{X(t), t \in T\} \) be a strict sense stationary process with finite second order moment and let \( Y(t) = a + bt + X(t) \), so now we check whether \( \{Y(t), t \in \)
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$T \} \text{ is a wide sense and strict sense stationary process.}$

The function $m(t) = E[Y(t)]$ will be a function of $t$ an hence $\{Y(t), t \in T\}$ is not a wide sense stationary process. Similarly it can be verified that the process $\{Y(t), t \in T\}$ is not a strict sense stationary process.

**Example 11.** Let $\{X_n, n = 1, 2, \ldots\}$ be uncorrelated random variables with $E[X_n] = k$, where $k$ is a constant and

\[
E[X_mX_n] = \begin{cases} 
\sigma^2 & \text{if } m = n \\
0 & \text{if } m \neq n
\end{cases}
\]

It can be verified that all the three conditions of wide sense stationary will be satisfied and hence it's a wide sense stationary process. This process is also called the white noise process.

If for a stochastic process $\{X(t), t \in T\}$ we assume it to have the ergodic property and be a wide sense stationary process then the mean of this process can be estimated from the time average i.e., $\hat{\mu}_T = \frac{1}{T} \int_{-T}^{T} X(t) \, dt$

**Example 12.** Consider the process $\{X(t), t \geq 0\}$, $X(t)$ is defined as:

\[X(t) = A \cos(\theta t) + B \sin(\theta t),\]

where, $A$ and $B$ are known to be uncorrelated random variables with mean and variance as 0 and $\sigma^2$ respectively and $\theta$ is a constant.

This, $X(t)$ is a random variable for fixed $t$

Now we check whether the given process is a wide sense stationary process:

For a fixed $t$, as mean of $A$ and $B$ is 0, $E(X(t)) = 0$.

\[C(s,t) = E(X(t)X(s)) - E(X(t))E(X(s)) = \sigma^2 \cos(\theta(t - s)), \]

which is a function of $(t - s)$.

\[E(X^2(t)) = \sigma^2.\]

From the above three conditions it can be concluded that the given process is wide sense stationary.

**Example 13.** This comes under time series forecasting.
Suppose we are given time series data $X_0, X_1, \ldots, X_n$.

Aim is to forecast for $X_{n+1}$ with the given $n+1$ data. By some function $f$ we can predict $X_{n+1}$ such that the mean square error is minimum. It turns out that the function $f$ is:

$$E[X_{n+1}/X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n].$$

If we assume that $X_i$’s are normally distributed then the above conditional expectation will be a linear function of the data of $X_0, X_1, \ldots, X_n$.

In general the above conditional expectation can be estimated using the estimator:

$$\hat{X}_{n+1} = a_0 x_0 + a_1 x_1 + \ldots + a_n * x_n$$

where $a_0, a_1, \ldots, a_n$ are parameters which are to be estimated. Now we can assume that the given time series satisfies the wide sense stationary and hence the parameters are invariant with time and therefore they can be estimated from the data itself.

**Example 14.** In this example we consider the Gaussian process, $\{W(t), t \geq 0\}$ which is a stochastic process wherein every random variable is normally distributed, i.e., $W(t) \sim$ normal distributed or we say that the joint distribution $\{W(t_1), W(t_2), \ldots, W(t_n)\} \sim$ multivariate normal distribution.

The distribution of the above random vector can be completely determined by the mean vector

$$E[W(t_1), W(t_2), \ldots, W(t_n)],$$

and the covariance matrix, $[C_{ij}]$, where $C_{ij} = \text{cov}(X_i, X_j)$, $i = 1, 2, 3, \ldots, n, j = 1, 2, 3, \ldots, n$

Now we determine whether the above process is the covariance or wide sense stationary.

If $\{W(t), t \geq 0\}$ satisfies the following 3 properties, i.e.,

1. $E[W(t)]$ is independent of $t$.
2. $E[W^2(t)] < \infty$.
3. $\text{cov}(W_{t_i}, W_{t_j})$ is a function of $|t_i - t_j|$.
then the given stochastic process will be wide sense stationary process.

Also then we can conclude that it will be a strict sense stationary process because it depends only on the joint distribution and mean vector and the covariance matrix.