DEFINITION 1. Let \( \{X(t), 0 \leq t \leq T\} \) be a stochastic process. Let \( \{X(t), 0 \leq t \leq T\} \) be adapted to the natural filtration \( \{\mathcal{F}(t), t \geq 0\} \) of Wiener process \( \{W(t), 0 \leq t \leq T\} \), i.e., \( X(t) \) be \( \mathcal{F}(t) \)-measurable. Define

\[
I(t) = \int_0^t X(u)dW(u), \quad 0 \leq t \leq T
\]

(1)
a stochastic integral with respect to a Wiener process. The above integral is called the Ito integral.

Ito Process

DEFINITION 2. Let \( \{W(t), t \geq 0\} \) be a Brownian motion and let \( \{\mathcal{F}(t), t \geq 0\} \) be an associated natural filtration. An Ito process is a stochastic process \( \{X(t), t \geq 0\} \) of the form

\[
X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du
\]

where \( X(0) \) is a non-random and \( \Delta(u) \) and \( \Theta(u) \) are adapted processes and \( \Delta(u) \) is mean square integrable.

It is now easy to write a stochastic differential equation form of an Ito process which is

\[
dX(t) = \Delta(t)dW(t) + \Theta(t)dt.
\]

All stochastic processes with no jumps are actually Ito processes. Ito integral and Brownian motion are examples of Ito processes.

If \( X(t) \) and \( W(t) \) are not stochastic processes, but rather deterministic functions. Assuming \( f(s) \) is a differential function and \( g(s) \) is a smooth function,

\[
\int_0^t g(s)df(s) = \int_0^t g(s)f'(s)ds
\]

which is a Reimann integral. But what happens if \( f(s) \) is not differentiable.

We can still define the integral. When \( f(s) \) is bounded variation, we can prove that the integral is well-defined. Further, if the function \( g(s) \) is extremely fluctuating at different points in time, the limit may still diverge.
Since \( f(s) \) has bounded variation, one can prove that the limit exists as long as \( g(s) \) is not varying too much and is given by

\[
\int_0^t g(s) \, df(s) = \lim_{n \to \infty} \sum_{i=0}^{n-1} g(s_i) \left( f(s_{i+1}) - f(s_i) \right).
\]

Remember that, in the integral (1), \( X(t) \) and \( W(t) \) are stochastic processes as well as \( W(t) \) is unbounded variation and is nowhere differentiable. Hence, it is different from Reimann integral. Now, we rewrite the Ito integral (1) in the above form as:

\[
I(t) = \lim_{n \to \infty} \sum_{i=0}^{n-1} X(s_i) \left( W(s_{i+1}) - W(s_i) \right).
\]

**Ito-Integrable**

A stochastic process \( \{X(t), t \geq 0\} \) is called Ito-integrable on \([0, t]\) if it satisfies

1. \( X(t) \) is adapted, i.e., \( X(s) \) is \( F(s) \) - measurable, \( 0 \leq s \leq t \).
2. \( \int_0^t E(X^2(s)) \, ds < \infty \).

**Example 3.** Consider the Ito Integral for simple integrand function. Consider the illustration with price may be negative. Let \( W(t) \) be the price per share of an asset at time (position) \( t \), \( X(t) \) be the number of shares taken in the asset at a time \( t \) and \( t_i \) be the trading dates of an asset. Assume \( X(t) \) be the simple process. It means \( X(t) \) is a constant in each \( [t_i, t_{i+1}] \). The gain/loss from trading at each time \( t \) can be viewed as \( I(t) \).

Since \( \{X(t), 0 \leq t \leq T\} \) is a simple process, it can be written as

\[
X(t, w) = \phi_0(w)1_{\{0\}}(t) + \lim_{p \to \infty} \sum_{i=1}^p \phi_i(w)1_{(t_{i-1}, t_i]}(t) \quad \forall \ w \in \Omega
\]

where \( \phi_i \) are bounded random variables such that \( \phi_0 \) is \( \mathcal{F}(0) \)-measurable, \( \phi_i \) is \( \mathcal{F}(t_{i-1}) \)-measurable and \( 0 < t_0 < t_1 < \ldots < t_p = T, p \in \mathbb{N} \). For a simple process \( \{X(t), t \in [0, T]\} \), the stochastic integral \( I(t) \) is defined by

\[
I(t) = \int_0^t X(s) \, dW(s) = \lim_{p \to \infty} \sum_{i=1}^p \phi_i(W(t_i) - W(t_{i-1})).
\]
In particular $X(t) = 1, \ t \in [0,T]$, hence $\phi_i = 1, \ 1 \leq i \leq p, \ p \in \mathcal{N}$, we have, for $t \in [0,T]$
\[ I(t) = \int_0^t dW(s) = \lim_{p \to \infty} \sum_{i=1}^{p} \left( W(t_i) - W(t_{i-1}) \right) = W(t). \]

**Example 4.** Consider Ito integral of a deterministic integrand. Let $\{W(t), t \geq 0\}$ be a Brownian motion and let $\Delta(t)$ be a non-random function of time. Define $I(t) = \int_0^t \Delta(s)dW(s)$.
\[ E(I(t)) = I(0) = 0 \quad \text{(martingale)} \]
\[ E(I^2(t)) = \int_0^t \Delta^2(s)ds \]
\[ \text{Var}(I(t)) = \int_0^t \Delta^2(s)ds/ \]
From moment generating function, we get
\[ E(e^{uI(t)}) = e^{\frac{1}{2}u^2 \int_0^t \Delta^2(s)ds}. \]
Hence, for each $t \geq 0$ the random variable $I(t)$ is normally distributed with mean zero and variance $\int_0^t \Delta^2(s)ds$
\[ i.e., I(t) \sim \mathcal{N} \left( 0, \int_0^t \Delta^2(s)ds \right). \]

**Example 5.** Evaluate
\[ \int_0^t W(1)dW(t), \ 0 \leq t \leq 1. \]
Note that, $W(1)$ is not adapted to the filtration $\sigma\{W(s), 0 < s \leq t\}, 0 \leq t \leq 1$. Hence, Ito integral does not exist. This example shows that, assumption of the integrand is adapted to the filtration $\{\mathcal{F}(t), t \geq 0\}$ is need to have existence of the Ito integral.

**Example 6.** Evaluate the Ito integral
\[ \int_0^T W(t)dW(t) \]
By using the definition, we get $0 < t_0 < t_1 < \ldots < t_n = T, n \in \mathbb{N}$.

$$\int_0^T W(t) dW(t) = \lim_{n \to \infty} \sum_{i=0}^{n-1} W(t_i) (W(t_{i+1}) - W(t_i)).$$

Note that, for each $i$, $W(t_i)$ and $W(t_{i+1}) - W(t_i)$ are independent variables and are having normal distributions. Let $\Pi$ be the set of all finite subdivisions of $\pi$ of the interval $[0, T]$ with $0 < t_0 < t_1 < \ldots < t_n = T$.

$$Q_\Pi = \sum_{j=0}^{n-1} (W(t_{i+1}) - W(t_i))^2$$

$$= \sum_{j=0}^{n-1} ((W(t_{i+1})^2 - (W(t_i)^2) - 2W(t_i)(W(t_{i+1}) - W(t_i)))$$

$$= (W(T))^2 - (W(0))^2 - 2 \sum_{i=0}^{n-1} W(t_i)(W(t_{i+1}) - W(t_i))$$

$$\int_0^T W(t) dW(t) = \frac{(W(T))^2 - T}{2}.$$

Hence,

$$\int_0^T W(s) dW(s) = \frac{(W(T))^2 - T}{2}.$$

We see that, unlike the Reimann integral, we have an extra term $T$. This arises from the quadratic variation of the Brownian motion which is finite. The above integral is defined not only for the upper limit of integration $T$ but also for every upper limit of integration between 0 and $t$. Construction of the Ito integral is similar to one of the Stieltjes integration. But instead of integrity with respect to a deterministic function (in Stieltjes integral) Ito’s integral with respect to a random function, more precisely path of a Wiener process.

Properties of Ito Integral

The following results hold by the Ito integral defined in equation (1).

1. The integral $I(t)$ is a martingale with respect to $\{F(t), t \geq 0\}$

2. $E(I(t)) = E(\int_0^t X(s)dW(s)) = 0$. 
3. \( E[(I(t))^2] = E[\int_0^t X^2(s)ds] = \int_0^t E(X^2(s))ds \) for all \( t \). This is called Ito isometry.

4. \( \text{Var}(I(t)) = \int_0^t E(X^2(s))ds \)

5. Quadratic variation is given by

\[
[I, I](t) = \int_0^t X^2(s)ds.
\]

The Ito integral is a random variable \( I(t) \), for all \( w \in \Omega \)

\[
I(t)(w) = \int_0^t X(s, w)dW(s, w) = \lim_{n \to \infty} \sum_{i=0}^{n-1} X(s_i, w)(W(s_{i+1}, w) - W(s_i, w)).
\]

Actually the convergence takes place only for a subsequence. Its integral depends on the sample path. Note that

\[
dW(t)dW(t) = dt
\]

\[
dtdt = 0
\]

\[
dW(t)dt = 0.
\]

\[
dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t)
\] (2)

The equation (2) is referred to as a stochastic differential equation. The interpretation of (2) tells us that the change \( dX(t) = X(t + \Delta t) - X(t) \) is caused by a change \( dt \) of time, with factor \( b(t, X(t)) \) in combination with a change \( dW(t) = W(t + \Delta t) - W(t) \) of Brownian motion with factor \( \sigma(t, X(t)) \).