A very important fact concerning linear PDEs is the superposition principle, which is stated below.

A linear PDE can be written in the form

$$ L[u] = f, \quad (1) $$

where $L[u]$ denotes a linear combination of $u$ and some of its partial derivatives, with coefficients which are given functions of the independent variables.

**Definition 1. (Superposition principle)** Let $u_1$ be a solution of the linear PDE

$$ L[u] = f_1 $$

and let $u_2$ be a solution of the linear PDE

$$ L[u] = f_2. $$

Then, for any any constants $c_1$ and $c_2$, $c_1u_1 + c_2u_2$ is a solution of

$$ L[u] = c_1f_1 + c_2f_2. $$

That is,

$$ L[c_1u_1 + c_2u_2] = c_1f_1 + c_2f_2. \quad (2) $$

In particular, when $f_1 = 0$ and $f_2 = 0$, (2) implies that if $u_1$ and $u_2$ are solutions of the homogeneous linear PDE $L[u] = 0$, then $c_1u_1 + c_2u_2$ will also be a solution of $L[u] = 0$.

**Example 2.** Observe that $u_1(x,y) = x^3$ is a solution of the linear PDE $u_{xx} - u_y = 6x$, and $u_2(x,y) = y^2$ is a solution of $u_{xx} - u_y = -2y$. Then, using superposition principle, it is easy to verify that $3u_1(x,y) - 4u_2(x,y)$ will be a solution of $u_{xx} - u_y = 18x + 8y$.

**Remark 3.** Note that the principle of superposition is not valid for nonlinear partial differential equations. This failure makes it difficult to form families of new solutions from an original pair of solutions.

**Example 4.** Consider the nonlinear first order PDE $u_xu_y - u(u_x + u_y) + u^2 = 0$. Note that $e^x$ and $e^y$ are two solutions of this equation. However, $c_1e^x + c_2e^y$ will not be a solution, unless $c_1 = 0$ or $c_2 = 0$.

**Solution.** Define $D[u] := (u_x - u)(u_y - u)$. For any $u, v \in C^1$, we have

$$ D[u + v] = (u_x + v_x - u - v)(u_y + v_y - u - v) $$

$$ = D[u] + D[v] + (u_y - u)(v_x - v) + (u_x - u)(v_y - v). $$
The computation shows that $D[u + v] \neq D[u] + D[v]$ in general. Taking $u = c_1 e^x$ and $v = c_2 e^y$, an easy computation shows that

$$D[c_1 e^x + c_2 e^y] = D[c_1 e^x] + D[c_2 e^y] + (c_1 e^x)(-c_2 e^y) = c_1 c_2 e^{x+y}.$$ 

Thus, $D[c_1 e^x + c_2 e^y] = 0$ only if $c_1 = 0$ or $c_2 = 0$.

1 Well-posed problems

A set of conditions was proposed by Hadamard (cf. [12]), who listed three requirements that must be met when formulating an initial and/or boundary value problem. A problem for which the PDE and the data lead to a solution is said to be well posed or correctly posed if the following three conditions are satisfied:

Hadamard’s conditions for a well-posed problem are:

1. The solution must exist.
2. The solution should be unique.
3. The solution should depend continuously on the initial and/or boundary data.

If it fails to meet these requirements, it is incorrectly posed.

The conditions (1)-(2) require that the equation plus the data for the problem must be such that one and only one solution exists. The third condition states that a small variation of the data for the problem should cause small variation in the solution. As data are generally obtained experimentally and may be subject to numerical approximations, we require that the solution be stable under small variations in initial and/or boundary values. That is, we cannot allow large variations to occur in the solution if the data are altered slightly.

A simple example of a ill posed problem is given below.

**Example 5.** Consider Cauchy’s problem for Laplace’s equation in $y \geq 0$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (3)$$

$$u(x, 0) = 0, \quad (4)$$

$$u_y(x, 0) = \frac{1}{n} \sin nx, \quad (5)$$

where $n$ is a positive integer, is not well-posed.
The solution is given by $u(x, y) = \frac{1}{ny} \sin(nx) \sinh(ny)$. Now, as $n \to \infty$, $u_y(x, 0) \to 0$ so that for large $n$ the Cauchy data $u(x, 0)$ and $u_y(x, 0)$ can be made arbitrarily small in magnitude. However, the solution $u(x, y)$ oscillates with an amplitude that grows exponentially like $e^{ny}$ as $n \to \infty$. Thus, arbitrarily small data can lead to arbitrarily large variation in solutions and hence the solution is unstable. This violates the condition (3) i.e., the continuous dependence of the solution on the data.

Boundary value problems are not well posed for hyperbolic and parabolic equations. This follows because these are, in general, equations whose solutions evolve in time and their behavior at later times is predicted by their previous states.

**Example 6.** Consider the hyperbolic equation

$$u_{xy} = 0 \text{ in } 0 < x < 1, \ 0 < y < 1$$

with the boundary conditions

\begin{align*}
  u(x, 0) &= f_1(x), \ u(x, 1) = f_2(x) \text{ for } 0 \leq x \leq 1, \\
  u(0, y) &= g_1(y), \ u(1, y) = g_2(y) \text{ for } 0 \leq y \leq 1.
\end{align*}

We shall show that this problem has no solution if the data are prescribed arbitrarily. Since $u_{xy} = 0$ implies that $u_x(x, y) = \text{constant}$, we have

$$u_x(x, 0) = u_x(x, 1).$$

In view of the given BC, we have

$$u_x(x, 0) = f_1'(x) \quad \text{and} \quad u_x(x, 1) = f_2'(x).$$

Thus, unless $f_1(x)$ and $f_2(x)$ are prescribed such that $f_1'(x) = f_2'(x)$, the BVP cannot be solved. Therefore, it is incorrectly posed.

2 Method of factorization

There is no general methods are available for obtaining the general solutions of second-order PDEs. Sometimes PDE of second-order can be factorized into two first-order equations. The equations

\begin{align*}
  u_{\xi\eta} &= 0, \\
  yu_{xx} + (x + y)u_{xy} + xu_{yy} &= 0.
\end{align*}
are examples of such equation. It is often much easier to factorize an equation when in its canonical form. But, we can often factorize equations with constant coefficients directly. The method of factorization can be a useful method of solution for hyperbolic and parabolic equations.

**Example 7.** The equation
\[ u_{xx} - u_{yy} + 4(u_x + u) = 0 \]
can be written as
\[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 2 \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 2 \right) u = 0. \]
It is equivalent to the pair of first order equations
\[ u_x - u_y + 2u = v, \]
and
\[ v_x + v_y + 2v = 0. \]

**Example 8.** The hyperbolic equation
\[ acu_{xy} + au_x + cu_y + u = 0 \]
can be written as
\[ \left( a \frac{\partial}{\partial x} + 1 \right) \left( c \frac{\partial}{\partial y} + 1 \right) u = 0. \]
It is equivalent to
\[ cu_y + u = v, \]
\[ av_x + v = 0. \]

**Note:** Unlike the case when the coefficients are constant, the differential operators need not commute.

**Practice Problems**

1. If \( u_1(x, y) = x^3 \) solves \( u_{xx} + u_{yy} = 2 \) and \( u_2(x, y) = c^3 + dy^3 \) solves \( u_{xx} + u_{yy} = 6cx + 6dy \) for real constants \( c \) and \( d \) then find a solution of \( u_{xx} + u_{yy} = ax + by + c \) for given real constants \( a \), \( b \) and \( c \).

2. Let \( u_1(x, y) \) be the solution to the Cauchy problem
\[ u_{xx} + u_{yy} = 0, \]
\[ u(x, 0) = f(x), \]
\[ u_y(x, 0) = g(x), \]
and let $u_2(x, y)$ be the solution of the following Cauchy problem

$$
\begin{align*}
    u_{xx} + u_{yy} &= 0, \\
    u(x, 0) &= f(x), \\
    u_y(x, 0) &= g(x) + \frac{1}{n} \sin(nx).
\end{align*}
$$

Show that $u_2 - u_1 = \frac{1}{n^2} \sinh(ny) \sin(nx)$ and the solution to the Cauchy problem for Laplace’s equation does not depend continuously on the initial data.

3. Show that the Dirichlet problem for the wave equation

$$
\begin{align*}
    u_{xx} - u_{yy} &= 0, \quad 0 < x < l, \quad 0 < y < T, \\
    u(0, y) &= u(l, y) = 0, \quad 0 \leq y \leq T, \\
    u(x, 0) &= u(x, T) = 0, \quad 0 \leq x \leq l,
\end{align*}
$$

is not wellposed.