NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 11

Brāhmasphuṭasiddhānta of Brahmagupta - Part 1

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Outline

- Introduction
- Twenty Logistics ; Cube root
- Rule of Proportion
- Mixtures; Interest Calculations
- Arithmetic and Geometric Progressions
- Plane figures :
  a. Triangles
  b. Right Triangles
  c. Quadrilaterals

Brahmagupta

- Brahmagupta described as *Gaṇakacakracūḍāmaṇi* (Jewel among the circle of Mathematicians) by Bhāskara - II.
- Brahmagupta holds a remarkable place in the history of Eastern Civilization. It was from his works that the Arabs learnt astronomy before they became acquainted with Ptolemy.
- Born in CE 598. Composed *Brāhmasphuṭasiddhānta* (24 chapters and a total 1008 verses) in CE 628. Commentary by Pṛthūdakasvāmim in CE 860

Mathematics in Brāhmasphuṭasiddhānta

- Chapter 12 on Arithmetic (which includes Geometry) and Chapter 18 on Algebra are the two chapters on mathematics. This lecture and the next one deal with arithmetic.
- There are 20 logistics: Addition, Subtraction, Multiplication, Division, Square, Square root, Cube, Cube root, 6 rules of reduction of fractions, rule of 3,5,7,9 & 11 and barter and 8 determinations : Mixture, Progression, Plane figure, Excavation, Stack, Saw, Mound and Shadow.

Elementary Operations; Cube ; Cube root

- Discusses elementary operations briefly. He deals first with fractions, reducing them to a common denominator etc., and multiplication and divisions.
- Cube is discussed based on $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Cube root extraction is discussed in verse 7. Same procedure as in $\bar{A}ryabhaṭiya$.

  छेदोऽघनाद्द्वितीयात् घनमूलकृतिस्त्रिसंगुणा सकृतिः ।
  शोध्या त्रिपूर्वगुणिता प्रथमात् घनतो घनो मूलम् ॥ ७ ॥

  “The divisor for the second non-cubic[digit] is thrice the square of the cubic-root. The square of the quotient multiplied by 3 and by the preceding must be subtracted from the next[non-cubic]; and the cube from the cubic[digit]: the root is [found].”
- Number written as $\cdots c, n_2 n_1\, c\, n_2 n_1 c$ where c : Cubic term; n_1: first non-cubic; n_2 second non-cubic. For example in 1771521, $1(c)7(n_2)7(n_1)1(c)5(n_2)2(n_1)1(c)$

Cube root Procedure

	$c\,n_2 n_1\,c\,n_2 n_1 c$	
	1 7 7 1 5 6 1	121
Subtract 1^3	1	Line of Cube root
Divide by $3 \cdot 1^2 = 3$	0 7 (2	
	0 6	
	1 7	
Subtract $3 \cdot 1 \cdot 2^2$	1 2	
	5 1	
Subtract 2^3	0 8	
Divide by $3 \cdot 12^2 = 432$	4 3 5 (1	
	4 3 2	
	3 6	
Subtract $3 \cdot 12 \cdot 1^2$	3 6	
	0 1	
Subtract 1^3	1	No rem. Cube root 121

Rule of proportion

$Pramāṇa \rightarrow A \rightarrow B \rightarrow Pramāṇaphala$
$Icchā \rightarrow C \rightarrow D \rightarrow Icchāphala$

Direct rule of three : $D = \frac{C \times B}{A}$

Inverse rule of three : $D = \frac{A \times B}{C}$

Direct Ex. Pṛthūdaka

A person gives away 108 (B) cows in 3 days (A). How many (D) does he bestow in a year and a month (390 days) (C)?

$D = \frac{C}{A} \times B = \frac{390}{3} \times 108 = 14040.$

Inverse Rule : Pṛthūdaka (adapted)

The measure of a certain quantity $= 10$ units(B), when unit $= 3\frac{1}{2} = \frac{7}{2}$ p (A) , where p is some fundamental unit.
How many measures (D) when unit $= 5\frac{1}{2} = \frac{11}{2}$ p (C)?

$D = \frac{A}{C} \times B = \frac{7}{11} \times 10 = \frac{70}{11} = 6\frac{4}{11}$.

Example Rule of 9. Pṛthūdaka

The price of 100 bricks of which the length, thickness and breadth respectively are 16, 8 and 10 is settled in 6 *dināra*s. We have received a 100,000 of other bricks, a quarter less in every dimension; say what we ought to pay.

16	8	10	100	6
12	6	$\frac{30}{4}$	100000	?

Answer

$$
\begin{aligned}
\frac{12}{16} \times \frac{6}{8} \times \frac{30/4}{10} \times \frac{100000}{100} \times 6 &= \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times 1000 \times 6 \\
&= \frac{27}{64} \times 6000 = \frac{27}{4} \times 375 \\
&= 2531\frac{1}{4}.
\end{aligned}
$$

In Section 2, Brahmagupta takes up problems involving ”Mixtures”.

Verse 15 deals with a problem involving calculation of interest in financial transactions.

Problems involving Mixtures

Let the interest on a principal, P for time t_0 be I_0. This interest I_0 is lent out at the same rate for further time, t_1. Let the interest on this be I_1. So, at the end of time t_1, the amount owed by the second borrower $= I_0 + I_1 = A_1$: “Mixed Amount”.

Given the principal P, first period of time t_0, second period of time t_1, mixed amount A_1; To find I_0 in Verse 15 :

कालप्रमाणघातः परकालहृतो द्विधाद्यमिश्रवधात्।
अन्यार्धकृतियुतात् पदमन्यार्धेन प्रमाणफलम् ॥१५॥

“The product of time and principal, divided by further time is twice set down. From the product of the one by the mixed amount added to the square of half the other, extract the square root; that root less half the second, is the interest of the principal.”

Expression for l_0 in the Verse and Rationale

$$l_0 = -\frac{Pt_0}{2t_1} + \sqrt{\left(\frac{Pt_0}{2t_1}\right)^2 + \left(\frac{Pt_0}{t_1}\right) \times A_1}$$

$$\text{Here} \quad A_1 = l_0 + l_1 = l_0 + \underbrace{l_0 \left(\frac{l_0}{P}\right)\left(\frac{t_1}{t_0}\right)}_{\text{Rule of 5}} \qquad \begin{array}{cc} P & l_0 \\ t_0 & t_1 \\ l_0 & ? \end{array}$$

$$\therefore \qquad l_0^2 \cdot \frac{t_1}{Pt_0} + l_0 = A_1$$

$$\therefore \qquad l_0^2 + \left(\frac{Pt_0}{t_1}\right) l_0 - A_1 \left(\frac{Pt_0}{t_1}\right) = 0$$

Expression for I_0 and an Example

$$\text{Hence,} \qquad I_0 = -\frac{Pt_0}{2t_1} + \sqrt{\left(\frac{Pt_0}{2t_1}\right)^2 + \left(\frac{Pt_0}{t_1}\right) \times A_1}$$

(The other root is negative and ruled out.)

Example: $P = 500$, $t_0 = 4$ months, $t_1 = 10$ months, Mixed amount is 78. What is the interest I_0 on $P(500)$ for 4 months?

$$\frac{Pt_0}{2t_1} = \frac{500 \times 4}{2 \times 10} = 100; \; I_0 = -100 + \sqrt{(100)^2 + 200 \times 78}$$

$$= -100 + 10\sqrt{256} = 60$$

Mixed quantities : Continued

Verse 16.

प्रक्षेपयोगहृतया लब्ध्या प्रक्षेपका गुणा लाभाः ।
ऊनाधिकोत्तराः तद्युतोनया स्वफलमूनयुतम् ॥ १६ ॥

"(i) The contributions, taken into the profit divided by the sum of the contributions, are the several gains: (ii) or, if there be subtractive or additive differences, with the profit increased or diminished by the differences; and the product thus has the corresponding difference subtracted or added"

Mixed quantities : Continued

(i) First rule: Let $C_1,\ C_2, \cdots$: contributions, sum $C = C_1 + C_2 + \cdots + C_n$; several gains: $P_1, P_2, \cdots$; Total $= P = P_1 + P_2 + \cdots$. Then,

$$P_i = \frac{C_i}{C} \times P$$

(ii) Second Rule: Let $P_i' = P_i + a_i$. $\quad P' = \sum P_i' = \sum P_i + \sum a_i = P + a$. Suppose P' is given. Then find $P = P' - a$. Then find

$$P_i = \frac{C_i}{C} \times P \quad \text{and} \quad P_i' = P_i + a_i$$

$+a_i$ is replaced by $-S_i$ and a by $-S$ when it is subtractive.

An Example and Solution

Example: 4 Colleges containing an equal number of pupils, were invited to partake of a feast. $\frac{1}{5}, \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ came from the respective colleges to the feast; and added to $1, 2, 3$ and 4 they were found to amount to 87.

Solution:

$$C_1 : C_2 : C_3 : C_4 = \frac{1}{5} : \frac{1}{2} : \frac{1}{3} : \frac{1}{4} = 12 : 30 : 20 : 15;$$

$$C = \sum C_i = 77$$

Solution Continued

Now given, $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4.$ $\quad a = \sum a_i = 10.$
$P' = \sum P_i' = \sum P_i + a_i = P + a = 87.$

Then, $P = P' - a = 87 - 10 = 77$. Then,

$$P_i = \frac{C_i}{C} \times P \ : P_1 = \frac{12}{77} \times 77 = 12; P_2 = 30; P_3 = 20; P_4 = 15.$$

$P_1' = P_1 + a_1 = 12 + 1 = 13$, $P_2' = P_2 + a_2 = 30 + 2 = 32$, $P_3' = P_3 + a_3 = 20 + 3 = 23$, $P_4' = P_4 + a_2 = 15 + 4 = 19$

Arithmetic Progession

Results same as in $\bar{A}ryabhaṭīya$.

In section 3. For an Arithmetic Progression: Verse 17 .

पदमेकहीनमुत्तरगुणितं संयुक्तमादिनान्त्यघनम् ।
आदियुतान्त्यघनार्द्धं मध्यघनं पदगुणं गुणितम् ॥ १७ ॥

"The period less one, multiplied by the common difference, being added to the first term, is the amount of the last. Half the sum of last and first terms is the mean amount which multiplied by the period, is the sum of whole."

Let first term $= a$, common difference $= d$, period (no. of terms)$= n$. So we have the A.P. :

$$a, a+d, a+2d, \cdots, a+(n-1)d.$$

Then

$$S = \text{sum} = \frac{[a+\{a+(n-1)d\}]}{2} \times n = na + \frac{n(n-1)d}{2}$$

where the factor multiplying n, is the average.

Geometrical progression is dealt with in the commentary. For finding the sum of a series increasing twofold or threefold etc. So we have the G.P. :

$$a, ar, ar^2, \cdots, ar^{n-1}$$

Geometric Progression

a : initial term; r : multiplier. Then

$$S = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{(r-1)}$$

To find r^n, the traditional procedure as in $chandas$ texts is given:

At the various stages, if the number is odd, subtract 1 and write 'm' (multiply); if the number is even, divide by 2 and write 's' (square). Go on till you exhaust (obtain 1). Below that is r.

Then to find r^n: Multiply by r, whenever there is 'm' and square the quantity when it is 's'.

Example:

n=17. odd

$$\begin{array}{lll} 17-1=16 & m & r^{17} \\ 16/2=8 & s & r^{16} \\ 8/2=4 & s & r^{8} \\ 4/2=2 & s & r^{4} \\ 2/2=1 & s & r^{2} \\ & r & r \end{array}$$

(In the last column, one goes upwards, that is, in reverse order.)

The reason is clear: when one is subtracting 1, one is dividing by r. when one is dividing by 2, one is finding the square root. Go on till one gets 1. Finally multiplier is r. Naturally to obtain r^n (quantity) we have to do it in the reverse order. Then

$$S = \frac{a(r^n - 1)}{(r - 1)} = \frac{(\text{Quantity} - 1) \times \text{Initial term}}{(\text{Multiplier} - 1)}$$

Number of terms in a A.P.

We come back to the arithmetical progressions. Given a, d, S in an arithmetic progression to find n (number of terms). Āryabhaṭa result in Verse 18:

उत्तरहीनाद्द्विगुणादिशेषवर्गं धनोत्तराष्टवधे ।
प्रक्षिप्य पदं शेषोनं द्विगुणोत्तरहृतं गच्छः ॥ १८ ॥

"Add the square of the difference between twice initial term and the common increase, to the product of the sum of the progression by 8 times the increase: the square root, less the foregoing remainder divided by twice the common increase, is the period."

Number of terms in a A.P.

It is given that

$$n = \frac{-(2a-d) + \sqrt{(2a-d)^2 + 8Sd}}{2d}$$

This follows from the quadratic equation satisfied by n:

$$n^2\frac{d}{2} + n(a - d/2) - S = 0$$

$$\therefore\ n = \frac{-(a-d/2) + \sqrt{(a-d/2)^2 + 4\cdot S\cdot d/2}}{d}$$

$$= \frac{-(2a-d) + \sqrt{(2a-d)^2 + 8Sd}}{2d}$$

(only '$+$' sign before the square root is relevant).

Sum and Sum of Sums

Results same as in *Āryabhaṭīya*

Verse 19: Sum $= S_1 = 1 + \cdots + n = \dfrac{n(n+1)}{2}$

$$\text{Sum of Sums } = S_2 = \sum_1^n \frac{r(r+1)}{2}$$

$$= \frac{\text{sum} \times (\text{period } + 2)}{3} = \frac{n(n+1)(n+2)}{2 \cdot 3}$$

Sum of Squares and Sum of Cubes

Results same as in $\bar{A}ryabhaṭ\bar{\imath}ya$ Verse 20:

$$1^2 + 2^2 + \cdots + n^2 = \frac{\text{sum} \times (2 \times \text{period} + 1)}{3}$$
$$= \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + \cdots + n^3 = (\text{sum})^2 = \frac{n^2(n+1)^2}{4}$$

Plane Figures: Triangles and Quadrilaterals

Verse 21.

स्थूलफलं त्रिचतुर्भुजबाहुप्रतिबाहुयोगदलघातः ।
भुजयोगार्द्धचतुष्टयभुजोनघातात् पदं सूक्ष्मम् ॥ २१ ॥

"The product of half the sides and countersides is the gross area of a triangle and a quadrilateral. Half the sum of the sides set down four times, and severally lessened by the sides, being multiplied together, the square root of the product is the exact area."

Gross Area of a Quadrlilateral

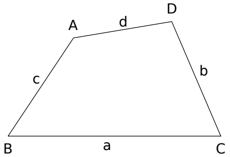


Figure: 1 :Quadrilateral

Sides: a, b, c, d

$s =$ Semi-perimeter $=$ Half sum of sides $= \frac{a+b+c+d}{2}$

$$\text{Gross area} = \frac{(a+d)}{2}\frac{(b+c)}{2}.$$

Exact Area

$$\text{Exact area} = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

The expression for the exact area is correct only for a cyclic quadrilateral (includes squares, rectangles, and isosceles trapezia). Proved in *Gaṇita-Yuktibhaṣā* of Jyeṣṭhadeva (1530 CE)

For a triangle, set $d = 0$. (One side is zero). Always true.

Segments आबाधाs of the Base and Perpendicular in a Triangle

Verse 22.

भुजकृत्यन्तरर्भूहृतहीनयुता भूर्द्विभाजिताबाधे ।
स्वाबाधावर्गोनात् भुजवर्गात् मूलमवलम्बः ॥ २२ ॥

“The difference of the squares of the sides being divided by the base, the quotient is added to and subtracted from the base; the sum and the remainder, divided by two are the segments. The square root, extracted from the difference of the side of its corresponding segment of the base, is the perpendicular.”

Segments and Perpendicular

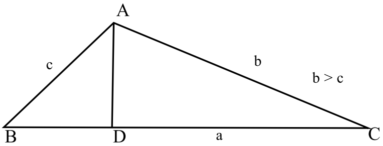
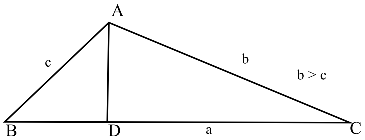


Figure: 2: Segments and the Perpendicular

$$\text{Segments: } CD = \frac{a + \frac{b^2 - c^2}{a}}{2}, \quad BD = \frac{a - \frac{b^2 - c^2}{a}}{2}$$

$$\text{Perpendicular: } AD = \sqrt{c^2 - BD^2} = \sqrt{b^2 - CD^2}$$

Proof



$$BD^2 = c^2 - AD^2$$
$$= c^2 - (b^2 - CD^2).$$
$$\therefore\ CD^2 - BD^2$$
$$= b^2 - c^2$$
Or $(CD - BD)(CD + BD)$
$$= b^2 - c^2.$$

As $CD + BD = BC = a, \quad CD - BD = \dfrac{b^2 - c^2}{a}$

Hence, $CD = \dfrac{a + \frac{b^2 - c^2}{a}}{2}, \quad BD = \dfrac{a - \frac{b^2 - c^2}{a}}{2};$ as stated.

Clearly, $AD = \sqrt{c^2 - BD^2} = \sqrt{b^2 - CD^2}.$

All the results follow from the theorem of a right triangle.

A Theorem for an Isosceles Trapezium

Verse 23.

अविषमचतुरस्रभुजप्रतिभुजवधयोर्युतेः पदं कर्णः ।
कर्णकृतिर्भूमुखयुतिदलवर्गोनपदं लम्बः ॥ २३ ॥

“In a quadrilateral but a general one, the square root of the sum of the products of the sides and countersides is the diagonal. Subtracting from the square of the diagonal, the square of half the sum of the base and summit, the square root of the remainder is the perpendicular.”

True for an “Isoceles trapezium”(or square, rectangle) where diagonals are equal.

Isosceles Trapezium

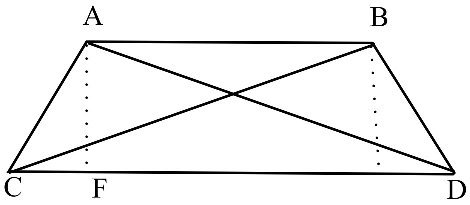
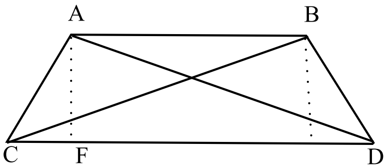


Figure: 3 : Isosceles Trapezium

In this case, $AD^2 = AF^2 + FD^2 = AC^2 - CF^2 + FD^2$ Summit $s = AB$, Base $b = CD$. $\therefore CF = (b - s)/2$.

Isosceles Trapezium



$$
\begin{aligned}
FD &= CD - CF \\
&= b - \left(\frac{b-s}{2}\right) \\
&= \frac{b+s}{2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore AD^2 &= AC^2 + \left(\frac{b+s}{2}\right)^2 - \left(\frac{b-s}{2}\right)^2 = AC^2 + b \cdot s \\
&= AC \cdot BD + AB \cdot CD, \text{ as } AC = BD. \\
AD &= BC = \sqrt{AC \cdot BD + AB \cdot CD}, \text{ as stated.}
\end{aligned}
$$

$$
AF^2 = AD^2 - FD^2 = (\text{Diag.})^2 - \left(\frac{\text{Base + Summit}}{2}\right)^2, \text{ as stated}
$$

Theorem of a right triangle

“Pythagoras” theorem: should be “*Bauddhāyana* theorem”.

Verse 24.

कर्णकृतेः कोटिकृतिं विशोध्य मूलं भुजो भुजस्य कृतिम् ।
प्रोह्य पदं कोटिः कोटिबाहुकृतियुतिपदं कर्णः ॥ २४ ॥

“Subtracting the square of the upright from the square of the diagonal, the square root of the remainder is the side or subtracting the square of the side, the root of the remainder is the upright; the root of the sum of the squares of the upright and the side is the diagonal.”

Theorem of a Right Triangle

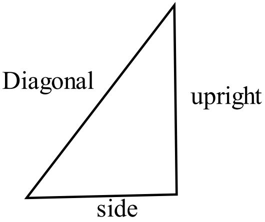


Figure: 4: Right Triangle

$$\text{Diagonal}^2 = \text{upright}^2 + \text{Side}^2.$$

Segments of the Diagonal and Perpendicular

Verse 25.

कर्णयुतावूर्ध्वाधरखण्डे कर्णावलम्बयोगे वा ।
स्वाबाधे स्वयुतिहृते द्विधा प्रथक्कर्णलम्बकगुणे ॥ २५ ॥

“At the intersection of the diagonals or the junction of a diagonal and a perpendicular, the upper and lower portions of the diagonal, or of the perpendicular and the diagonal, are the quotients of those lines taken into the corresponding segment of the base and divided by the complement of the segments.”

Segments and Portions

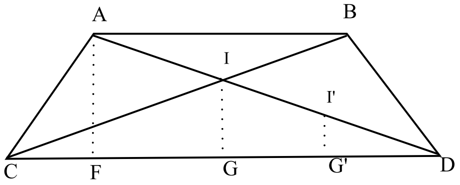
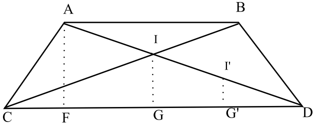


Figure: 5 : Segments of Diagonal

$$\frac{ID}{GD} = \frac{AD}{AF} = \frac{AI}{FG} = \frac{AD}{DF}$$
$$\therefore ID = GD \cdot \frac{AD}{DF}$$
$$= \frac{\text{Corresp. segment of the base } \times \text{Diagonal}}{\text{Comple. of segment (greater segment of base)}}$$

Segments of Diagonal and Portion of Perpendicular



Similarly,

$$AI = FG \cdot \frac{AD}{DF}$$

In fact, this is true for any point I' on the diagonal.

$$I'D = \frac{G'D \cdot AD}{DF}$$

$$= \frac{\text{(Corr. Segment of base)} \times \text{Diag.}}{\text{Complement of the less segment}}$$

Denominator is the greater segment. Similarly,

$$IG = GD \times \frac{AF}{DF} \qquad \text{etc.,}$$

Circumradius of a Triangle, Cyclic quadrilateral

Verse 26a. and 27

अविषमपार्श्वभुजगुणकर्णो द्विगुणावलम्बकविभक्तः ।
हृदयं . . . ॥ २६ ॥

“The diagonal of a quadrilateral other than a general one being multiplied by the flank and divided by twice the perpendicular, is the central line (circumradius) . . . .”

त्रिभुजस्य वधोभुजयोर्द्विगुणितलम्बोद्धृतो हृदयरज्जुः ।
सा द्विगुणा त्रिचतुर्भुजकोणस्पृग्वृत्तविष्कम्भः ॥ २७ ॥

Circumradius of a Triangle, Cyclic quadrilateral

"The product of the two sides of a triangle divided by twice the perpendicular, is the central line; and the double of this is the diameter of the exterior circle."

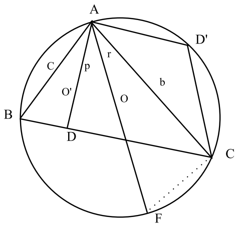


Figure: 6: Circumradius of Triangle, Cyclic quad.

Circumradius of a Triangle

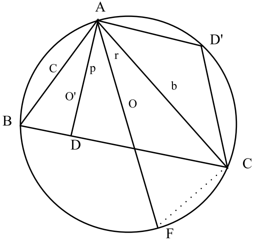
We take up verse 27 first. AD is perpendicular to BC. In the triangle ACF, $A\hat{C}F = 90^\circ$. In triangle ABD, $A\hat{D}B = 90^\circ$. Also $A\hat{F}C = A\hat{B}D$. $\therefore$ Triangles ABD and AFC are similar.

$$\therefore \frac{AB}{AD} = \frac{AF}{AC} \qquad \therefore \frac{c}{p} = \frac{2r}{b}$$

$$\therefore \text{Circumradius}, r = \frac{b \cdot c}{2p} = \frac{\text{Product of two sides}}{2 \times \text{Perpendicular}}$$

Circumradius of a Cyclic quadrilateral

Verse 26a.



Now $ABCD$ is a cyclic quadrilateral, with $AC = b$ as its diagonal. c is its flank. $AD = p$ is the perpendicular.

$$\therefore r = \frac{bc}{2p} = \frac{\text{Diagonal} \times \text{flank}}{2(\text{perpendicular})}$$

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