Lecture 3  Surfaces and Integral Curves

In Lecture 3, we recall some geometrical concepts that are essential for understanding the nature of solutions of partial differential equations to be discussed in the subsequent lectures.

**Surface: A surface is the locus of a point moving in space with two degrees of freedom.** Generally, we use *implicit* and *explicit* representations for describing such a locus by mathematical formulas.

In the implicit representation we describe a surface as a set
$$S = \{(x, y, z) \mid F(x, y, z) = 0\},$$
i.e., a set of points \((x, y, z)\) satisfying an equation of the form \(F(x, y, z) = 0\).

Sometimes we can solve such an equation for one of the coordinates in terms of the other two, say for \(z\) in terms of \(x\) and \(y\). When this is possible we obtain an explicit representation of the form \(z = f(x, y)\).

**Example 1.** A sphere of radius 1 and center at the origin has the implicit representation
$$x^2 + y^2 + z^2 - 1 = 0.$$

When this equation is solved for \(z\) it leads to two solutions:
$$z = \sqrt{1 - x^2 - y^2} \quad \text{and} \quad z = -\sqrt{1 - x^2 - y^2}.$$

The first equation gives an explicit representation of the upper hemisphere and the second of the lower hemisphere.

We now describe here a class of surfaces more general than surfaces obtained as graphs of functions. For simplicity, we restrict the discussion to the case of three dimensions.

Let \(\Omega \subset \mathbb{R}^3\) and let \(F(x, y, z) \in C^1(\Omega)\), where \(C^1(\Omega) := \{F(x, y, x) \in C(\Omega) : F_x, F_y, F_z \in C(\Omega)\}\). We know the gradient of \(F\), denoted by \(\nabla F\), is a vector valued function defined by
$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right).$$

One can visualize \(\nabla F\) as a field of vectors (vector fields), with one vector, \(\nabla F\), emanating from each point \((x, y, z) \in \Omega\). Assume that
$$\nabla F(x, y, z) \neq (0, 0, 0), \quad \forall x \in \Omega.$$  \(\quad (1)\)
This means that the partial derivatives of $F$ do not vanish simultaneously at any point of $\Omega$.

**Definition 2. (Level surface)** The set

$$S_c = \{(x, y, z) \mid (x, y, z) \in \Omega \text{ and } F(x, y, z) = c\},$$

for some appropriate value of the constant $c$, is a surface in $\Omega$. This surface is called a level surface of $F$.

**Note:** When $\Omega \subset \mathbb{R}^2$, the set $S_c = \{(x, y) \mid (x, y) \in \Omega \text{ and } F(x, y) = c\}$ is called a level curve in $\Omega$.

Let $(x_0, y_0, z_0) \in \Omega$ and set $c = F(x_0, y_0, z_0)$. The equation

$$F(x, y, z) = c$$

represents a surface in $\Omega$ passing through the point $(x_0, y_0, z_0)$. For different values of $c$, (2) represents different surfaces in $\Omega$. Each point of $\Omega$ lies on exactly one level surface of $F$. Any two points $(x_0, y_0, z_0)$ and $(x_1, y_1, z_1)$ of $\Omega$ lie on the same level surface if and only if

$$F(x_0, y_0, z_0) = F(x_1, y_1, z_1).$$

Thus, one may visualize $\Omega$ as being laminated by the level surfaces of $F$. The equation (2) represents one parameter family of surfaces in $\Omega$.

**Example 3.** Take $\Omega = \mathbb{R}^3 \setminus (0,0,0)$ and let $F(x, y, z) = x^2 + y^2 + z^2$. Then

$$\nabla F(x, y, z) = (2x, 2y, 2z).$$

Note that the condition (1) is satisfied $\forall (x, y, z) \in \Omega$. The level surfaces of $F$ are spheres with center at the origin.

**Example 4.** Take $\Omega = \mathbb{R}^3$. Then $\nabla F(x, y, z) = (0,0,1)$. The condition (1) is satisfied at every point of $\Omega$. The level surfaces are planes parallel to the $(x,y)$-plane.

Consider the surface given by the equation (2) and let the point $(x_0, y_0, z_0)$ lie on this surface. We now ask the following question: Is it possible to describe $S_c$ by an equation of the form

$$z = f(x, y),$$

so that $S_c$ is the graph of $f$? This is equivalent to asking whether it is possible to solve (2) for $z$ in terms of $x$ and $y$. An answer to this question is contained in the following theorem.
THEOREM 5. (Implicit Function Theorem)
If $F$ is defined within a sphere containing the point $(x_0, y_0, z_0)$, where $F(x_0, y_0, z_0) = 0$, $F_z(x_0, y_0, z_0) \neq 0$, and $F_x, F_y$ and $F_z$ are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines $z = f(x, y)$ near the point $(x_0, y_0, z_0)$.

EXAMPLE 6. Consider the unit sphere

$$x^2 + y^2 + z^2 = 1. \tag{4}$$

Note that the point $(0, 0, 1)$ lies on this surface and $F_z(0, 0, 1) = 2$. By the implicit function theorem, we can solve (4) for $z$ near the point $(0, 0, 1)$. In fact, we have

$$z = \sqrt{1 - x^2 - y^2}, \quad x^2 + y^2 < 1. \tag{5}$$

In the upper half space $z > 0$, (4) and (5) describe the same surface.

The point $(0, 0, -1)$ is also an on the surface (4) and $F_z(0, 0, -1) = -2$. Near $(0, 0, -1)$, we have

$$z = -\sqrt{1 - x^2 - y^2}, \quad x^2 + y^2 < 1. \tag{6}$$

In the lower half space $z < 0$, (4) and (6) represents the same surface.

On the other hand, at the point $(1, 0, 0)$, we have $F_z(1, 0, 0) = 0$. Clearly, it is not possible to solve (4) for $z$ in terms of $x$ and $y$ near this point.

Note that the set of points satisfying the equations

$$F(x, y, z) = c_1, \quad G(x, y, z) = c_2 \tag{7}$$

must lie on the intersection of these two surfaces. If $\nabla F$ and $\nabla G$ are not colinear at any point of the domain $\Omega$, where both $F$ and $G$ are defined, i.e.,

$$\nabla F(x, y, z) \times \nabla G(x, y, z) \neq 0, \quad (x, y, z) \in \Omega, \tag{8}$$

then the intersection of the two surfaces given by (7) is always a curve.

Since

$$\nabla F \times \nabla G = \left( \frac{\partial(F, G)}{\partial(y, z)}, \frac{\partial(F, G)}{\partial(z, x)}, \frac{\partial(F, G)}{\partial(x, y)} \right), \tag{9}$$

where the Jacobian

$$\frac{\partial(F, G)}{\partial(y, z)} = \frac{\partial F}{\partial y} \frac{\partial G}{\partial z} - \frac{\partial F}{\partial z} \frac{\partial G}{\partial y}.$$ 

The condition (8) means that at every point of $\Omega$ at least one of the Jacobian on the right side of (9) is different from zero.
Example 7. Let
\[ F(x, y, z) = x^2 + y^2 - z, \quad G(x, y, z) = z. \]

Note that \( \nabla F = (2x, 2y, -1) \) and \( \nabla G = (0, 0, 1) \). It is easy to see that if \( \Omega = \mathbb{R}^3 \) with the z-axis removed, then the condition (8) is satisfied in \( \Omega \). The pair of equations
\[ x^2 + y^2 - z = 0, \quad z = 1 \]
represents a circle which is the intersection of the paraboloidal surface represented by the first equation and the plane represented by the second equation.

**Systems of Surfaces:** A one-parameter system of surfaces is represented by the equation of the form
\[ f(x, y, z, c) = 0. \quad (10) \]
Consider the system of surfaces described by the equation
\[ f(x, y, z, c + \delta c) = 0, \quad (11) \]
corresponding to the slightly different value \( c + \delta c \).

Note that these two surfaces will intersect in a curve whose equations are (10) and (11). This curve may be considered to be intersection of the equations
\[ f(x, y, z, c) = 0, \quad \lim_{\delta c \to 0} \frac{f(x, y, z, c + \delta c) - f(x, y, z, c)}{\delta c}. \]
The limiting curve described by the set of equations
\[ f(x, y, z, c) = 0, \quad \frac{\partial}{\partial c} f(x, y, z, c) = 0. \quad (12) \]
is called the characteristic curve (cf. [10]) of (10).

**Remark 8.** Geometrically, it is the curve on the surface (10) approached by the intersection of (10) and (11) as \( \delta c \to 0 \). Note that as \( c \) varies, the characteristic curve (12) trace out a surface whose equation is of the form
\[ g(x, y, z) = 0. \]

**Definition 9.** (Envelope of one-parameter system)
The surface determined by eliminating the parameter \( c \) between the equations
\[ f(x, y, z, c) = 0, \quad \frac{\partial}{\partial c} f(x, y, z, c) = 0 \]
is called the envelope of the one-parameter system \( f(x, y, z, c) = 0 \).
Example 10. Consider the equation
\[ x^2 + y^2 + (z - c)^2 = 1. \]
This equation represents the family of spheres of unit radius with centers on the z-axis. Set
\[ f(x, y, z, c) = x^2 + y^2 + (z - c)^2 - 1. \]
Then \( \frac{\partial f}{\partial c} = z - c \). The set of equations
\[ x^2 + y^2 + (z - c)^2 = 1, \quad z = c \]
describe the characteristic curve to the surface. Eliminating the parameter \( c \), the envelope of this family is the cylinder
\[ x^2 + y^2 = 1. \]

Now consider the two parameter system of surfaces defined by the equation
\[ f(x, y, z, c, d) = 0, \quad (13) \]
where \( c \) and \( d \) are parameters.

In a similar way, the characteristics curve of the surface (13) passes through the point defined by the equations
\[ f(x, y, z, c, d) = 0, \quad \frac{\partial}{\partial c} f(x, y, z, c, d) = 0, \quad \frac{\partial}{\partial d} f(x, y, z, c, d) = 0. \]
This point is called the characteristics point of the two-parameter system (13). As the parameters \( c \) and \( d \) vary, this point generates a surface which is called the envelope of the surfaces (13).

Definition 11. (Envelope of two-parameter system)
The surface obtained by eliminating \( c \) and \( d \) from the equations
\[ f(x, y, z, c, d) = 0, \quad \frac{\partial}{\partial c} f(x, y, z, c, d) = 0, \quad \frac{\partial}{\partial d} f(x, y, z, c, d) = 0 \]
is called the envelope of the two-parameter system \( f(x, y, z, c, d) = 0 \).

Example 12. Consider the equation
\[ (x - c)^2 + (y - d)^2 + z^2 = 1, \]
where \( c \) and \( d \) are parameters. Observe that
\[ (x - c)^2 + (y - d)^2 + z^2 = 1, \quad x - c = 0, \quad y - d = 0. \]
The characteristics points of the two-parameter system (13) are \((c, d, \pm 1)\). Eliminating \( c \) and \( d \), the envelope is the pair of parallel planes \( z = \pm 1 \).
Integral Curves of Vector Fields: Let \( V(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)) \) be a vector field defined in some domain \( \Omega \subset \mathbb{R}^3 \) satisfying the following two conditions:

- \( V \neq 0 \) in \( \Omega \), i.e., the component functions \( P, Q \) and \( R \) of \( V \) do not vanish simultaneously at any point of \( \Omega \).
- \( P, Q, R \in C^1(\Omega) \).

**Definition 13.** A curve \( C \) in \( \Omega \) is an integral curve of the vector field \( V \) if \( V \) is tangent to \( C \) at each of its points.

**Example 14.**
1. The integral curves of the constant vector fields \( V = (1, 0, 0) \) are lines parallel to the \( x \)-axis (see Fig. 1.1).
2. The integral curves of \( V = (y, -x, 0) \) are circles parallel to the \((x, y)\)-plane and centered on the \( z \)-axis (see Fig. 1.1).

![Figure 1.1: Integral curves of \( V = (1, 0, 0) \) and \( V = (y, -x, 0) \)](image)

**Remark 15.** In physics, if \( V \) is a force field, the integral curves of \( V \) are called lines of force. If \( V \) is the velocity of the fluid flow, the integral curves of \( V \) are called lines of flow. These are the paths of motion of the fluid particles.

With \( V = (P, Q, R) \), associate the system of ODEs:

\[
\frac{dx}{dt} = P(x, y, z), \quad \frac{dy}{dt} = Q(x, y, z), \quad \frac{dz}{dt} = R(x, y, z). \tag{14}
\]

A solution \((x(t), y(t), z(t))\) of the system (14), defined for \( t \) in some interval \( I \), may be regarded as a curve in \( \Omega \). We call this curve a solution curve of the system (14). Every solution curve of the system (14) is an integral curve of the vector field \( V \). Conversely, if \( C \) is an integral curve of \( V \), then there is a parametric representation

\[
x = x(t), \quad y = y(t), \quad z = z(t); \quad t \in I,
\]
of $C$ such that $(x(t), y(t), z(t))$ is a solution of the system of equations (14). Thus, every integral curve of $V$, if parametrized appropriately, is a solution curve of the associated system of equations (14).

It is customary to write the systems (14) in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \quad (15)$$

**Example 16.** The systems associated with the vector fields $V = (x, y, x)$ and $V = (y, -x, 0)$, respectively, are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}, \quad (16)$$

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}. \quad (17)$$

Note that the zero which appears in the denominator of (17) should not be disturbing. It simply means that $dz/dx = dz/dy = dz/dt = 0$.

Before we discuss the method of solutions of (15), let us introduce some basic definitions and facts (cf. [11]).

**Definition 17.** Two functions $\phi(x, y, z), \psi(x, y, z) \in C^1(\Omega)$ are functionally independent in $\Omega \subset \mathbb{R}^3$ if

$$\nabla \phi(x, y, z) \times \nabla \psi(x, y, z) \neq 0, \quad (x, y, z) \in \Omega. \quad (18)$$

Geometrically, condition (18) means that $\nabla \phi$ and $\nabla \psi$ are not parallel at any point of $\Omega$.

**Definition 18.** A function $\phi \in C^1(\Omega)$ is called a first integral of the vector field $V = (P, Q, R)$ (or its associated system (15)) in $\Omega$, if at each point of $\Omega$, $V$ is orthogonal to $\nabla \phi$, i.e.

$$V \cdot \nabla \phi = 0 \quad \Rightarrow \quad P \frac{\partial \phi}{\partial x} + Q \frac{\partial \phi}{\partial y} + R \frac{\partial \phi}{\partial z} = 0 \quad in \ \Omega.$$

**Theorem 19.** Let $\phi_1$ and $\phi_2$ be any two functionally independent first integrals of $V$ in $\Omega$. Then the equations

$$\phi(x, y, z) = c_1, \quad \phi_2(x, y, z) = c_2 \quad (19)$$

describe the collection of all integrals of $V$ in $\Omega$. 
If \( \phi(x, y, z) \) is a first integral of \( \mathbf{V} \) and \( f(\phi) \) is a \( C^1 \) function of single variable \( \phi \) then 
\[ w(x, y, z) = f(\phi(x, y, z)) \]
is also a first integral of \( \mathbf{V} \). This follows from the fact that
\[
P \frac{\partial w}{\partial x} + Q \frac{\partial w}{\partial y} + R \frac{\partial w}{\partial z} = P f' \frac{\partial \phi}{\partial x} + Q f' \frac{\partial \phi}{\partial y} + R f' \frac{\partial \phi}{\partial z} = f' \left( P \frac{\partial \phi}{\partial x} + Q \frac{\partial \phi}{\partial y} + R \frac{\partial \phi}{\partial z} \right) = 0.
\]
Similarly, if \( f(u, v) \) is a \( C^1 \) function of two variables \( \phi_1 \) and \( \phi_2 \) and if \( \phi_1(x, y, z) \) and \( \phi_2(x, y, z) \) are any two first integrals of \( \mathbf{V} \) then 
\[ w(x, y, z) = f(\phi_1(x, y, z), \phi_2(x, y, z)) \]
is also a first integral of \( \mathbf{V} \).

**Example 20.** Let \( \mathbf{V} = (1, 0, 0) \) be a vector field and let \( \Omega = \mathbb{R}^3 \). A first integral of \( \mathbf{V} \) is a solution of the equation
\[ \phi_x = 0. \]
Any function of \( y \) and \( z \) only is a solution of this equation. For example,
\[ \phi_1 = y, \quad \phi_2 = z \]
are two solutions which are functionally independent. The integral curves of \( \mathbf{V} \) are described by the equations
\[ y = c_1, \quad z = c_2, \]
and are straight lines parallel to the \( x \)-axis.

**Example 21.** Let \( \mathbf{V} = (y, -x, 0) \) be a vector field and let \( \Omega = \mathbb{R}^3 \setminus z \)-axis. A first integral of \( \mathbf{V} \) is a solution of the equation
\[ y\phi_x - x\phi_y = 0. \]
It is easy to verify that
\[
\phi_1(x, y, z) = x^2 + y^2, \quad \phi_2(x, y, z) = z
\]
are two functionally independent first integrals of \( \mathbf{V} \). Therefore, the integral curves of \( \mathbf{V} \) in \( \Omega \) are given by
\[ x^2 + y^2 = c_1, \quad z = c_2. \]
The above equations describe circles parallel to the \((x, y)\)-plane and centered on the \( z \)-axis (see the second figure of Fig 1.1).
Practice Problems 3

1. Find a vector $V(x, y, z)$ normal to the surface $z = \sqrt{x^2 + y^2 + (x^2 + y^2)^{3/2}}$.

2. If $\nabla f(x, y, z)$ is always parallel to the vector $(x, y, z)$, show that $f$ must assume equal values at the points $(0, 0, a)$ and $(0, 0, -a)$.

3. Find $\nabla F$, where $F(x, y, z) = z^2 - x^2 - y^2$. Find the largest set in which grad $F$ does not vanish?

4. Find a vector normal to the surface $z^2 - x^2 - y^2 = 0$ at the point $(1, 0, 1)$.

5. If possible, solve the equation $z^2 - x^2 - y^2 = 0$ in terms of $x, y$ near the following points: (a) $(1, 1, \sqrt{2})$; (b) $(1, 1, \sqrt{2})$; (c) $(0, 0, 0)$.

6. Find the integral curves of the following vector fields: (a) $V = (x, 0, -z)$, (b) $V = (x^2, -y^3, 0)$, (c) $V = (2, 3y^2, 0)$.

7. Let $u(x, y, z)$ be a first integral of $V$ and let $C$ be an integral curve of $V$ given by $x = x(t), y = y(t), z = z(t); t \in I$.

Show that $C$ must lie on some level surface of $u$. [Hint: Compute $\frac{d}{dt} \{u(x(t), y(t), z(t))\}$].

8. If $V$ be the vector field given by $V = (x, y, z)$ and let $\Omega$ be the octant $x > 0, y > 0, z > 0$. Show that $u_1(x, y, z) = \frac{y}{z}$ and $u_2(x, y, z) = \frac{z}{x}$ are functionally independent first integrals of $V$ in $\Omega$.