Lecture XIX - Deck Transformations

Given a covering projection $p : \tilde{X} \rightarrow X$, the deck transformations are, roughly speaking, the symmetries of the covering space. Thus it should not come as a surprise that they play a crucial part in the theory of covering spaces. In this lecture all spaces are assumed to be connected and locally path connected.

**Definition 19.1 (Deck transformations):** Let $p : \tilde{X} \rightarrow X$ be a covering projection. A deck transformation is a homeomorphism $\tilde{\phi} : \tilde{X} \rightarrow \tilde{X}$ such that $p \circ \tilde{\phi} = p$, that is to say $\tilde{\phi}$ is a lift of $p$.

**Examples 19.1:**

(i) For the covering space $\exp : \mathbb{R} \rightarrow S^1$ given by $\exp(t) = \exp(2\pi it)$ the deck transformations are the maps $T_n : \mathbb{R} \rightarrow \mathbb{R}$; $T_n(x) = x + n$, $n \in \mathbb{Z}$.

(ii) For the two sheeted covering $p : S^n \rightarrow \mathbb{R}P^n$ the deck transformations are the identity map and the antipodal map.

The following theorem summarizes the most basic properties of the group of deck transformations.

**Theorem 19.1:** Let $p : \tilde{X} \rightarrow X$ be a covering projection and $\tilde{\phi}$ be a deck transformation. Then

(i) $\tilde{\phi}$ is uniquely determined by its value at one point of $\tilde{X}$

(ii) $\tilde{\phi}(\tilde{x}_0) \in p^{-1}(x_0)$ whenever $\tilde{x}_0 \in x_0$.

(iii) If $\tilde{\phi}(\tilde{x}_1) = \tilde{x}_2$, where $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ then

$$p_* \pi_1(\tilde{X}, \tilde{x}_1) = p_* \pi_1(\tilde{X}, \tilde{x}_2) \quad (19.1)$$

(iv) Conversely if (1) holds then there exists a unique deck transformation $\tilde{\phi}$ such that $\tilde{\phi}(\tilde{x}_1) = \tilde{x}_2$.

**Proof:** Statement (i) follows from the uniqueness of lifts. Statement (ii) follows immediately from the definition. To prove (iii) apply the lifting criterion (necessity) to both $\phi$ and $\phi^{-1}$. To prove (iv) apply lifting criterion (sufficiency) to get continuous functions $\phi : \tilde{X} \rightarrow \tilde{X}$ and $\psi : \tilde{X} \rightarrow \tilde{X}$ such that

$$p \circ \phi = p, \ \phi(\tilde{x}_1) = \tilde{x}_2; \ \ p \circ \psi = p, \ \psi(\tilde{x}_2) = \tilde{x}_1.$$ 

Then $\phi \circ \psi$ and $\psi \circ \phi$ are both lifts of the map $p : \tilde{X} \rightarrow X$ such that

$$\phi \circ \psi(\tilde{x}_2) = \tilde{x}_2, \ \ \psi \circ \phi(\tilde{x}_1) = \tilde{x}_1.$$ 

The identity map on $\tilde{X}$ is also a lift of $p$ with these initial conditions. By uniqueness, we see that both $\phi \circ \psi$ and $\psi \circ \phi$ must be the identity map on $\tilde{X}$ proving that $\phi$ and $\psi$ are homeomorphisms. The uniqueness clause follows from the uniqueness of lifts. \qed
Remark: If \( \phi : \tilde{X} \to X \) is a continuous map such that \( p \circ \phi = p \), then prove that \( \phi \) is a homeomorphism in the following cases:

(i) \( \pi_1(\tilde{X}) \) is a finite group
(ii) \( p_*\pi_1(\tilde{X}, \tilde{x_0}) \) has finite index in \( \pi_1(X, x_0) \)
(iii) \( \tilde{X} \) is a regular cover of \( X \). Is this true in general? The point is that if \( H \) is a subgroup of \( G \) and \( gHg^{-1} \subset H \) then it follows \( gHg^{-1} = H \) in case \( H \) is finite or has finite index or is normal.

Definition 19.2: The set of deck transformations of a covering projection \( p : \tilde{X} \to X \) forms a group under composition of maps denoted by \( \text{Deck}(\tilde{X}, X) \).

Action of \( \text{Deck}(\tilde{X}, X) \) on the fibers \( p^{-1}(x_0) \): We fix a base point \( x_0 \in X \). Since each deck transformation is a bijection, it is a permutation of the fiber \( p^{-1}(x_0) \) and so acts on \( p^{-1}(x_0) \) as a group of permutations:

\[
(\phi, \tilde{x}_0) \mapsto \phi(\tilde{x}_0)
\]

We study this action closely and relate it to the action of \( \pi_1(X, x_0) \) on the fiber \( p^{-1}(x_0) \). We first look at the case of regular coverings.

Theorem 19.2: The covering \( p : \tilde{X} \to X \) is a regular covering if and only if the action of \( \text{Deck}(\tilde{X}, X) \) is transitive on \( p^{-1}(x_0) \).

Proof: Let \( \tilde{x}_1 \) and \( \tilde{x}_2 \) be two arbitrary points of \( p^{-1}(x_0) \). The action of \( \text{Deck}(\tilde{X}, X) \) is transitive on \( p^{-1}(x_0) \) if and only if there is a \( \phi \in \text{Deck}(\tilde{X}, X) \) carrying \( \tilde{x}_1 \) to \( \tilde{x}_2 \), which is the case if and only if (19.1) holds. This in turn implies that the conjugacy class

\[
\left\{ p_*\pi_1(\tilde{X}, \tilde{x}_0) : \tilde{x}_0 \in p^{-1}(x_0) \right\}
\]

reduces to a singleton and conversely, in other words, if and only if the covering is regular. \( \square \)

We now relate the (perhaps intransitive) action of \( \text{Deck}(\tilde{X}, X) \) on \( p^{-1}(x_0) \) with the transitive action of \( \pi_1(X, x_0) \) on \( p^{-1}(x_0) \). Pick \( \phi \in \text{Deck}(\tilde{X}, X) \) and \( \phi(\tilde{x}_1) = \tilde{x}_2 \). Then on the one hand (19.1) must hold while since \( p_*\pi_1(\tilde{X}, \tilde{x}_1) = \text{stab} \tilde{x}_1 \) (for the action of \( \pi_1(X, x_0) \)), we have on the other hand

\[
\text{stab} \tilde{x}_1 = \text{stab} \tilde{x}_2 = g(\text{stab} \tilde{x}_1)g^{-1},
\]

(19.2)

for some \( g \in \pi_1(X, x_0) \). In fact (19.2) states that \( g \) belongs to the normalizer

\[
N(\text{stab} \tilde{x}_1) = N(p_*(\pi_1(\tilde{X}, \tilde{x}_1))) \subset \pi_1(X, x_0).
\]

This suggests that we must relate \( \phi \) to the element \( g \in N(p_*(\pi_1(\tilde{X}, \tilde{x}_1))) \). However since there may be several such elements \( g \) it is expedient to define the map in the opposite direction.

Let \( g \in N(p_*(\pi_1(\tilde{X}, \tilde{x}_1))) \subset \pi_1(X, x_0) \) and \( \tilde{x}_1 \cdot g = \tilde{x}_2 \). Then (19.1) holds since \( g \) is in the normalizer of \( \text{stab} \tilde{x}_1 \). There is a unique \( \phi_g \in \text{Deck}(\tilde{X}, X) \) such that \( \phi_g(\tilde{x}_1) = \tilde{x}_2 = \tilde{x}_1 \cdot g \). The map

\[
\psi : N(p_*(\pi_1(\tilde{X}, \tilde{x}_1))) \to \text{Deck}(\tilde{X}, X), \quad g \mapsto \phi_g
\]

(19.3)

is a homomorphism. To see that it is surjective, let \( \phi \in \text{Deck}(\tilde{X}, X) \). There is a \( g \in \pi_1(X, x_0) \) such that

\[
\tilde{x}_1 \cdot g = \phi(\tilde{x}_1)
\]

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then stab $\tilde{x}_1$ and stab $\phi(\tilde{x}_1)$ are conjugate by $g$ but they are also equal by (iii) of Theorem (19.1), whereby we conclude $g$ is in the normalizer $N(p*(\pi_1(\tilde{X}, \tilde{x}_1)))$ and $\phi = \phi_g$. To determine the kernel of $\psi$, observe that $\phi_g = \text{id}$ if and only if

$$\phi_g(\tilde{x}_1) = \tilde{x}_1 \cdot g$$

that is, if and only if $g \in \text{stab } \tilde{x}_1$. But stab $\tilde{x}_1 = p*(\pi_1(\tilde{X}, \tilde{x}_1))$. Summarizing these observations,

**Theorem 19.3:** We the group isomorphism

$$\text{Deck}(\tilde{X}, X) \cong N(p*(\pi_1(\tilde{X}, \tilde{x}_1))/p*(\pi_1(\tilde{X}, \tilde{x}_1))).$$

**Corollary 19.4:** If $p : \tilde{X} \longrightarrow X$ is a regular covering then

$$\text{Deck}(\tilde{X}, X) \cong \pi_1(X, x_0)/p*(\pi_1(\tilde{X}, \tilde{x}_1)).$$

**Corollary 19.5:** If $\tilde{X}$ is a simply connected covering of $X$ then

$$\text{Deck}(\tilde{X}, X) \cong \pi_1(X, x_0).$$

**Corollary 19.6:** $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$

**Existence of a simply connected covering space:** Despite being an important theme, we shall not discuss this in any detail in this elementary course but make a few remarks about it. Most of the spaces that we shall encounter are reasonably well-behaved and indeed many of them such $SO(n, \mathbb{R})$, $S^3$ and the projective spaces are smooth manifolds. Given the existence of a simply connected covering - called a universal covering, one can develop a Galois correspondence for covering spaces which asserts the existence of a unique (upto isomorphism) covering corresponding to each conjugacy class of subgroups of $\pi_1(X, x_0)$.

**Definition 19.3:** Let us consider a fixed connected topological space $X$ with a specified base point $x_0 \in X$. A homomorphism between two coverings $p : (Y, y_0) \longrightarrow (X, x_0)$ and $q : (Z, z_0) \longrightarrow (X, x)$ is a surjective continuous map $r : (Y, y_0) \longrightarrow (Z, z_0)$ such that $q \circ r = p$ or diagrammatically,

$$
\begin{array}{ccc}
(Y, y_0) & \xrightarrow{r} & (Z, z_0) \\
\downarrow{p} & & \downarrow{q} \\
(X, x_0)
\end{array}
$$

The definition enables us to form a category of coverings of a given space $X$ with a specified base point $x_0 \in X$. To obtain a satisfactory theory one must impose some additional assumption on $X$ such as local connectedness. In other words $r$ is a lift of $p$ with respect to the covering map $q$. The universal covering is then defined in terms of a universal property.

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$^4$Actually the notion of a universal covering is more general than the notion of a simply connected coverings but the two notions coincide for all reasonable spaces and certainly for all spaces that we shall deal with.
Definition 19.4: The universal covering is a covering \( e : (E, e_0) \rightarrow (X, x_0) \) such that for every covering \( p : (Y, y_0) \rightarrow (X, x_0) \) there is a unique homomorphism \( \psi : (E, e_0) \rightarrow (Y, y_0) \), that is a continuous surjection \( \psi \) such that \( p \circ \psi = e \).

The universal covering if it exists is unique and one can establish the existence of a universal covering for a reasonable nice class of topological spaces \( X \).

Exercises

1. Suppose that \( G \) and \( \tilde{G} \) are topological groups and \( p : \tilde{G} \rightarrow G \) is a covering projection that is also a group homomorphism then \( \ker p = \text{Deck}(\tilde{G}, G) \).

2. Determine the deck transformations for the covering
\[
\sin : \mathbb{C} - \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\} \rightarrow \mathbb{C} - \{\pm1\}
\]

3. Determine the deck transformations for the covering
\[
p : \mathbb{C} - \{\pm1, \pm2\} \rightarrow \mathbb{C} - \{\pm2\}
\]
given by \( p(z) = z^3 - 3z \). Show that this covering is not regular. Hint: Use Riemann’s removable singularities theorem to show that a deck transformation must be analytic on the whole plane.

4. If \( p \) is a prime, what can you say about the group of deck transformations of a \( p \)-sheeted covering space?

5. Show using the universal property that the universal covering, if it exists is unique upto isomorphism of covering projections.