Lecture 23 : Solutions of Cubic and Quartic Equations

Objectives

(1) Cardano's method for roots of cubic equations.
(2) Lagrange's method for roots of quartic equations.
(3) Ferrari's method for roots of quartic equations.

Keywords and Phrases : Cubic equations, quartic equations.

In this section we present algorithms for finding roots of cubic and quartic polynomials over any field F of characteristic different from 2 and 3. This is to make sure that irreducible cubics and quartics are separable.

Cubic polynomials

Cardano published Tartaglia's method to find roots of cubic polynomials in 1545. This is known as Cardano's method. We may assume that the given cubic is of the form $f(x) = x^3 + px + q$ since a general cubic can be transformed into this form without changing its splitting field. One begins by introducing two unknowns u and v. Put $x = u + v$ into $f(x) = 0$ to get

$$u^3 + v^3 + 3u^2v + 3uv^2 + p(u + v) + q = u^3 + v^3 + q + (3uv + p)(u + v) = 0.$$

We set $u^3 + v^3 + q = 0$ and $3uv + p = 0$. Hence $v = -p/3u$. Put this into the first equation to get

$$u^6 + qu^3 - p^3/27 = 0.$$

This is a quadratic equation in u^3. Put $D = -(4p^3 + 27q^2)$. By the quadratic formula we get

$$u^3 = \frac{-q \pm \sqrt{q^2 + (4p^3/27)}}{2} = -\frac{q}{2} \pm \sqrt{-D/108}.$$

Set $A = -q/2 + \sqrt{-D/108}$ and $B = -q/2 - \sqrt{-D/108}$. By symmetry of u and v, we set $u^3 = A$ and $v^3 = B$. Let ω be a primitive cube root of unity. Then

$$u = \sqrt[3]{A},\ \ \omega\sqrt[3]{A},\ \ \omega^2\sqrt[3]{A},\ \text{ and }\ v = \sqrt[3]{B},\ \ \omega\sqrt[3]{B},\ \ \omega^2\sqrt[3]{B}.$$

We must choose cube roots of A and B in such a way that $\sqrt[3]{A}\sqrt[3]{B} = -p/3$. Having chosen these we see that the three roots of $f(x)$ are

$$\sqrt[3]{A} + \sqrt[3]{B},\ \ \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B},\ \ \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}.$$

Example 23.1. Consider the cubic $f(x) = x^3 - 3x + 1$. Reducing modulo 2, we see that $f(x)$ is irreducible over $\mathbb{Q}$. The discriminant of $f(x)$ is $D = -81$. Hence

$$A = -q/2 + \sqrt{-D/108} = \exp(2\pi i/3), \quad \text{and} \quad B = \exp(-2\pi i/3).$$

Substitute these values of A and B into the formula for the roots, we see that the three roots of $f(x)$ are $2\cos(2\pi/9),\ 2\cos(8\pi/9)$ and $2\cos(14\pi/9)$.

Let $f(x) = x^3 + px + q \in \mathbb{R}[x]$. If disc $(f) < 0$, then cube roots of A and B can be chosen to be real. In this case

$$\begin{aligned} r_1 &= \sqrt[3]{A} + \sqrt[3]{B} \in \mathbb{R}, \\ r_2 &= -\frac{\sqrt[3]{A} + \sqrt[3]{B}}{2} + i\sqrt{3}\left(\frac{\sqrt[3]{A} - \sqrt[3]{B}}{2}\right), \\ r_3 &= \overline{r_2}. \end{aligned}$$

If $D =$ disc $(f(x)) > 0$ then $A = -q/2 + i\sqrt{D/108}$ and $B = \overline{A}$. Suppose that $\sqrt[3]{A} = a + ib$ then due to $uv = -p/3$ we have $\sqrt[3]{B} = a - ib$. Hence the roots of $f(x)$ are $r_1 = 2a$, $r_2 = -a - b\sqrt{3}$ and $r_3 = -a + b\sqrt{3}$.

Notice that in this case, all the roots are real. However, they are expressed in terms of complex numbers. It can be proved that the roots cannot be expresseed in terms of real radicals. Historically, this is called the **irreducible case**. This fact forced mathematicians to accept complex numbers.

Quartic polynomials

We present Lagrange's method for the roots of a quartic polynomial. We continue with the assumption that F has characteristics different from $2, 3$. Consider a general quartic polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$. We put $y = x - a/4$ to get the polynomial $g(y) = y^4 + py^2 + qy + r$. Let r_1, r_2, r_3, r_4 be roots of $g(y)$. Consider the quantities

$$\theta_1 = (r_1 + r_2)(r_3 + r_4),\ \theta_2 = (r_1 + r_3)(r_2 + r_4),\ \theta_3 = (r_1 + r_4)(r_2 + r_3).$$

The cubic polynomial whose roots are θ_1, θ_2 and θ_3 is called the **resolvent cubic** of the quartic polynomial. It turns out to be the polynomial

$$h(x) = x^3 - 2px^2 + (p^2 - 4r)x + q^2.$$

Using the relation $r_1 + r_2 + r_3 + r_4 = 0$ we get

$$\begin{aligned} r_1 + r_2 &= \sqrt{-\theta_1}, \qquad & r_3 + r_4 &= -\sqrt{-\theta_1} \\ r_1 + r_3 &= \sqrt{-\theta_2}, \qquad & r_2 + r_4 &= -\sqrt{-\theta_2} \\ r_1 + r_4 &= \sqrt{-\theta_3}, \qquad & r_2 + r_3 &= -\sqrt{-\theta_3}. \end{aligned}$$

One can show that $\sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3} = -q$. Hence two of the square roots determine the third. Adding the three equations on the left and using the fact that $r_1 + r_2 + r_3 + r_4 = 0$, we get

$$\begin{aligned} 2r_1 &= \sqrt{-\theta_1} + \sqrt{-\theta_2} + \sqrt{-\theta_3}, \\ 2r_2 &= \sqrt{-\theta_1} - \sqrt{-\theta_2} - \sqrt{-\theta_3}, \\ 2r_3 &= -\sqrt{-\theta_1} + \sqrt{-\theta_2} - \sqrt{-\theta_3}, \\ 2r_4 &= -\sqrt{-\theta_1} - \sqrt{-\theta_2} + \sqrt{-\theta_3}. \end{aligned}$$

This shows that the roots of the resolvent cubic determine the roots of the quartic.

Proposition 23.2. *The discriminant of the quartic* $g(y) = y^4 + py^2 + qy + r$ *and its resolvent cubic* $h(x) = x^3 - 2px^2 + (p^2 - 4r)x + q^2$ *are equal.*

Proof. The differences of the roots of the resolvent cubic are:

$$\theta_1 - \theta_2 = (r_2 - r_3)(r_4 - r_1),\ \theta_1 - \theta_3 = (r_2 - r_4)(r_3 - r_1),\ \theta_2 - \theta_3 = (r_3 - r_4)(r_2 - r_1).$$

Hence the quartic and the resolvent cubic have same discriminant. □

Remark 23.3. In the literature, we find that the term resolvent cubic is also used for the cubic whose roots are

$$t_1 = r_1 r_2 + r_3 r_4,\ t_2 = r_1 r_3 + r_2 r_4, \text{ and } t_3 = r_1 r_4 + r_2 r_3.$$

It can be shown that this cubic is $r(x) = x^3 - px^2 - 4rx + 4pr - q^2$ and $h(x)$ and $r(x)$ have equal discriminant and the same splitting field over F.

Ferrari's method for solving quartic equations

Consider the general quartic equation

$$x^4 + bx^3 + cx^2 + dx + e = 0.$$

Rewrite this as $x^4 + bx^3 = -cx^2 - dx - e$. Now complete the square to get

$$\left(x^2 + \frac{bx}{2}\right)^2 = \left(\frac{b^2}{4} - c\right)x^2 - dx - e.$$

Let y be another variable and consider the equation:

$$\begin{aligned}\left(x^2 + \frac{bx}{2} + \frac{y}{2}\right)^2 &= \left(\frac{b^2}{4} - c\right)x^2 - dx - e + y\left(x^2 + \frac{bx}{2}\right) + \frac{y^2}{4} \\ &= x^2\left(\frac{b^2}{4} - c + y\right) + x\left(\frac{by}{2} - d\right) + \frac{y^2}{4} - e \qquad (1)\end{aligned}$$

The right hand side of the last equation is a square of a linear polynomial in x if and only if its discriminant is zero. i.e.

$$(\tfrac{1}{2}by - d)^2 - 4(\tfrac{1}{4}y^2 - e)(\tfrac{1}{4}b^2 - c + y) = 0.$$

Therefore

$$y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2 = 0.$$

Let y be any root of this cubic and substitute it in the equation (1) to get

$$x^2 + \frac{1}{2}bx + \frac{1}{2}y = \pm mx + n \qquad (2)$$

Notice that the roots of the equation (2) are the roots of the given quartic.

Proposition 23.4. *Let x_1, x_2, x_3, x_4 be the roots of*

$$f = x^4 + bx^3 + cx^2 + dx + e = 0.$$

Then $y_1 = x_1x_2 + x_3x_4$, $y_2 = x_1x_3 + x_2x_4$, $y_3 = x_1x_4 + x_2x_3$ are roots of resolvent cubic $g(y) = y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2$.