NPTEL Course
on
Power Quality in Power Distribution Systems

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Chapter 1

SINGLE PHASE CIRCUITS: POWER DEFINITIONS AND ITS COMPONENTS (Lectures 1-8)

1.1 Introduction

The definitions of power and its various components are very important to understand quantitative and qualitative power quality aspects in power system [1]–[5]. This is not only necessary from the point of view of conceptual clarity but also very much required for practical applications such as metering, quantification of active, reactive power, power factor and other power quality parameters in power system. These aspects become more important when power system is not ideal i.e. it deals with unbalance, harmonics, faults and fluctuations in frequency. We therefore, in this chapter explore the concept and fundamentals of single phase system with some practical applications and illustrations.

1.2 Power Terms in a Single Phase System

Let us consider a single-phase system with sinusoidal system voltage supplying a linear load as shown in Fig. 1.1. The voltage and current are expressed as below.

\[
v(t) = \sqrt{2}V \sin \omega t \\
i(t) = \sqrt{2}I \sin(\omega t - \phi)
\]

(1.1)

The instantaneous power can be computed as,

\[
p(t) = v(t) \, i(t) = VI \left[ 2 \sin \omega t \sin(\omega t - \phi) \right] \\
= VI \left[ \cos \phi - \cos(2\omega t - \phi) \right] \\
= VI \cos \phi(1 - \cos 2\omega t) - VI \sin \phi \sin 2\omega t \\
= P(1 - \cos 2\omega t) - Q \sin 2\omega t \\
= p_{\text{active}}(t) - p_{\text{reactive}}(t)
\]

(1.2)
Fig. 1.1 A single phase system

Here, \( P = \frac{1}{T} \int_{t_0}^{T+t_1} p(t) \, dt \) = average value of \( p_{\text{active}}(t) \). This is called as average active power. The reactive power \( Q \) is defined as,

\[
Q \triangleq \max \{p_{\text{reactive}}(t)\}
\]

(1.3)

It should be noted that the way \( Q \) is defined is different from \( P \). The \( Q \) is defined as maximum value of the second term of (1.2) and not an average value of the second term. This difference should always kept in mind.

Equation (1.2) shows that instantaneous power can be decomposed into two parts. The first term has an average value of \( VI \cos \phi \) and an alternating component of \( VI \cos 2\omega t \), oscillating at twice the line frequency. This part is never negative and therefore is called unidirectional or dc power. The second term has an alternating component \( VI \sin \phi \sin 2\omega t \) oscillating at twice frequency with a peak value of \( VI \sin \phi \). The second term has zero average value. The equation (1.2) can further be written in the following form.

\[
p(t) = VI \cos \phi - VI \cos(2\omega t - \phi)
\]

\[
= \bar{p}(t) + \tilde{p}(t)
\]

\[
= P_{\text{average}} + P_{\text{oscillation}}
\]

\[
= p_{\text{useful}} + p_{\text{nonuseful}}
\]

(1.4)

With the above definitions of \( P \) and \( Q \), the instantaneous power \( p(t) \) can be re-written as following.

\[
p(t) = P(1 - \cos 2\omega t) - Q \sin 2\omega t
\]

(1.5)
Example 1.1 Consider a sinusoidal supply voltage $v(t) = \sqrt{2} \times 230 \sin \omega t$ supplying a linear load of impedance $Z_L = 12 + j13 \ \Omega$ at $\omega = 2\pi f$ radian per second, $f = 50$ Hz. Express current $i(t)$ as a function of time. Based on $v(t)$ and $i(t)$ determine the following.

1. Instantaneous power $p(t)$, instantaneous active power $p_{\text{active}}(t)$ and instantaneous reactive power $p_{\text{reactive}}(t)$

2. Compute average real power $P$, reactive power $Q$, apparent power $S$, and power factor $pf$.

3. Repeat the above when load is $Z_L = 12 - j13 \ \Omega$, $Z_L = 12 \ \Omega$, and $Z_L = j13 \ \Omega$

4. Comment upon the results.

Solution: A single phase circuit supplying linear load is shown Fig. 1.1. In general, the current in the circuit is given as,

$$i(t) = \sqrt{2}I \sin(\omega t - \phi)$$

where $\phi = \tan^{-1}(X/R_L)$, and $I = (V/|Z_L|)$

Case 1: When load is inductive, $Z_L = 12 + j13 \ \Omega$

$$|Z_L| = \sqrt{R_L^2 + X_L^2} = \sqrt{12^2 + 13^2} = 17.692 \ \Omega$$

and $I = 230/17.692 = 13 \ \text{A}$

$\phi = \tan^{-1}(X/R) = \tan^{-1}(13/12) = 47.29^\circ$

Therefore we have,

$$v(t) = \sqrt{2} \times 230 \sin \omega t$$

$$i(t) = \sqrt{2} \times 13 \sin(\omega t - 47.29^\circ)$$

The instantaneous power is given as,

$$p(t) = VI \cos \phi(1 - \cos 2\omega t) - VI \sin \phi \sin 2\omega t$$

$$= 230 \times 13 \cos 47.29^\circ(1 - \cos(2 \times 314t)) - 230 \times 13 \sin 47.29^\circ \sin(2 \times 314t)$$

$$= 2028.23(1 - \cos(2 \times 314t)) - 2196.9 \sin(2 \times 314t)$$

$$= p_{\text{active}}(t) - p_{\text{reactive}}(t)$$

The above implies that,

$$p_{\text{active}}(t) = 2028.23(1 - \cos(2 \times 314t))$$

$$p_{\text{reactive}}(t) = 2196.9 \sin(2 \times 314t)$$

3
Average real power \( (P) \) is given as,

\[
P = \frac{1}{T} \int_{0}^{T} p(t) \, dt
\]

\[
P = VI \cos \phi = 230 \times 13 \times \cos 47.29^\circ = 2028.23 \text{ W}
\]

Reactive power \( (Q) \) is given as maximum value of \( p_{\text{reactive}} \), and equals to \( VI \sin \phi \) as given below.

\[
Q = VI \sin \phi = 230 \times 13 \times \sin 47.2906^\circ = 2196.9 \text{ VAr}
\]

Apparant power, \( S = \sqrt{P^2 + Q^2} = 230 \times 13 = 2990 \text{ VA} \)

Power factor \( = \frac{P}{S} = \frac{2028.23}{2990} = 0.6783 \)

For this case, the voltage, current and various components of the power are shown in Fig. 1.2. As seen from the figure the current lags the voltage due to inductive load. The \( p_{\text{active}} \) has an offset of 2028.23 W, which is also indicated as \( P \) in the right bottom graph. The \( p_{\text{reactive}} \) has zero average value and its maximum value is equal to \( Q \), which is 2196.9 VArs.

![Fig. 1.2 Case 1: Voltage, current and various power components](image-url)
Case 2: When load is Capacitive, \( Z_L = 12 - j13 \) that implies \(|Z_L| = \sqrt{12^2 + 13^2} = 17.692\Omega\), and \( I = 230/17.692 = 13 \text{ A} \), \( \phi = \tan^{-1}(-13/12) = -47.2906^\circ \).

\[
v(t) = \sqrt{2} \times 230 \sin \omega t \\
i(t) = \sqrt{2} \times 13 \sin(\omega t + 47.2906^\circ) \\
p(t) = VI \cos \phi (1 - \cos 2\omega t) - VI \sin \phi \sin 2\omega t \\
= 230 \times 13 \cos(-47.2906^\circ)(1 - \cos(2 \times 314t)) - 230 \times 13 \sin(-47.2906^\circ) \sin(2 \times 314t) \\
= 2028.23(1 - \cos(2 \times 314t)) + 2196.9 \sin(2 \times 314t)
\]

\[
p_{\text{active}}(t) = 2028.23(1 - \cos(2 \times 314t)) \\
p_{\text{reactive}}(t) = -2196.9 \sin(2 \times 314t)
\]

\[
P = VI \cos \phi = 230 \times 13 \times \cos 47.2906^\circ = 2028.23 \text{ Watt} \\
Q = VI \sin \phi = 230 \times 13 \times \sin(-47.2906^\circ) = -2196.9 \text{ VAr} \\
S = VI = \sqrt{P^2 + Q^2} = 230 \times 13 = 2990 \text{ VA}
\]

For Case 2, the voltage, current and various components of the power are shown in Fig. 1.3. The explanation given earlier also holds true for this case.

![Graphs showing voltage, current, and power components](image-url)
Case 3: When load is resistive, $Z_L = 12 \, \Omega$, $|Z_L| = 12 \, \Omega$, $I = (230/12) = 19.167 \, \text{A}$, and $\phi = 0^\circ$. Therefore, we have

$$v(t) = \sqrt{2} \times 230 \sin \omega t$$
$$i(t) = \sqrt{2} \times 19.167 \sin \omega t$$
$$p(t) = 230 \times 19.167 \cos 0^\circ(1 - \cos(2 \times 314t)) - 230 \times 19.167 \sin 0^\circ \sin(2 \times 314t)$$
$$= 4408.33(1 - \cos(2 \times 314t)) - 0$$

$$p_{\text{active}}(t) = 4408.33(1 - \cos(2 \times 314t))$$
$$p_{\text{reactive}}(t) = 0$$

$$P = VI \cos \phi = 230 \times 19.167 \times \cos 0^\circ = 4408.33 \, \text{W}$$
$$Q = VI \sin \phi = 230 \times 19.167 \times \sin 0^\circ = 0 \, \text{VA}$$
$$S = VI = \sqrt{P^2 + Q^2} = 230 \times 19.167 = 4408.33 \, \text{VA}$$

Power factor $= \frac{4408.33}{4408.33} = 1$

For Case 3, the voltage, current and various components of the power are shown in Fig. 1.4. Since the load is resistive, as seen from the graph $p_{\text{reactive}}$ is zero and $p(t)$ is equal to $p_{\text{active}}$. The average value of $p(t)$ is real power ($P$), which is equal to 4408.33 W.

![Graph of voltage, current, active power, reactive power, and average power for Case 3](image-url)
Case 4: When the load is purely reactive, $Z_L = j 13 \, \Omega$, $|Z_L| = 13 \, \Omega$, $I = \frac{230}{13} = 17.692 \, \text{A}$, and $\phi = 90^\circ$. Therefore, we have

\begin{align*}
    v(t) &= \sqrt{2} \times 230 \sin \omega t \\
    i(t) &= \sqrt{2} \times 17.692 \sin(\omega t - 90^\circ) \\
    p(t) &= 230 \times 17.692 \cos 90^\circ(1 - \cos(2 \times 314t)) - 230 \times 17.692 \sin 90^\circ \sin(2 \times 314t) \\
        &= 0 - 4069 \sin(2 \times 314t)
\end{align*}

\begin{align*}
    p_{active}(t) &= 0 \\
    p_{reactive}(t) &= 4069 \sin(2 \times 314t)
\end{align*}

\begin{align*}
    P &= VI \cos \phi = 230 \times 17.692 \times \cos 90^\circ = 0 \, \text{W} \\
    Q &= VI \sin \phi = 230 \times 17.692 \times \sin 90^\circ = 4069 \, \text{VAR} \\
    S &= VI = \sqrt{P^2 + Q^2} = 230 \times 17.692 = 4069 \, \text{VA}
\end{align*}

Power factor $= \frac{0}{4069} = 0$

For Case 4, the voltage, current and various components of the power are shown in Fig. 1.5. The load in this case is purely reactive, hence their is no average component of $p(t)$. The maximum value of $p(t)$ is same as $p_{reactive}(t)$ or $Q$, which is equal to 4069 VARs.
1.3 Sinusoidal Voltage Source Supplying Non-linear Load Current

The load current is now considered as nonlinear load such as single-phase rectifier load. The voltage and current are expressed as following.

\[ v(t) = \sqrt{2}V \sin \omega t \]
\[ i(t) = \sqrt{2} \sum_{n=1}^{\infty} I_n \sin(n \omega t - \phi_n) \]  

(1.6)

The instantaneous power is therefore given by,

\[ p(t) = v(t) i(t) = \sqrt{2}V \sin \omega t \sqrt{2} \sum_{n=1}^{\infty} I_n \sin(n \omega t - \phi_n) \]
\[ = V \sum_{n=1}^{\infty} [I_n \sin(\omega t - \phi_n) - \cos \phi_n \sin 2n \omega t] \]
\[ = V [I_1 \sin(\omega t - \phi_1)] + V \sum_{n=2}^{\infty} [I_n \sin(\omega t - \phi_n)] \]  

(1.7)

Note that \( 2 \sin A \sin B = \cos(A - B) - \cos(A + B) \), using this, Eqn. (1.7) can be re-written as the following.

\[ p(t) = V I_1 [\cos \phi_1 - \cos(2 \omega t - \phi_1)] - V I_1 \sin \phi_1 \sin 2\omega t \]
\[ + V \sum_{n=2}^{\infty} I_n [(\cos \phi_n - \cos(2n \omega t - \phi_n)) - \sin \phi_n \sin 2n \omega t] \]
\[ = V I_1 \cos \phi_1 (1 - \cos 2\omega t) - V I_1 \sin \phi_1 \sin 2\omega t \]
\[ + \sum_{n=2}^{\infty} V I_n [\cos \phi_n (1 - \cos 2n \omega t) - \sin \phi_n \sin 2n \omega t] \]
\[ = A + B \]  

(1.8)

In above equation, average active power \( P \) and reactive power \( Q \) are given by,

\[ P = P_1 = \text{average value of } p(t) = V I_1 \cos \phi_1 \]
\[ Q = Q_1 = \text{peak value of second term in } A = V I_1 \sin \phi_1 \]  

(1.9)

The apparent power \( S \) is given by

\[ S = V I \]
\[ S = V \sqrt{[I_1^2 + I_2^2 + I_3^2 + ...]} \]  

(1.10)
Equation (1.10) can be re-arranged as given below.

\[
S^2 = V^2 I_1^2 + V^2 [I_2^2 + I_3^2 + I_4^2 + ...] \\
= (V I_1 \cos \phi_1)^2 + (V I_1 \sin \phi_1)^2 + V^2 [I_2^2 + I_3^2 + I_4^2 + .....] \\
= P^2 + Q^2 + H^2
\]

(1.11)

In above equation, \( H \) is known as harmonic power and represents \( VAs \) corresponding to harmonics and is equal to,

\[
H = V \sqrt{[I_2^2 + I_3^2 + I_4^2 + .....]}
\]

(1.12)

The following points are observed from description.

1. \( P \) and \( Q \) are dependent on the fundamental current components
2. \( H \) is dependent on the current harmonic components
3. Power components \(-V I \cos 2\omega t \) and \( V I_1 \sin \phi_1 \sin 2\omega t \) are oscillating components and can be eliminated using appropriately chosen capacitors and inductors
4. There are other terms in (1.10), which are functions of multiple integer of fundamental frequency are reflected in ‘B’ terms of Eqn. (1.8). These terms can be eliminated using tuned LC filters.

This is represented by power tetrahedron instead of power triangle (in case of voltage and current of sinusoidal nature of fundamental frequency). In this context, some important terms are defined here.

**Displacement Factor or Fundamental Power Factor** (DPF) is denoted by \( \cos \phi_1 \) and is cosine angle between the fundamental voltage and current.

**Distortion Factor** (DF) is defined as ratio of fundamental apparent power \((V_1 I_1)\) to the total apparent power \((V I)\).

\[
\text{Distortion factor (DF)} = \frac{\sqrt{(P_1^2 + Q_1^2)}}{S} \\
= \frac{\sqrt{V^2 I_1^2 \cos^2 \phi_1 + V^2 I_1^2 \sin^2 \phi_1}}{V I} \]

\[
= \frac{\sqrt{V^2 I_1^2}}{V I} \\
= \frac{I_1}{I} = \cos \gamma
\]

(1.13)

The **Power Factor** \((p_f)\) is defined as ratio of average active power to the total apparent power \((V I)\) and is expressed as,
Power Factor \( (pf) \) = \( \frac{P}{S} \)

\[ = \frac{V I_1 \cos \phi_1}{V I} \]

\[ = \left( \frac{I_1}{I} \right) \cos \phi_1 \]

\[ = \cos \gamma \cos \phi_1 \] (1.14)

The equation (1.14) shows that power factor becomes less by a factor of \( \cos \gamma \). This is due to the presence of the harmonics in the load current. The nonlinear load current increases the ampere rating of the conductor for same amount of active power transfer with increased VA rating. Such kind of load is not desired in power system.

Example 1.2 Consider following single phase system supplying a rectifier load as given in Fig. 1.6. Given a supply voltage, \( v(t) = \sqrt{2} \times 230 \sin \omega t \) and source impedance is negligible, draw the voltage and current waveforms. Express current using Fourier series. Based on that determine the following.

1. Plot instantaneous power \( p(t) \).
2. Plot components of \( p(t) \) i.e. \( p_{\text{active}}(t) \), \( p_{\text{reactive}}(t) \).
3. Compute average real power, reactive power, apparent power, power factor, displacement factor (or fundamental power factor) and distortion factor.
4. Comment upon the results in terms of VA rating and power output.

![Fig. 1.6 A single phase system with non-linear load](image)

**Solution:** The above system has been simulated using MATLAB/SIMULINK. The supply voltage and current are shown in Fig. 1.7. The current waveform is of the square type and its Fourier
series expansion is given below.

\[ i(t) = \sum_{n=2h+1}^{\infty} \frac{4I_{dc}}{n\pi} \sin(n\omega t) \quad \text{where} \quad h = 0, 1, 2 \ldots \]

The instantaneous power is therefore given by,

\[ p(t) = v(t) i(t) = \sqrt{2}V \sin \omega t \sum_{n=2h+1}^{\infty} \frac{4I_{dc}}{n\pi} \sin(n\omega t). \quad (1.15) \]

By expansion of the above equation, the average active power \( P \) and reactive power \( Q \) are given as below.

\[
P = P_1 = \text{average value of } P_{\text{active}}(t) \text{ or } p(t) = V I_1 \cos \phi_1 \\
= V I_1 \quad (\text{since, } \phi_1 = 0, \cos \phi_1 = 1, \sin \phi_1 = 0) \\
Q = Q_1 = \text{peak value of } P_{\text{reactive}}(t) = V I_1 \sin \phi_1 = 0
\]

![Supply voltage, current and instantaneous power waveforms](image)

Fig. 1.7 Supply voltage, current and instantaneous power waveforms

The rms value of fundamental and rms value of the total source current are given below.

\[
I_{rms} = I_d = 103.5 \text{ A} \\
I_1 = \frac{2\sqrt{2}}{\pi} I_d = 93.15 \text{ A}
\]
The real power \((P)\) is given by
\[
P = V I_1
= V \times \frac{2\sqrt{2}}{\pi} I_d = 21424.5 \text{ W}.
\]

The reactive power \((Q)\) is given by
\[
Q = Q_1 = 0.
\]

The apparent power \((S)\) is given by
\[
S = V I_{rms}
= V I_d = 23805 \text{ VA}.
\]

The distortion factor \((\cos \gamma)\) is,
\[
DF = \frac{\sqrt{P_1^2 + Q_1^2}}{S}
= \frac{I_1}{I_{rms}} = \cos \gamma = 0.9.
\]

The displacement factor \((\cos \phi_1)\) is,
\[
DPF = \cos \phi_1 = 1.
\]

Therefore power factor is given by,
\[
pf = \frac{P}{S} = \frac{P_1}{S_1} \frac{S_1}{S}
= \cos \gamma \cos \phi_1
= DF \times DPF = 0.9 \text{ (lag)}
\]

1.4 Non-sinusoidal Voltage Source Supplying Non-linear Loads

The voltage source too may have harmonics transmitted from generation or produced due to non-linear loads in presence of feeder impedance. In this case, we shall consider generalized case of non-sinusoidal voltage source supplying nonlinear loads including dc components. These voltages and currents are represented as,
\[
v(t) = V_{dc} + \sum_{n=1}^{\infty} \sqrt{2}V_n \sin(n\omega t - \phi_{vn})
\]  \hspace{1cm} (1.16)

and
\[
i(t) = I_{dc} + \sum_{n=1}^{\infty} \sqrt{2}I_n \sin(n\omega t - \phi_{in})
\]  \hspace{1cm} (1.17)
Therefore, instantaneous power \( p(t) \) is given by,

\[
p(t) = [V_{dc} + \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n \omega t - \phi_{vn})][I_{dc} + \sum_{n=1}^{\infty} \sqrt{2} I_n \sin(n \omega t - \phi_{in})]
\]  

(1.18)

\[
p(t) = V_{dc} I_{dc} + V_{dc} \sum_{n=1}^{\infty} \sqrt{2} I_n \sin(n \omega t - \phi_{in}) + I_{dc} \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n \omega t - \phi_{vn})
\]

\[
+ \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n \omega t - \phi_{vn}) \sum_{n=1}^{\infty} \sqrt{2} I_n \sin(n \omega t - \phi_{in})
\]  

(1.19)

\[
p(t) = p_{dc-de} + p_{dc-ac} + p_{ac-de} + p_{ac-ac}
\]  

(1.20)

The I term \( (p_{dc-de}) \) contribute to power from dc components of voltage and current. Terms II \( (p_{dc-ac}) \) and III \( (p_{ac-de}) \) are result of interaction of dc and ac components of voltage and current. In case, there are no dc components all these power components are zero. In practical cases, dc components are very less and the first three terms have negligible value compared to IV term. Thus, we shall focus on IV \( (p_{ac-ac}) \) term which correspond to ac components present in power system. The IV term can be written as,

\[
IV^{th} \text{ term} = p_{ac-ac} = \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n \omega t - \phi_{vn}) \sum_{h=1}^{\infty} \sqrt{2} I_h \sin(h \omega t - \phi_{ih})
\]  

(1.21)

where \( n = h = 1, 2, 3..., \) similar frequency terms will interact. When \( n \neq h, \) dissimilar
The terms in A of above equation form similar frequency terms and terms in B form dissimilar frequency terms, we shall denote them by $p_{ac-ac-nn}$ and $p_{ac-ac-nh}$. Thus,

$$p_{ac-ac-nn}(t) = \sum_{n=1}^{\infty} V_n I_n 2 \sin(n\omega t - \phi_v) \sin(n\omega t - \phi_i)$$  \hspace{1cm} (1.23)$$

and

$$p_{ac-ac-nh}(t) = \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n\omega t - \phi_v) \sum_{h=1, h \neq n}^{\infty} \sqrt{2} I_h \sin(h\omega t - \phi_i)$$  \hspace{1cm} (1.24)$$
Now, let us simplify $p_{ac-ac-nn}$ in

\[
p_{ac-ac-nn}(t) = \sum_{n=1}^{\infty} V_n I_n [\cos(\phi_n - \phi_{vn}) - \cos(2n\omega t - \phi_{in} - \phi_{vn})] \]

\[
= \sum_{n=1}^{\infty} V_n I_n [\cos(\phi_n) - \cos(2n\omega t - (\phi_{in} - \phi_{vn}) - 2\phi_{vn})] \]

\[
= \sum_{n=1}^{\infty} V_n I_n [\cos(\phi_n) - \cos (2n\omega t - 2\phi_{vn}) - \phi_n] \]

\[
= \sum_{n=1}^{\infty} [V_n I_n \cos \phi_n \{1 - \cos(2n\omega t - 2\phi_{vn})\}]

- V_n I_n \sin \phi_n \sin(2n\omega t - 2\phi_{vn}) \]  

(1.25)

where $\phi_n = (\phi_{in} - \phi_{vn}) =$ phase angle between $n^{th}$ harmonic current and voltage.

\[
p_{ac-ac-nn}(t) = \sum_{n=1}^{\infty} [V_n I_n \cos \phi_n \{1 - \cos(2n\omega t - 2\phi_{vn})\}]

- \sum_{n=1}^{\infty} [V_n I_n \sin \phi_n \sin(2n\omega t - 2\phi_{vn})] \]  

(1.26)

Therefore, the instantaneous power is given by,

\[
p(t) = p_{dc-dc} + p_{dc-ac} + p_{ac-de} + p_{ac-ac-nn} + p_{ac-ac-nh}

\]

\[
p(t) = V_{dc} I_{dc} + V_{dc} \sum_{n=1}^{\infty} \sqrt{2} I_n \sin(n\omega t - \phi_{in}) + I_{dc} \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n\omega t - \phi_{vn})

+ \sum_{n=1}^{\infty} [V_n I_n \cos \phi_n \{1 - \cos(2n\omega t - \phi_{vn})\}]

- \sum_{n=1}^{\infty} [V_n I_n \sin \phi_n \sin(2n\omega t - \phi_{in})] \]  

(1.27)

1.4.1 Active Power

Instantaneous active power, $p_{active}(t)$ is expressed as,

\[
p_{active}(t) = V_{dc} I_{dc} + \sum_{n=1}^{\infty} [V_n I_n \cos \phi_n \{1 - \cos(2n\omega t - \phi_{vn})\}] \]  

(1.28)
It has non-negative value with some average component, giving average active power. Therefore,

\[
P = \frac{1}{T} \int_{0}^{T} p(t) \, dt \]

\[
= V_{dc}I_{dc} + \sum_{n=1}^{\infty} V_n I_n \cos \phi_n. \tag{1.29}
\]

Reactive power can be defined as

\[
q(t) = p_{\text{reactive}}(t) = -\sum_{n=1}^{\infty} [V_n I_n \sin \phi_n \sin(2n\omega t - 2\phi_vn)] \tag{1.30}
\]

resulting in

\[
Q \triangleq \text{max of (1.30) magnitude} \nonumber
\]

\[
= \sum_{n=1}^{\infty} V_n I_n \sin \phi_n. \tag{1.31}
\]

From (1.29)

\[
P = P_{dc} + \sum_{n=1}^{\infty} V_n I_n \cos \phi_n
\]

\[
= P_{dc} + V_1 I_1 \cos \phi_1 + V_2 I_2 \cos \phi_2 + V_3 I_3 \cos \phi_3 + \ldots
\]

\[
= P_{dc} + P_1 + P_2 + P_3 + \ldots
\]

\[
= P_{dc} + P_1 + P_H \tag{1.32}
\]

In above equation,

\[P_{dc} = \text{Average active power corresponding to the dc components} \]

\[P_1 = \text{Average fundamental active power} \]

\[P_H = \text{Average Harmonic active power} \]

Average fundamental active power \((P_1)\) can also be found from fundamentals of voltage and current i.e.,

\[
P_1 = \frac{1}{T} \int_{0}^{T} v_1(t) i_1(t) \, dt \tag{1.33}
\]

and harmonic active power \((P_H)\) can be found

\[
P_H = \sum_{n=1}^{\infty} V_n I_n \cos \phi_n = P - P_1 \tag{1.34}
\]
1.4.2 Reactive Power

The reactive power or Budeanu’s reactive power (Q) can be found by summing maximum value of each term in (1.30). This is given below.

\[
Q = \sum_{n=1}^{\infty} V_n I_n \sin \phi_n
\]

\[
= V_1 I_1 \sin \phi_1 + V_2 I_2 \sin \phi_2 + V_3 I_3 \sin \phi_3 + \ldots
\]

\[
= Q_1 + Q_2 + Q_3 + \ldots
\]

\[
= Q_1 + Q_H
\]

(1.35)

Usually this reactive power is referred as Budeanu’s reactive power, and sometimes we use subscript B’ to indicate that i.e.,

\[
Q_B = Q_{1B} + Q_{HB}
\]

(1.36)

The remaining dissimilar terms of (1.27) are accounted using \( p_{rest}(t) \). Therefore, we can write,

\[
p(t) = p_{dc} - p_{dc} + p_{active(t)} + p_{reactive(t)} + p_{rest(t)}
\]

\[
\text{similar frequency terms} + \text{non-similar frequency terms}
\]

(1.37)

where,

\[
p_{dc} - p_{dc} = V_{dc} I_{dc}
\]

\[
p_{active(t)} = \sum_{n=1}^{\infty} [V_n I_n \cos \phi_n \{1 - \cos(2n \omega t - 2 \phi_{vn})\}]
\]

\[
p_{reactive(t)} = -\sum_{n=1}^{\infty} [V_n I_n \sin \phi_n \sin(2n \omega t - 2 \phi_{vn})]
\]

\[
p_{rest(t)} = V_{dc} \sum_{n=1}^{\infty} \sqrt{2} I_n \sin(n \omega t - \phi_{in}) + I_{dc} \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n \omega t - \phi_{vn})
\]

\[+ \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n \omega t - \phi_{vn}) \sum_{m=1,m \neq n}^{\infty} \sqrt{2} I_m \sin(m \omega t - \phi_{im})
\]

(1.38)

1.4.3 Apparent Power

The scalar apparent power which is defined as product of rms value of voltage and current, is expressed as following.

\[
S = VI
\]

\[
= \sqrt{V_{dc}^2 + V_1^2 + V_2^2 + \cdots} \sqrt{I_{dc}^2 + I_1^2 + I_2^2 + \cdots}
\]

\[
= \sqrt{V_{dc}^2 + V_1^2 + V_H^2} \sqrt{I_{dc}^2 + I_1^2 + I_H^2}
\]

(1.39)
Where,

\[ V_H^2 = V_2^2 + V_3^2 + \cdots = \sum_{n=2}^{\infty} V_n^2 \]

\[ I_H^2 = I_2^2 + I_3^2 + \cdots = \sum_{n=2}^{\infty} I_n^2 \]  \hspace{1cm} (1.40)

\( V_H \) and \( I_H \) are denoted as harmonic voltage and harmonic current respectively. Expanding (1.39) we can write

\[ S^2 = V^2 I^2 \]

\[ = (V_{dc}^2 + V_1^2 + V_H^2)(I_{dc}^2 + I_1^2 + I_H^2) \]

\[ = V_{dc}^2 I_{dc}^2 + V_{dc}^2 I_1^2 + V_{dc}^2 I_H^2 + V_1^2 I_{dc}^2 + V_1^2 I_1^2 + V_1^2 I_H^2 + V_H^2 I_{dc}^2 + V_H^2 I_1^2 + V_H^2 I_H^2 \]

\[ = V_{dc}^2 I_{dc}^2 + V_1^2 I_1^2 + V_1^2 I_H^2 + V_{dc}^2 (I_1^2 + I_H^2) + I_{dc}^2 (V_1^2 + V_H^2) + V_1^2 I_H^2 + V_H^2 I_1^2 \]

\[ = S_{dc}^2 + S_1^2 + S_H^2 + S_D^2 \]

\[ = S_1^2 + S_N^2 \]  \hspace{1cm} (1.41)

In above equation, the term \( S_N \) is as following.

\[ S_N^2 = V_{dc}^2 I_1^2 + V_{dc}^2 I_H^2 + V_1^2 I_{dc}^2 + V_1^2 I_1^2 + V_1^2 I_H^2 + V_H^2 I_{dc}^2 + V_H^2 I_1^2 + V_H^2 I_H^2 + I_{dc}^2 I_1^2 + I_{dc}^2 I_H^2 \]  \hspace{1cm} (1.42)

Practically in power systems dc components are negligible. Therefore neglecting the contribution of \( V_{dc} \) and \( I_{dc} \) associated terms in (1.42), the following is obtained.

\[ S_N^2 = I_1^2 V_H^2 + V_1^2 I_H^2 + V_1^2 I_H^2 \]

\[ = D_V^2 + D_I^2 + S_H^2 \]  \hspace{1cm} (1.43)

The terms \( D_I \) and \( D_V \) in (1.43) are known as apparent powers due to distortion in current and voltage respectively. These are given below.

\[ D_V = I_1 V_H \]

\[ D_I = V_1 I_H \]  \hspace{1cm} (1.44)

These are further expressed in terms of THD components of voltage and current, as given below.

\[ THD_V = \frac{V_H}{V_1} \]

\[ THD_I = \frac{I_H}{I_1} \]  \hspace{1cm} (1.45)

From (1.45), the harmonic components of current and voltage are expressed below.

\[ V_H = THD_V V_1 \]

\[ I_H = THD_I I_1 \]  \hspace{1cm} (1.46)
Using (1.44) and (1.46),

\[ \begin{align*}
D_V &= V_1 I_1 THD_V = S_1 THD_V \\
D_I &= V_1 I_1 THD_I = S_1 THD_I \\
S_H &= V_H I_H = S_1 THD_I THD_V
\end{align*} \] (1.47)

Therefore using (1.43) and (1.47), \( S_N \) could be expressed as following.

\[ S_N^2 = S_1^2 (THD_I^2 + THD_V^2 + THD_I^2 THD_V^2) \] (1.48)

Normally in power system, \( THD_V << THD_I \), therefore,

\[ S_N \approx S_1 D_I \] (1.49)

The above relationship shows that as the THD content in voltage and current increases, the non fundamental apparent power \( S_N \) increases for a given useful transmitted power. This means there are more losses and hence less efficient power network.

### 1.4.4 Non Active Power

Non active power is denoted by \( N \) and is defined as per following equation.

\[ S^2 = P^2 + N^2 \] (1.50)

This power includes both fundamental as well as non fundamental components, and is usually computed by knowing active power \( (P) \) and apparent power \( (S) \) as given below.

\[ N = \sqrt{S^2 - P^2} \] (1.51)

### 1.4.5 Distortion Power

Due to presence of distortion, the total apparent power \( S \) can also be written in terms of active power \( (P) \), reactive power \( (Q) \) and distortion power \( (D) \)

\[ S^2 = P^2 + Q^2 + D^2. \] (1.52)

Therefore,

\[ D = \sqrt{S^2 - P^2 - Q^2}. \] (1.53)

### 1.4.6 Fundamental Power Factor

Fundamental power factor is defined as ratio of fundamental real power \( (P_1) \) to the fundamental apparent power \( (S_1) \). This is given below.

\[ pf_1 = \cos \phi_1 = \frac{P_1}{S_1} \] (1.54)

The fundamental power factor as defined above is also known as displacement power factor.
1.4.7 Power Factor

Power factor for the single phase system considered above is the ratio of the total real power \( P \) to the total apparent power \( S \) as given by the following equation.

\[
pf = \frac{P}{S} = \frac{P_1 + P_H}{\sqrt{S_1^2 + S_N^2}} = \left(1 + \frac{P_H}{P_1}\right) \frac{P_1}{\sqrt{1 + (S_N/S_1)^2 S_1}}
\]

Substituting \( S_N \) from (1.48), the power factor can further be simplified to the following equation.

\[
pf = \frac{(1 + P_H/P_1)}{\sqrt{1 + THD_I^2 + THD_V^2 + THD_I^2 THD_V^2}} \cdot pf_1
\]

Thus, we observe that the power factor of a single phase system depends upon fundamental \( P_1 \) and harmonic active power \( P_H \), displacement factor\( DPF = pf_1 \) and THDs in voltage and current. Further, we note following points.

1. \( P/S \) is also called as utilization factor indicator as it indicates the usage of real power.
2. The term \( S_N/S_1 \) is used to decide the overall degree of harmonic content in the system.
3. The flow of fundamental power can be characterized by measurement of \( S_1, P_1, pf_1, \) and \( Q_1 \).

For a practical power system \( P_1 \gg P_H \) and \( THD_V \ll THD_I \), the above expression of power factor is further simplified as given below.

\[
pf = \frac{pf_1}{\sqrt{1 + THD_I^2}}
\]

**Example 1.3** Consider the following voltage and current in single phase system.

\[
\begin{align*}
v_s(t) &= \sqrt{2} \times 230 \sin(\omega t) + \sqrt{2} \times 50 \sin(3\omega t - 30^\circ) \\
i(t) &= 2 + \sqrt{2} \times 10 \sin(\omega t - 30^\circ) + \sqrt{2} \times 5 \sin(3\omega t - 60^\circ)
\end{align*}
\]

Determine the following.

(a) Active power, \( P \)
(b) Reactive power, \( Q \)
(c) Apparent power, \( S \)
(d) Power factor, \( pf \)
**Solution:** Here the source is non-sinusoidal and is feeding a non-linear load. The instantaneous power is given by,

\[ p(t) = v(t) i(t) \]

\[ p(t) = \left\{ V_{dc} + \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n\omega t - \phi_{vn}) \right\} \{ I_{dc} + \sum_{n=1}^{\infty} \sqrt{2} I_n \sin(n\omega t - \phi_{in}) \} \]

(a) The active power ‘P’ is given by,

\[ P = \frac{1}{T} \int_{0}^{T} p(t) \, dt \]

\[ = P_{dc} + V_1 I_1 \cos \phi_1 + V_2 I_2 \cos \phi_2 + \ldots + V_n I_n \cos \phi_n \quad (1.58) \]

where,

\[ \phi_n = \phi_{in} - \phi_{vn} \]

\[ P_{dc} = V_{dc} I_{dc} \]

\[ P_1 = V_1 I_1 \cos \phi_1 \]

\[ P_H = \sum_{n=2}^{\infty} V_n I_n \cos \phi_n \]

Here, \( V_{dc} = 0 \), \( V_1 = 230 \, V \), \( \phi_{v1} = 0 \), \( V_3 = 50 \, V \), \( \phi_{v3} = 30^\circ \), \( I_{dc} = 2 \, A \), \( I_1 = 10 \, A \), \( \phi_{i1} = 30^\circ \), \( I_3 = 5 \, A \), \( \phi_{i3} = 60^\circ \). Therefore, \( \phi_1 = \phi_{i1} - \phi_{v1} = 30^\circ \) and \( \phi_3 = \phi_{i3} - \phi_{v3} = 30^\circ \).

Substituting these values in (1.58), the above equation gives,

\[ P = 0 \times 2 + 230 \times 10 \times \cos 30^\circ + 50 \times 5 \times \cos 30^\circ = 2208.36 \, W. \]

(b). The reactive power \( (Q) \) is given by,

\[ Q = \sum_{n=1}^{\infty} V_n I_n \sin \phi_n \]

\[ = V_1 I_1 \sin \phi_1 + V_2 I_2 \sin \phi_2 + \ldots + V_n I_n \sin \phi_n \]

\[ = 230 \times 10 \times \sin 30^\circ + 50 \times 5 \times \sin 30^\circ = 1275 \, \text{VAR}. \]

(c). The Apparent power \( S \) is given by,

\[ S = V_{rms} I_{rms} \]

\[ = \sqrt{V_{dc}^2 + V_1^2 + V_2^2 + \ldots V_n^2} \sqrt{I_{dc}^2 + I_1^2 + I_2^2 + \ldots I_n^2} \]

\[ = \sqrt{V_{dc}^2 + V_1^2 + V_2^2} \sqrt{I_{dc}^2 + I_1^2 + I_2^2} \]

where,

\[ V_H = \sqrt{V_3^2 + \ldots V_n^2} \]

\[ I_H = \sqrt{I_3^2 + \ldots I_n^2} \]
Substituting the values of voltage and current components, the apparent power $S$ is computed as following.

\[
S = \sqrt{0 + 230^2 + 50^2} \sqrt{2^2 + 10^2 + 5^2} \\
= 235.37 \times 11.357 = 2673.31 \text{ VA}
\]

(d). The power factor is given by

\[
\text{pf} = \frac{P}{S} = \frac{2208.36}{2673.31} = 0.8261 \text{ lag}
\]

**Example 1.4** Consider following system with distorted supply voltages,

\[
v(t) = V_{dc} + \sum_{n=1}^{\infty} \sqrt{2} V_n \frac{n^2}{n^2} \sin(n \omega t - \phi_{vn})
\]

with $V_{dc} = 10 V$, $V_n/n^2 = 230\sqrt{2}/n^2$ and $\phi_{vn} = 0$ for $n = 1, 3, 5, 7, \ldots$

The voltage source supplies a nonlinear current of,

\[
i(t) = I_{dc} + \sum_{n=1}^{\infty} \sqrt{2} I_n \frac{n}{n} \sin(n \omega t - \phi_{in}).
\]

with $I_{dc} = 2 A$, $I_n = 20/n$ A and $\phi_{in} = n \times 30^\circ$ for $n = 1, 3, 5, 7, \ldots$

Compute the following.

1. Plot instantaneous power $p(t)$, $p_{\text{active}}(t)$, $p_{\text{reactive}}(t)$, $P_{dc}$, and $p_{\text{rest}}(t)$.

2. Compute $P$, $P_1$, $P_H$ ($= P_3 + P_5 + P_7 + \ldots$).

3. Compute $Q$, $Q_1$, $Q_H$ ($= Q_3 + Q_5 + Q_7 + \ldots$).

4. Compute $S$, $S_1$, $S_H$, $N$, $D$.

5. Comment upon each result.

**Solution:** Instantaneous power is given as following.
\[ p(t) = v(t) i(t) = \left( 10 + \sum_{n=1,3,5}^{\infty} \frac{230 \sqrt{2}}{n^2} \sin(n\omega t) \right) \left( 2 + \sum_{n=1,3,5}^{\infty} \frac{20 \sqrt{2}}{n} \sin(n\omega t - 30^\circ) \right) \]

\[ = \frac{20}{I} + 10 \sum_{n=1,3,5}^{\infty} \frac{20 \sqrt{2}}{n} \sin(n\omega t - 30^\circ) + 2 \sum_{n=1,3,5}^{\infty} \frac{230 \sqrt{2}}{n^2} \sin(n\omega t) \]

\[ + \left( \sum_{n=1,3,5}^{\infty} \frac{230 \sqrt{2}}{n^2} \sin(n\omega t) \right) \left( \sum_{n=1,3,5}^{\infty} \frac{20 \sqrt{2}}{n} \sin(n\omega t - 30^\circ) \right) \]

\[ = \frac{20}{I} + \sum_{n=1,3,5}^{\infty} \frac{200 \sqrt{2}}{n} \sin(n\omega t - 30^\circ) + \sum_{n=1,3,5}^{\infty} \frac{460 \sqrt{2}}{n^2} \sin(n\omega t) \]

\[ + \sum_{n=1,3,5}^{\infty} \frac{4600}{n^3} (\cos(30^\circ n)(1 - \cos 2n\omega t) - \sin (2n\omega t) \sin(30^\circ n)) \]

\[ + \left( \sum_{n=1,3,5}^{\infty} \frac{230 \sqrt{2}}{n^2} \sin(n\omega t) \right) \left( \sum_{h=1,3,5; h \neq n}^{\infty} \frac{20 \sqrt{2}}{h} \sin(h\omega t - 30^\circ) \right) \]

**1. Computation of** \( p(t), p_{active}(t), p_{reactive}(t), P_{dc} ,\) **and** \( p_{rest}(t) \)

\[ p_{dc--dc}(t) = 20 \text{ W} \]

\[ p_{active}(t) = \sum_{n=1,3,5}^{\infty} \frac{4600}{n^3} \cos n30^\circ(1 - \cos 2n\omega t) \]

\[ p_{reactive}(t) = - \sum_{n=1,3,5}^{\infty} \frac{4600}{n^3} \sin(n30^\circ) \sin(2n\omega t) \]

\[ p_{rest}(t) = \sum_{n=1,3,5}^{\infty} \frac{200 \sqrt{2}}{n} \sin(n\omega t - 30^\circ) + \sum_{n=1,3,5}^{\infty} \frac{460 \sqrt{2}}{n^2} \sin(n\omega t) \]

\[ + \left( \sum_{n=1,3,5}^{\infty} \frac{230 \sqrt{2}}{n^2} \sin(n\omega t) \right) \left( \sum_{h=1,3,5; h \neq n}^{\infty} \frac{20 \sqrt{2}}{h} \sin(h\omega t - 30^\circ) \right) \]
2. Computation of $P$, $P_1$, $P_H$

\[ P = \frac{1}{T} \int_0^T p(t)dt \]
\[ = 20 + \sum_{n=1,3,5}^{\infty} \frac{4600}{n^3} \cos(30^\circ n) \]
\[ = 20 + 4600 \cos 30^\circ + \sum_{n=3,5,7\ldots}^{\infty} \frac{4600}{n^3} \cos(30^\circ n) \]
\[ = 20 + 3983.71 + (-43.4841) \]
\[ = P_{dc} + P_1 + P_H \]

Thus,
Active power contributed by dc components of voltage and current, $P_{dc} = 20$ W.

Active power contributed by fundamental frequency components of voltage and current, $P_1 = 3983.71$ W.

Active power contributed by harmonic frequency components of voltage and current, $P_H = -43.4841$ W.

3. Computation of $Q$, $Q_1$, $Q_H$

\[ Q = \sum_{n=1,3,5}^{\infty} \frac{4600}{n^3} \sin(30^\circ n) \]
\[ = 4600 \sin 30^\circ + \sum_{n=3,5,7\ldots}^{\infty} \frac{4600}{n^3} \sin(30^\circ n) \]
\[ = 2300 + 175.7548 \text{ VArs} \]
\[ = Q_1 + Q_H \]

The above implies that, $Q_1 = 4600 \text{ VArs}$ and $Q_H = \sum_{n=3,5,7\ldots}^{\infty} \frac{4600}{n^3} \sin(30^\circ n) = 175.7548 \text{ VArs}$.

4. Computation of Apparent Powers and Distortion Powers
The apparent power $S$ is expressed as following.

\[
V_{\text{rms}} = \sqrt{V_{dc}^2 + V_1^2 + V_3^2 + V_5^2 + V_7^2 + V_9^2 + \ldots} \\
= \sqrt{10^2 + 230^2 + (230/3)^2 + (230/5)^2 + (230/7)^2 + (230/9)^2 + \ldots} \\
= 231.87 V \quad \text{(up to } n = 9) \\
I_{\text{rms}} = \sqrt{I_{dc}^2 + I_1^2 + I_3^2 + I_5^2 + I_7^2 + I_9^2 + \ldots} \\
= \sqrt{2^2 + 20^2 + (20/3)^2 + (20/5)^2 + (20/7)^2 + (20/9)^2 + \ldots} \\
= 21.85 A \quad \text{(up to } n = 9) 
\]

The apparent power, $S = V_{\text{rms}} I_{\text{rms}} = 231.87 \times 21.85 = 5066.36$ VA.  
Fundamental apparent power, $S_1 = V_1 \times I_1 = 4600$ VA.  
Apparent power contributed by harmonics $S_H = V_H \times I_H$

\[
V_H = \sqrt{V_3^2 + V_5^2 + V_7^2 + V_9^2 + \ldots} \\
= \sqrt{(230/3)^2 + (230/5)^2 + (230/7)^2 + (230/9)^2 + \ldots} \\
= 27.7 V \quad \text{(up to } n = 9) \\
I_H = \sqrt{I_3^2 + I_5^2 + I_7^2 + I_9^2 + \ldots} \\
= \sqrt{(20/3)^2 + (20/5)^2 + (20/7)^2 + (20/9)^2 + \ldots} \\
= 8.57 A \quad \text{(up to } n = 9). 
\]

Therefore the harmonic apparent power, $S_H = V_H \times I_H = 237.5$ VA.

Non active power, $N = \sqrt{S^2 - P^2} = \sqrt{5067^2 - 3960.2^2} = 3160.8$ VAr \quad (up to \quad n=9) 
Distortion Power $D = \sqrt{S^2 - P^2 - Q^2} = \sqrt{5067^2 - 3960.2^2 - 2475.77^2} = 1965.163$ VAr \quad (up to \quad n=9).

**Displacement power factor** \((\cos \phi_1)\)

\[
\cos \phi_1 = \frac{P_1}{S_1} = \frac{3983.7}{(230)(20)} = 0.866 \text{ lagging} 
\]

**Power factor** \((\cos \phi)\)

\[
\cos \phi = \frac{P}{S} = \frac{3960.217}{5067} = 0.781 \text{ lagging} 
\]

The voltage, current, various powers and power factor are plotted in the Fig. 1.8, verifying above values.
Fig. 1.8 Various powers

References


Chapter 2

THREE PHASE CIRCUITS: POWER DEFINITIONS AND VARIOUS COMPONENTS
(Lectures 9-18)

2.1 Three-phase Sinusoidal Balanced System

Usage of three-phase voltage supply is very common for generation, transmission and distribution of bulk electrical power. Almost all industrial loads are supplied by three-phase power supply for its advantages over single phase systems such as cost and efficiency for same amount of power usage. In principle, any number of phases can be used in polyphase electric system, but three-phase system is simpler and giving all advantages of polyphase system is preferred. In previous section, we have seen that instantaneous active power has a constant term \( VI\cos\phi \) as well pulsating term \( VI\cos(2\omega t - \phi) \). The pulsating term does not contribute to any real power and thus increases the VA rating of the system.

In the following section, we shall study the various three-phase circuits such as balanced, unbalanced, balanced and unbalanced harmonics and discuss their properties in details [1]–[5].

2.1.1 Balanced Three-phase Circuits

A balanced three-phase system is shown in Fig. 2.1 below.

Three-phase balanced system is expressed using following voltages and currents.

\[
\begin{align*}
v_a(t) &= \sqrt{2}V \sin(\omega t) \\
v_b(t) &= \sqrt{2}V \sin(\omega t - 120^\circ) \\
v_c(t) &= \sqrt{2}V \sin(\omega t + 120^\circ)
\end{align*}
\] (2.1)
In (2.1) and (2.2) subscripts $a, b$ and $c$ are used to denote three phases which are balanced. Balanced three-phase means that the magnitude ($V$) is same for all three phases and they have a phase shift of $-120^\circ$ and $120^\circ$. The balanced three phase system has certain interesting properties. These will be discussed in the following section.

2.1.2 Three Phase Instantaneous Active Power

Three phase instantaneous active power in three phase system is given by,

$$p_{3\phi}(t) = p(t) = v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t)$$

$$= p_a + p_b + p_c$$  \hspace{1cm} (2.3)

In above equation, $p_a(t), p_b(t)$ and $p_c(t)$ are expressed similar to single phase system done previously. These are given below.

$$p_a(t) = V I \cos \phi \{1 - \cos 2\omega t\} - V I \sin \phi \sin 2\omega t$$

$$p_b(t) = V I \cos \phi \{1 - \cos(2\omega t - 120^\circ)\} - V I \sin \phi \sin 2(\omega t - 120^\circ)$$  \hspace{1cm} (2.4)

$$p_c(t) = V I \cos \phi \{1 - \cos(2\omega t + 120^\circ)\} - V I \sin \phi \sin 2(\omega t + 120^\circ)$$

Adding three phase instantaneous powers given in (2.4), we get the three-phase instantaneous power as below.

$$p(t) = 3 V I \cos \phi - V I \cos \phi \{\cos 2\omega t + \cos(2\omega t - 120^\circ) + \cos(2\omega t + 120^\circ)\}$$

$$- V I \sin \phi \{\sin 2\omega t + \sin(2\omega t - 120^\circ) + \sin(2\omega t + 120^\circ)\}$$  \hspace{1cm} (2.5)

Summation of terms in curly brackets is always equal to zero. Hence,

$$p_{3\phi}(t) = p(t) = 3VI \cos \phi.$$  \hspace{1cm} (2.6)
This is quite interesting result. It indicates for balanced three-phase system, the total instantaenous power is equal to the real power or average active power \(P\), which is constant. This is the reason we use 3-phase system. It does not involve the pulsating or oscillating components of power as in case of single phase systems. Thus it ensures less VA rating for same amount of power transfer.

Here, total three-phase reactive power can be defined as sum of maximum value of \(p_{\text{reactive}}(t)\) terms in (2.4). Thus,

\[
Q = Q_a + Q_b + Q_c = 3VI \sin \phi.
\]  

(2.7)

Is there any attempt to define instantaneous reactive power \(q(t)\) similar to \(p(t)\) such that \(Q\) is average value of that term \(q(t)\)? H. Akagi et al. published paper [6], in which authors defined term instantaneous reactive power. The definition was facilitated through \(\alpha\beta0\) transformation. Briefly it is described in the next subsection.

### 2.1.3 Three Phase Instantaneous Reactive Power

H. Akagi et al. [6] attempted to define instantaneous reactive power \(q(t)\) using \(\alpha\beta0\) transformation. This transformation is described below.

The abc coordinates and their equivalent \(\alpha\beta0\) coordinates are shown in the Fig. 2.2 below.

![Fig. 2.2 A abc to \(\alpha\beta0\) transformation](image)

Resolving \(a, b, c\) quantities along the \(\alpha\beta\) axis we have,

\[
v_\alpha = \sqrt{\frac{2}{3}} (v_a - \frac{v_b}{2} - \frac{v_c}{2})
\]  

(2.8)

\[
v_\beta = \sqrt{\frac{2}{3}} \sqrt{\frac{3}{2}} (v_b - v_c)
\]  

(2.9)

Here, \(\sqrt{\frac{2}{3}}\) is a scaling factor, which ensures power invariant transformation. Along with that, we define zero sequence voltage as,
\[ v_0 = \sqrt{\frac{2}{3}} \sqrt{\frac{1}{2}} (v_a + v_b + v_c) \] (2.10)

Based on Eqns.(4.60)-(2.10) we can write the above equations as follows.

\[
\begin{bmatrix}
  v_0(t) \\
v_\alpha(t) \\
v_\beta(t)
\end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix}
  \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
  \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
  0 & \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}}
\end{bmatrix} \begin{bmatrix}
  v_a(t) \\
v_b(t) \\
v_c(t)
\end{bmatrix}
\] (2.11)

\[
\begin{bmatrix}
  v_0 \\
v_\alpha \\
v_\beta
\end{bmatrix} = [A_{0\alpha\beta}] \begin{bmatrix}
  v_a \\
v_b \\
v_c
\end{bmatrix}
\]

The above is known as Clarke-Concordia transformation. Thus, \(v_a, v_b\) and \(v_c\) can also be expressed in terms of \(v_0, v_\alpha\) and \(v_\beta\) by pre-multiplying (2.11) by matrix \([A_{0\alpha\beta}]^{-1}\), we have

\[
\begin{bmatrix}
  v_a \\
v_b \\
v_c
\end{bmatrix} = [A_{0\alpha\beta}]^{-1} \begin{bmatrix}
  v_0 \\
v_\alpha \\
v_\beta
\end{bmatrix}
\]

It will be interesting to learn that

\[
[A_{0\alpha\beta}]^{-1} = [A_{abc}] = \left(\sqrt{\frac{2}{3}} \begin{bmatrix}
  \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
  \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
  0 & \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}}
\end{bmatrix}\right)^{-1}
\]

\[
[A_{0\alpha\beta}]^{-1} = \left(\sqrt{\frac{2}{3}} \begin{bmatrix}
  \frac{1}{\sqrt{2}} & 1 & 0 \\
  \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \\
  \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix}\right) = [A_{0\alpha\beta}]^T = [A_{abc}]
\] (2.12)

Similarly, we can write down instantaneous symmetrical transformation for currents, which is given below.

\[
\begin{bmatrix}
  i_0 \\
i_\alpha \\
i_\beta
\end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix}
  \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
  \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
  0 & \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}}
\end{bmatrix} \begin{bmatrix}
  i_a \\
i_b \\
i_c
\end{bmatrix}
\] (2.13)

Now based on ‘0αβ’ transformation, the instantaneous active and reactive powers are defined as follows. The three-phase instantaneous power \(p(t)\) is expressed as the dot product of 0αβ components of voltage and currents such as given below.
\[ p(t) = v_\alpha i_\alpha + v_\beta i_\beta + v_0 i_0 \]
\[
= \frac{2}{3} \left[ \left( v_a - \frac{v_b}{2} - \frac{v_c}{2} \right) \left( i_a - \frac{i_b}{2} - \frac{i_c}{2} \right) - \frac{\sqrt{3}}{2} (v_b - v_c) \frac{\sqrt{3}}{2} (i_b - i_c) \right] \\
+ \frac{1}{\sqrt{2}} \left( v_a + v_b + v_c \right) \frac{1}{\sqrt{2}} \left( i_b + i_c \right) \\
= v_\alpha i_\alpha + v_\beta i_\beta + v_0 i_0 
\]

Now what about instantaneous reactive power? Is there any concept defining instantaneous reactive power? In 1983-84, authors H. Akagi have attempted to define instantaneous reactive power using stationary \( \alpha \beta 0 \) frame, as illustrated below. In [6], the instantaneous reactive power \( q(t) \) is defined as the cross product of two mutual perpendicular quantities, such as given below.

\[
q(t) = v_\alpha \times i_\beta + v_\beta \times i_\alpha \\
q(t) = v_\alpha i_\beta - v_\beta i_\alpha \\
= \frac{2}{3} \left[ \left( v_a - \frac{v_b}{2} - \frac{v_c}{2} \right) \frac{\sqrt{3}}{2} (i_b - i_c) - \frac{\sqrt{3}}{2} (v_b - v_c) \left( i_a - \frac{i_b}{2} - \frac{i_c}{2} \right) \right] \\
= \frac{2\sqrt{3}}{3} \left[ \left( -v_b + v_c \right) i_a + \left( v_a - \frac{v_b}{2} - \frac{v_c}{2} + \frac{v_b}{2} - \frac{v_c}{2} \right) i_b + \left( v_b - v_c \right) i_c \right] \\
= -\frac{1}{\sqrt{3}} \left[ \left( v_b - v_c \right) i_a + \left( v_c - v_a \right) i_b + \left( v_a - v_b \right) i_c \right] \\
= -\left[ v_0 i_a + v_\alpha i_b + v_\beta i_c \right] / \sqrt{3} 
\]

This is also equal to the following.

\[
q(t) = \frac{1}{\sqrt{3}} \left[ \left( i_b - i_c \right) v_a + \left( -\frac{i_b}{2} + \frac{i_c}{2} - i_a + \frac{i_c}{2} + \frac{i_b}{2} \right) v_b + \left( -\frac{i_b}{2} - \frac{i_c}{2} + i_a - \frac{i_b}{2} - \frac{i_c}{2} \right) v_c \right] \\
= \frac{1}{\sqrt{3}} \left[ \left( i_b - i_c \right) v_a + \left( i_c - i_a \right) v_b + \left( i_a - i_b \right) v_c \right] 
\]

### 2.1.4 Power Invariance in abc and \( \alpha \beta 0 \) Coordinates

As a check for power invariance, we shall compute the energy content of voltage signals in two transformations. The energy associated with the \( abc0 \) system is given by \( (v_a^2 + v_b^2 + v_c^2) \) and the energy associated with the \( \alpha \beta 0 \) components is given by \( (v_0^2 + v_\alpha^2 + v_\beta^2) \). The two energies must be equal to ensure power invariance in two transformations. It is proved below. Using, (2.11) and
squares of the respective components, we have the following.

\[ v^2_\alpha = \left[ \frac{\sqrt{2}}{3} \left( v_a - \frac{v_b}{2} - \frac{v_c}{2} \right) \right]^2 \]

\[ v^2_\alpha = \frac{2}{3} \left\{ v^2_a + \frac{v^2_b}{4} + \frac{v^2_c}{4} - \frac{2v_a v_b}{2} + \frac{2v_b v_c}{4} - \frac{2v_a v_c}{2} \right\} \]

\[ = \frac{2}{3} v^2_a + \frac{v^2_b}{6} + \frac{v^2_c}{6} - \frac{2v_a v_b}{3} + \frac{v_b v_c}{3} - \frac{2v_a v_c}{3} \]  \hspace{1cm} (2.17)

Similary we can find out square of \( v_\beta \) term as given below.

\[ v^2_\beta = \left[ \frac{\sqrt{3}}{2} \sqrt{\frac{2}{3}} (v_b - v_c) \right]^2 \]

\[ = \frac{1}{2} (v^2_b + v^2_c - 2v_b v_c) \]

\[ = \frac{v^2_b}{2} + \frac{v^2_c}{2} - v_b v_c \]  \hspace{1cm} (2.18)

Adding (2.17) and (2.18), we find that,

\[ v^2_\alpha + v^2_\beta = \frac{2}{3} \left( v^2_a + v^2_b + v^2_c - v_c v_b - v_b v_c - v_c v_a \right) \]

\[ = \left( v^2_a + v^2_b + v^2_c \right) - \left( \frac{v^2_a}{3} + \frac{v^2_b}{3} + \frac{v^2_c}{3} + \frac{2v_a v_b}{3} + \frac{2v_b v_c}{3} + \frac{2v_a v_c}{3} \right) \]

\[ = \left( v^2_a + v^2_b + v^2_c \right) - \frac{1}{3} (v_a + v_b + v_c)^2 \]

\[ = \left( v^2_a + v^2_b + v^2_c \right) - \left\{ \frac{1}{\sqrt{3}} (v_a + v_b + v_c) \right\}^2 \]  \hspace{1cm} (2.19)

Since \( v_0 = \frac{1}{\sqrt{3}} (v_a + v_b + v_c) \), the above equation, (2.19) can be written as,

\[ v^2_\alpha + v^2_\beta + v^2_0 = v^2_a + v^2_b + v^2_c. \]  \hspace{1cm} (2.20)

From the above it is implies that the energy associated with the two systems remain same instant to instant basis. In general the instantaneous power \( p(t) \) remain same in both transformations. This is proved below.
Using (2.14), following can be written.

\[
p(t) = v_a i_a + v_\beta i_\beta + v_\omega i_\omega
\]

\[
p(t) = \begin{bmatrix} v_0 & v_\alpha & v_\beta \end{bmatrix}^T \begin{bmatrix} i_0 \\ i_\alpha \\ i_\beta \end{bmatrix}
\]

\[
= \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix}^T \begin{bmatrix} v_a & v_b & v_c \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}
\]

\[
= \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix}^T \begin{bmatrix} v_a & v_b & v_c \end{bmatrix}^{-1} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}
\]

\[
= \begin{bmatrix} v_a & v_b & v_c \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}
\]

\[
= v_a i_a + v_b i_b + v_c i_c
\]

(2.21)

From (2.12), \([A_{abc}]\) and its inverse are as following.

\[
[A_{abc}] = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}
\]

and

\[
[A_{abc}]^{-1} = [A_{abc}]^T = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}
\]

(2.22)

2.2 Instantaneous Active and Reactive Powers for Three-phase Circuits

In the previous section instantaneous active and reactive powers were defined using \(\alpha\beta0\) transformation. In this section we shall study these powers for various three-phase circuits such as three-phase balanced, three-phase unbalanced, balanced three-phase with harmonics and unbalanced three-phase with harmonics. Each case will be considered and analyzed.
2.2.1 Three-Phase Balance System

For three-phase balanced system, three-phase voltages have been expressed by equation (2.1). For these phase voltages, the line to line voltages are given as below.

\[ v_{ab} = \sqrt{3} \sqrt{2} v \sin (\omega t + 30^\circ) \]
\[ v_{bc} = \sqrt{3} \sqrt{2} v \sin (\omega t - 90^\circ) \]
\[ v_{ca} = \sqrt{3} \sqrt{2} v \sin (\omega t + 150^\circ) \] (2.23)

The above relationship between phase and line to line voltages is also illustrated in Fig. 2.3. For the above three-phase system, the instantaneous power \( p(t) \) can be expressed using (2.21) and it is equal to,

\[ p(t) = v_a i_a + v_b i_b + v_c i_c \]
\[ = v_a i_a + v_b i_b + v_0 i_0 \]
\[ = 3 V I \cos \phi \] (2.24)

The instantaneous reactive power \( q(t) \) is as following.

\[
q(t) = [\sqrt{3} \sqrt{2} V \sin (\omega t - 90^\circ) \sqrt{2} I \sin (\omega t - \phi) \\
+ \sqrt{3} \sqrt{2} V \sin (\omega t + 150^\circ) \sqrt{2} I \sin (\omega t - 120^\circ - \phi) \\
+ \sqrt{3} \sqrt{2} V \sin (\omega t + 30^\circ) \sqrt{2} I \sin (\omega t + 120^\circ - \phi)]/\sqrt{3} \\
= -\sqrt{3} V I [\cos (90^\circ - \phi) - \cos (2\omega t - 90^\circ - \phi) \\
+ \cos (90^\circ - \phi) - \cos (2\omega t - 30^\circ - \phi) \\
+ \cos (90^\circ - \phi) - \cos (2\omega t + 150^\circ - \phi)]/\sqrt{3} \\
= -\sqrt{3} V I [3 \sin \phi - \cos (2\omega t - \phi + 30^\circ) - \cos (2\omega t - \phi + 30^\circ + 120^\circ) \\
- \cos (2\omega t - \phi + 30^\circ - 120^\circ)]/\sqrt{3} \\
= -VI [3 \sin \phi - 0] \\
q(t) = -3VI \sin \phi \] (2.25)
The above value of instantaneous reactive power is same as defined by Budeanu’s [1] and is given in equation (2.7). Thus, instantaneous reactive power given in (2.15) matches with the conventional definition of reactive power defined in (2.7). However the time varying part of second terms of each phase in (2.4) has no relevance with the definition given in (2.15).

Another interpretation of line to line voltages in (2.15) is that the voltages $v_{ab}$, $v_{bc}$ and $v_{ca}$ have $90^\circ$ phase shift with respect to voltages $v_c$, $v_a$ and $v_b$ respectively. These are expressed as below.

$$v_{ab} = \sqrt{3}v_c - 90^\circ$$
$$v_{bc} = \sqrt{3}v_a - 90^\circ$$
$$v_{ca} = \sqrt{3}v_b - 90^\circ$$  \hspace{1cm} (2.26)

In above equation, $v_c - 90^\circ$ implies that $v_c$ lags $v_c$ by $90^\circ$. Analyzing each term in (2.15) contributes to,

$$v_{bc}i_a = \sqrt{3}v_a - 90^\circ \cdot i_a$$
$$= \sqrt{3} \sqrt{2} V \sin (\omega t - 90^\circ) \cdot \sqrt{2} I \sin (\omega t - \phi)$$
$$= \sqrt{3} V I \left[ \cos (90^\circ - \phi) - \cos (2\omega t - 90^\circ - \phi) \right]$$
$$= \sqrt{3} V I \left[ \sin \phi - \cos \left\{ 90^\circ + (2\omega t - \phi) \right\} \right]$$
$$= \sqrt{3} V I \left[ \sin \phi + \sin (2\omega t - \phi) \right]$$
$$\sqrt{3} = V I \left[ \sin \phi + \cos \phi \sin 2\omega t \sin \phi \right]$$

Similarly,

$$v_{ca}i_b = \sqrt{3} = V I \left[ \sin \phi \left( 1 - \cos \left( \frac{2\omega t - 2\pi}{3} \right) \right) \right]$$
$$+ V I \cos \phi \sin 2 \left( \frac{\omega t - 2\pi}{3} \right)$$

$$v_{ab}i_c = \sqrt{3} = V I \left[ \sin \phi \left( 1 - \cos \left( \frac{2\omega t + 2\pi}{3} \right) \right) \right]$$
$$+ V I \cos \phi \sin 2 \left( \frac{\omega t + 2\pi}{3} \right)$$  \hspace{1cm} (2.27)

Thus, we see that the role of the coefficients of $\sin \phi$ and $\cos \phi$ have reversed. Now if we take average value of (2.27), it is not equal to zero but $VI \sin \phi$ in each phase. Thus three-phase reactive power will be $3VI \sin \phi$. The maximum value of second term in (2.27) represents active average power i.e., $VI \cos \phi$. However, this is not normally convention about the notation of the powers. But, important contribution of this definition is that average reactive power could be defined as the average value of terms in (2.27).

### 2.2.2 Three-Phase Unbalance System

Three-phase unbalance system is not uncommon in power system. Three-phase unbalance may result from single-phasing, faults, different loads in three phases. To study three-phase system
with fundamental unbalance, the voltages and currents are expressed as following.

\[ v_a = \sqrt{2} V_a \sin (\omega t - \phi_{va}) \]
\[ v_b = \sqrt{2} V_b \sin (\omega t - 120^\circ - \phi_{vb}) \]
\[ v_c = \sqrt{2} V_c \sin (\omega t + 120^\circ - \phi_{vc}) \]  

and,

\[ i_a = \sqrt{2} I_a \sin (\omega t - \phi_{ia}) \]
\[ i_b = \sqrt{2} I_b \sin (\omega t - 120^\circ - \phi_{ib}) \]
\[ i_c = \sqrt{2} I_c \sin (\omega t + 120^\circ - \phi_{ic}) \]  

For the above system, the three-phase instantaneous power is given by,

\[ p_{3\phi}(t) = p(t) = v_a i_a + v_b i_b + v_c i_c \]
\[ = \sqrt{2} V_a \sin (\omega t - \phi_{va}) \sin (\omega t - \phi_{ia}) \]
\[ + \sqrt{2} V_b \sin (\omega t - 120^\circ - \phi_{vb}) \sqrt{2} I_b \sin (\omega t - 120^\circ - \phi_{ib}) \]
\[ + \sqrt{2} V_c \sin (\omega t + 120^\circ - \phi_{vc}) \sqrt{2} I_c \sin (\omega t + 120^\circ - \phi_{ic}) \]  

Simplifying above expression we get,

\[ p_{3\phi}(t) = V_a I_a \cos \phi_a \{1 - \cos (2\omega t - 2\phi_{va})\} \]
\[ - V_a I_a \sin \phi_a \sin (2\omega t - 2\phi_{va}) \]
\[ + V_b I_b \cos \phi_b \{1 - \cos \{2(\omega t - 120^\circ) - 2\phi_{vb}\}\} \]
\[ - V_b I_b \sin \phi_b \sin \{2(\omega t - 120^\circ) - 2\phi_{vb}\} \]
\[ + V_c I_c \cos \phi_c \{1 - \cos \{2(\omega t + 120^\circ) - 2\phi_{vc}\}\} \]
\[ - V_c I_c \sin \phi_c \sin \{2(\omega t + 120^\circ) - 2\phi_{vc}\} \]  

where \( \phi_a = (\phi_{ia} - \phi_{va}) \)

Therefore,

\[ p_{3\phi}(t) = p_{a,\text{active}} + p_{b,\text{active}} + p_{c,\text{active}} + p_{a,\text{reactive}} + p_{b,\text{reactive}} + p_{c,\text{reactive}} \]
\[ = \bar{p}_a + \bar{p}_b + \bar{p}_c + \bar{p}_a + \bar{p}_b + \bar{p}_c \]  

where,

\[ \bar{p}_a = P_a = V_a I_a \cos \phi_a \]
\[ \bar{p}_b = P_b = V_b I_b \cos \phi_b \]
\[ \bar{p}_c = P_c = V_c I_c \cos \phi_c \]  

and

\[ \tilde{p}_a = -V_a I_a \cos (2\omega t - \phi_a - 2\phi_{va}) \]
\[ \tilde{p}_b = -V_b I_b \cos (2\omega t - 240^\circ - \phi_b - 2\phi_{vb}) \]
\[ \tilde{p}_c = -V_c I_c \cos (2\omega t + 240 - \phi_c - 2\phi_{vc}) \]  

36
Also it is noted that,
\[ p_a + p_b + p_c = v_a i_a + v_b i_b + v_c i_c = P \] (2.35)

and,
\[ \tilde{p}_a + \tilde{p}_b + \tilde{p}_c = -V_a I_a \cos(2\omega t - \phi_{va} - \phi_{ib}) \\
- V_b I_b \cos \{2(\omega t - 120) - \phi_{vb} - \phi_{ib}\} \\
- V_c I_c \cos \{2(\omega t + 120) - \phi_{vc} - \phi_{ic}\} \neq 0 \]

This implies that, we no longer get advantage of getting constant power, \(3V I \cos \phi\) from interaction of three-phase voltages and currents. Now, let us analyze three phase instantaneous reactive power \(q(t)\) as per definition given in (2.15).

\[
q(t) = -\frac{1}{\sqrt{3}}(v_b - v_c)i_a + (v_c - v_a)i_b + (v_a - v_b)i_c \\
= -\frac{2}{\sqrt{3}} \left[ \left\{ V_b \sin(\omega t - 120^\circ - \phi_{vb}) - V_c \sin(\omega t + 120^\circ - \phi_{vc}) \right\} I_a \sin(\omega t - \phi_{ia}) \\
+ \left\{ V_c \sin(\omega t - 120^\circ - \phi_{vc}) - V_a \sin(\omega t - \phi_{va}) \right\} \sqrt{2} I_b \sin(\omega t - 120^\circ - \phi_{ib}) \\
+ \left\{ V_a \sin(\omega t - 120^\circ - \phi_{va}) - V_c \sin(\omega t + 120^\circ - \phi_{vc}) \right\} \sqrt{2} I_c \sin(\omega t + 120^\circ - \phi_{ic}) \right] \] (2.36)

From the above,
\[
\sqrt{3}q(t) = -\left[ V_b I_a \left\{ \cos(\phi_{ia} - 120^\circ - \phi_{vb}) - \cos(2\omega t - 120^\circ - \phi_{ia} - \phi_{vb}) \right\} \\
V_c I_a \left\{ \cos(\phi_{ia} + 120^\circ - \phi_{vc}) - \cos(2\omega t + 120^\circ - \phi_{ia} - \phi_{vc}) \right\} \\
+ V_c I_b \left\{ \cos(\phi_{ib} + 240^\circ - \phi_{vc}) - \cos(2\omega t + \phi_{ib} - \phi_{vc}) \right\} \\
- V_a I_b \left\{ \cos(\phi_{ib} - 120^\circ - \phi_{va} - \phi_{ib}) \right\} \\
+ V_a I_c \left\{ \cos(\phi_{ic} - 120^\circ - \phi_{va} - \phi_{ic}) \right\} \\
- V_b I_c \left\{ \cos(\phi_{ic} - 240^\circ - \phi_{vb} - \phi_{ic}) \right\} \right] \] (2.37)

Now looking this expression, we can say that
\[
\frac{1}{T} \int_0^T q(t) dt = -\frac{1}{\sqrt{3}} \left[ V_b I_a \cos(\phi_{ia} - \phi_{vb} - 120^\circ) \\
- V_c I_a \cos(\phi_{ia} - \phi_{vc} + 120^\circ) \\
+ V_c I_b \cos(\phi_{ib} + 240^\circ - \phi_{vc}) \\
- V_a I_b \cos(\phi_{ib} - 120^\circ - \phi_{va}) \\
+ V_a I_c \cos(\phi_{ic} - 120^\circ - \phi_{va}) \\
- V_b I_c \cos(\phi_{ic} - 240^\circ - \phi_{vb}) \right] \\
= \overline{q}_a(t) + \overline{q}_b(t) + \overline{q}_c(t) \neq V_a I_a \cos \phi_a + V_b I_b \cos \phi_b + V_c I_c \cos \phi_c \] (2.38)
Hence the definition of instantaneous reactive power does not match to that defined by Budeanue’s reactive power [1] for three-phase unbalanced circuit. If only voltages or currents are distorted, the above holds true as given below. Let us consider that only currents are unbalanced, then

\[ v_a(t) = \sqrt{2}V \sin(\omega t) \]
\[ v_b(t) = \sqrt{2}V \sin(\omega t - 120^\circ) \] (2.39)
\[ v_c(t) = \sqrt{2}V \sin(\omega t + 120^\circ) \]

and

\[ i_a(t) = \sqrt{2}I_a \sin(\omega t - \phi_a) \]
\[ i_b(t) = \sqrt{2}I_b \sin(\omega t - 120^\circ - \phi_b) \] (2.40)
\[ i_c(t) = \sqrt{2}I_c \sin(\omega t + 120^\circ - \phi_c) \]

And the instantaneous reactive power is given by,

\[ q(t) = \frac{1}{\sqrt{3}}[v_{bc}i_a + v_{ca}i_b + v_{ab}i_c] \]
\[ = \frac{1}{\sqrt{3}}[\sqrt{3}v_a\angle - \pi/2 \cdot i_a + \sqrt{3}v_b\angle - \pi/2 \cdot i_b + \sqrt{3}v_c\angle - \pi/2 \cdot i_c] \]
\[ = -[\sqrt{2}V \sin(\omega t - \pi/2)\sqrt{2}I_a \sin(\omega t - \phi_{ia}) + \sqrt{2}V \sin(\omega t - 120^\circ - \pi/2)\sqrt{2}I_b \sin(\omega t - 120^\circ - \phi_{ib}) + \sqrt{2}V \sin(\omega t + 120^\circ + \pi/2)\sqrt{2}I_c \sin(\omega t + 120^\circ - \phi_{ic})] \]
\[ = -(VI_a \cos(\pi/2 - \phi_{ia}) - \cos\{\pi/2 - (2\omega t - \phi_{ia})\} + VI_b \cos(\pi/2 - \phi_{ib}) - \cos(2\omega t - 240^\circ - \pi/2 - \phi_{ib}) + VI_c \cos(\pi/2 - \phi_{ic}) - \cos(2\omega t + 240^\circ - \pi/2 - \phi_{ic})] \]
\[ = -(V I_a \sin\phi_{ia} + V I_b \sin\phi_{ib} + V I_c \sin\phi_{ic}) + V I_a \sin(2\omega t - \phi_{ia}) + V I_b \sin(2\omega t - 240^\circ - \phi_{ib}) + V I_c \sin(2\omega t + 240^\circ - \phi_{ic})] \]

Thus,

\[ Q = \frac{1}{T} \int_0^T q(t) dt = -(V I_a \sin\phi_{ia} + V I_b \sin\phi_{ib} + V I_c \sin\phi_{ic}) \] (2.41)

Which is similar to Budeanue’s reactive power.

The oscillating term of \( q(t) \) which is equal to \( \tilde{q}(t) \) is given below.

\[ \tilde{q}(t) = V I_a \sin(2\omega t - \phi_{ia}) + V I_b \sin(2\omega t - 240^\circ - \phi_{ib}) + V I_c \sin(2\omega t + 240^\circ - \phi_{ic}) \] (2.42)

which is not similar to what is being defined as reactive component of power in (2.4).

### 2.3 Symmetrical components

In the previous section, the fundamental unbalance in three phase voltage and currents have been considered. Ideal power systems are not designed for unbalance quantities as it makes power system components over rated and inefficient. Thus, to understand unbalance three-phase systems,
a concept of symmetrical components introduced by C. L. Fortescue, will be discussed. In 1918, C. L. Fortescue, wrote a paper [7] presenting that an unbalanced system of \( n \)-related phasors can be resolved into \( n \) system of balanced phasors, called the symmetrical components of the original phasors. The \( n \) phasors of each set of components are equal in length and the angles. Although, the method is applicable to any unbalanced polyphase system, we shall discuss about three phase systems.

For the discussion of symmetrical components, a complex operator denoted as \( a \) is defined as,

\[
a = 1 \angle 120^\circ = e^{\frac{2\pi}{3}} = \cos \frac{2\pi}{3} + j \sin \frac{2\pi}{3}
\]

\[
= -\frac{1}{2} + j\frac{\sqrt{3}}{2}
\]

\[
a^2 = 1 \angle 240^\circ = 1 \angle -120^\circ = e^{\frac{4\pi}{3}} = e^{-\frac{2\pi}{3}} = \cos \frac{4\pi}{3} + j \sin \frac{4\pi}{3}
\]

\[
= -\frac{1}{2} - j\frac{\sqrt{3}}{2}
\]

\[
a^3 = 1 \angle 360^\circ = e^{2\pi} = 1
\]

Also note an interesting property relating \( a \), \( a^2 \) and \( a^3 \),

\[
a + a^2 + a^3 = 0.
\] (2.43)

These quantities i.e., \( a \), \( a^2 \) and \( a^3 = 1 \) also represent three phasors which are shifted by \( 120^\circ \) from each other. This is shown in Fig. 2.4.

Knowing the above and using Fortescue theorem, three unbalanced phasor of a three phase unbalanced system can be resolved into three balanced system phasors.

1. Positive sequence components are the composed of three phasors equal in magnitude and displaced from each other by 120 degrees in phase and having a phase sequence of original phasors.
2. Negative sequence components consist of three phasors equal in magnitude, phase shift of 
$-120^\circ$ and $120^\circ$ between phases and with phase sequence opposite to that of the original phasors.

3. Zero sequence components consist of three phasors equal in magnitude with zero phase shift
from each other.

Positive sequence components: $\overline{V}_{a1}, \overline{V}_{b1}, \overline{V}_{c1}$
Negative sequence components: $\overline{V}_{a2}, \overline{V}_{b2}, \overline{V}_{c2}$
Zero sequence components: $\overline{V}_{a0}, \overline{V}_{b0}, \overline{V}_{c0}$

Thus we can write,

$$\begin{align*}
\overline{V}_a &= \overline{V}_{a1} + \overline{V}_{a2} + \overline{V}_{a0} \\
\overline{V}_b &= \overline{V}_{b1} + \overline{V}_{b2} + \overline{V}_{b0} \\
\overline{V}_c &= \overline{V}_{c1} + \overline{V}_{c2} + \overline{V}_{c0}
\end{align*}$$

(2.44)

Graphically, these are represented in Fig. 2.5. Thus if we add the sequence components of each
phase vectorially, we shall get $\overline{V}_a$, $\overline{V}_b$ and $\overline{V}_s$ as per (2.44). This is illustrated in Fig. 2.6.

Now knowing all these preliminaries, we can proceed as following. Let $\overline{V}_{a1}$ be a reference phasor,
therefore $\overline{V}_{b1}$ and $\overline{V}_{c1}$ can be written as,

$$\begin{align*}
\overline{V}_{b1} &= a^2 \overline{V}_{a1} = \overline{V}_{a1} \angle -120^\circ \\
\overline{V}_{c1} &= a \overline{V}_{a1} = \overline{V}_{a1} \angle 120^\circ
\end{align*}$$

(2.45)

Similarly $\overline{V}_{b2}$ and $\overline{V}_{c2}$ can be expressed in terms of $\overline{V}_{a2}$ as following.

$$\begin{align*}
\overline{V}_{c2} &= a^2 \overline{V}_{b2} = \overline{V}_{a2} \angle -120^\circ \\
\overline{V}_{b2} &= a \overline{V}_{a2} = \overline{V}_{a2} \angle 120^\circ
\end{align*}$$

(2.46)
Fig. 2.6 Unbalanced phasors as vector sum of positive, negative and zero sequence phasors

The zero sequence components have same magnitude and phase angle and therefore these are expressed as,

\[
V_{b0} = V_{c0} = V_{a0} \quad (2.47)
\]

Using (2.45), (2.46) and (2.47) we have,

\[
V_a = V_{a0} + V_{a1} + V_{a2} \quad (2.48)
\]

\[
V_b = V_{b0} + V_{b1} + V_{b2} \\
= V_{a0} + a^2 V_{a1} + a V_{a2} \quad (2.49)
\]

\[
V_c = V_{c0} + V_{c1} + V_{c2} \\
= V_{a0} + a V_{a1} + a^2 V_{a2} \quad (2.50)
\]

Equations (2.48)-(2.50) can be written in matrix form as given below.

\[
\begin{bmatrix}
V_a \\
V_b \\
V_c
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 \\
1 & a^2 & a \\
1 & a & a^2
\end{bmatrix}
\begin{bmatrix}
V_{a0} \\
V_{a1} \\
V_{a2}
\end{bmatrix} \quad (2.51)
\]

Premultiplying by inverse of matrix \([A_{abc}]\) which is equal to

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & a^2 & a \\
1 & a & a^2
\end{bmatrix}
\]

the symmetrical com-
ponents are expressed as given below.

\[
\begin{bmatrix}
V_{a0} \\
V_{a1} \\
V_{a2}
\end{bmatrix}
= \frac{1}{3}
\begin{bmatrix}
1 & 1 & 1 \\
1 & a & a^2 \\
1 & a^2 & a
\end{bmatrix}
\begin{bmatrix}
V_a \\
V_b \\
V_c
\end{bmatrix}
\]
(2.52)

\[
= [A_{012}]
\begin{bmatrix}
V_a \\
V_b \\
V_c
\end{bmatrix}
\]

The symmetrical transformation matrices \(A_{abc}\) and \(A_{012}\) are related as following.

\[
[A_{012}] = [A_{abc}]^{-1} = 3 [A_{abc}]^*
\]
(2.53)

From (2.52), the symmetrical components can therefore be expressed as phase voltages as following.

\[
V_{a0} = \frac{1}{3}(V_a + V_b + V_c)
\]
\[
V_{a1} = \frac{1}{3}(V_a + aV_b + a^2V_c)
\]
\[
V_{a2} = \frac{1}{3}(V_a + a^2V_b + aV_c)
\]
(2.54)

The other component i.e., \(V_{b0}, V_{c0}, V_{b1}, V_{c1}, V_{b2}, V_{c2}\) can be found from \(V_{ao}, V_{a1}, V_{a2}\). It should be noted that quantity \(V_{ao}\) does not exist if sum of unbalanced phasors is zero. Since sum of line to line voltage phasors i.e., \(V_{ab} + V_{bc} + V_{ca} = (V_a - V_b) + (V_b - V_c) + (V_c - V_a)\) is always zero, hence zero sequence voltage components are never present in the line voltage, regardless of amount of unbalance. The sum of the three phase voltages, i.e., \(V_a + V_b + V_c\) is not necessarily zero and hence zero sequence voltage exists.

Similarly sequence components can be written for currents. Denoting three phase currents by \(I_a, I_b, \) and \(I_c\) respectively, the sequence components in matrix form are given below.

\[
\begin{bmatrix}
I_{a0} \\
I_{a1} \\
I_{a2}
\end{bmatrix}
= \frac{1}{3}
\begin{bmatrix}
1 & 1 & 1 \\
1 & a & a^2 \\
1 & a^2 & a
\end{bmatrix}
\begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix}
\]
(2.55)

Thus,

\[
I_{a0} = \frac{1}{3}(I_a + I_b + I_c)
\]
\[
I_{a1} = \frac{1}{3}(I_a + aI_b + a^2I_c)
\]
\[
I_{a2} = \frac{1}{3}(I_a + a^2I_b + aI_c)
\]
In three-phase, 4-wire system, the sum of line currents is equal to the neutral current ($I_n$). thus,

$$I_n = \frac{1}{3}(I_a + I_b + I_c)$$

$$= 3I_{a0} \tag{2.56}$$

This current flows in the fourth wire called neutral wire. Again if neutral wire is absent, then zero sequence current is always equal to zero irrespective of unbalance in phase currents. This is illustrated below.

![Diagram](image)

Fig. 2.7 Various three phase systems (a) Three-phase three-wire system (b) Three-phase four-wire system

In 2.7(b), $i_n$ may or may not be zero. However neutral voltage ($V_{Nn}$) between the system and load neutral is always equal to zero. In 2.7(a), there is no neutral current due to the absence of the neutral wire. But in this configuration the neutral voltage, $V_{Nn}$, may or may not be equal to zero depending upon the unbalance in the system.

**Example 2.1** Consider a balanced $3 \phi$ system with following phase voltages.

$$V_a = 100\angle 0^\circ$$  
$$V_b = 100\angle -120^\circ$$  
$$V_c = 100\angle 120^\circ$$

Using (2.54), it can be easily seen that the zero and negative sequence components are equal to zero, indicating that there is no unbalance in voltages. However the converse may not apply. Now consider the following phase voltages. Compute the sequence components and show that the energy associated with the voltage components in both system remain constant.

$$V_a = 100\angle 0^\circ$$  
$$V_b = 150\angle -100^\circ$$  
$$V_c = 75\angle 100^\circ$$
Solution Using (2.54), sequence components are computed. These are:

\[ V_{a0} = \frac{1}{3}(V_a + V_b + V_c) \]
\[ = 31.91 \angle -50.48^\circ V \]
\[ V_{a1} = \frac{1}{3}(V_a + aV_b + a^2V_c) \]
\[ = 104.16 \angle 4.7^\circ V \]
\[ V_{a2} = \frac{1}{3}(V_a + a^2V_b + aV_c) \]
\[ = 28.96 \angle 146.33^\circ V \]

If you find energy content of two frames that is abc and 012 system, it is found to be constant.

\[ E_{abc} = k [V_a^2 + V_b^2 + V_c^2] = k.(381.25) \]
\[ E_{012} = k [3V_{a0}^2 + V_{a1}^2 + V_{a2}^2] = k.(381.25) \]

Thus, \[ E_{abc} = E_{012} \] with \( k \) some constant of proportionality.

The invariance of power can be further shown by following proof.

\[ \overline{S}_v = P + jQ = [ V_a \quad V_b \quad V_c ] [ I_a \quad I_b \quad I_c]^* \]
\[ = \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}^T \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix}^* \]
\[ = \begin{bmatrix} [A_{abc}] \begin{bmatrix} V_{a0} \\ V_{a1} \\ V_{a2} \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} [A_{abc}] \begin{bmatrix} I_{a0} \\ I_{a1} \\ I_{a2} \end{bmatrix} \end{bmatrix}^* \]
\[ = \begin{bmatrix} V_{a0} \\ V_{a1} \\ V_{a2} \end{bmatrix}^T [A_{abc}]^T [A_{abc}]^* \begin{bmatrix} I_{a0} \\ I_{a1} \\ I_{a2} \end{bmatrix}^* \] \hspace{1cm} (2.57)

The term \( \overline{S}_v \) is referred as vector or geometric apparent power. The difference between will be given in the following. The transformation matrix \( [A_{abc}] \) has following properties.

\[ [A_{abc}]^{-1} = \frac{1}{3} [A_{abc}]^T \] and \hspace{1cm} (2.58)
\[ [A_{abc}]^* = [A_{abc}] \]
Therefore using (2.58), (2.57) can be written as the following.

\[
{\mathbf{S}} = P + jQ = \begin{bmatrix} V_a^0 \\ V_a^1 \\ V_a^2 \end{bmatrix}^T \begin{bmatrix} I_{a0} \\ I_{a1} \\ I_{a2} \end{bmatrix}^* \\
= 3 \begin{bmatrix} V_a^0 \\ V_a^1 \\ V_a^2 \end{bmatrix}^T \begin{bmatrix} I_{a0} \\ I_{a1} \\ I_{a2} \end{bmatrix}^*
\]

Equation (2.59) indicates that power invariance holds true in both \(abc\) and \(012\) components. But, this is true on phasor basis. Would it be true on the time basis? In this context, concept of instantaneous symmetrical components will be discussed in the latter section. The equation (2.59) further implies that,

\[
{\mathbf{S}} = P + jQ = V_aI_a^* + V_bI_b^* + V_cI_c^* \\
= 3 \left[ V_aI_a^* + V_bI_b^* + V_cI_c^* \right]
\]

(2.60)

The power terms in (2.60) accordingly form positive sequence, negative sequence and zero sequence powers denoted as following. The positive sequence power is given as,

\[
P^+ = V_aI_a^1 \cos \phi_a^1 + V_bI_b^1 \cos \phi_b^1 + V_cI_c^1 \cos \phi_c^1 \\
= 3V_aI_a^1 \cos \phi_a^1.
\]

(2.61)

Negative sequence power is expressed as,

\[
P^- = 3V_aI_a^2 \cos \phi_a^2.
\]

(2.62)

The zero sequence power is

\[
P^0 = 3V_aI_a^0 \cos \phi_a^0.
\]

(2.63)

Similarly, sequence reactive power are denoted by the following expressions.

\[
Q^+ = 3V_aI_a^1 \sin \phi_a^1 \\
Q^- = 3V_aI_a^2 \sin \phi_a^2 \\
Q^0 = 3V_aI_a^0 \sin \phi_a^0
\]

(2.64)

Thus, following holds true for active and reactive powers.

\[
P = P_a + P_b + P_c = P_0 + P_1 + P_2 \\
Q = Q_a + Q_b + Q_c = Q_0 + Q_1 + Q_2
\]

(2.65)
Here, positive sequence, negative sequence and zero sequence apparent powers are denoted as the following.

\[ S^+ = |S_+| = \sqrt{P^2 + Q^2} = 3V_{a1}I_{a1} \]
\[ S^- = |S_-| = \sqrt{P^- + Q^-} = 3V_{a2}I_{a2} \]
\[ S^0 = |S_0| = \sqrt{P^{02} + Q^{02}} = 3V_{a0}I_{a0} \]

(2.66)

The scalar value of vector apparent power \((S_v)\) is given as following.

\[ S_v = |S_a + S_b + S_c| = |S^0 + S^+ + S^-| = |(P_a + P_b + P_c) + j(Q_a + Q_b + Q_c)| \]
\[ = \sqrt{P^2 + Q^2} \]

(2.67)

Similarly, arithematic apparent power \((S_A)\) is defined as the algebraic sum of each phase or sequence apparent power, i.e.,

\[ S_A = |S_a| + |S_b| + |S_c| = |P_a + jQ_a| + |P_b + jQ_b| + |P_c + jQ_c| \]
\[ = \sqrt{P_a^2 + Q_a^2} + \sqrt{P_b^2 + Q_b^2} + \sqrt{P_c^2 + Q_c^2} \]

(2.68)

In terms of sequence components apparent power,

\[ S_A = |S^0| + |S^+| + |S^-| = |P^0 + jQ^0| + |P^+ + jQ^+| + |P^- + jQ^-| \]
\[ = \sqrt{P_0^2 + Q_0^2} + \sqrt{P_+^2 + Q_+^2} + \sqrt{P_-^2 + Q_-^2} \]

(2.69)

Based on these two definitions of the apparent powers, the power factors are defined as the following.

\[ \text{Vector apparent power} = p_{f_v} = \frac{P}{S_v} \]

(2.70)

\[ \text{Arithematic apparent power} = p_{f_A} = \frac{P}{S_A} \]

(2.71)
Example 2.2 Consider a 3-phase 4 wire system supplying resistive load, shown in Fig. 2.8 below. Determine power consumed by the load and feeder losses.

\[
\text{Power dissipated by the load } = \frac{(\sqrt{3}V)^2}{R} = \frac{3V^2}{R}
\]

The current flowing in the line \( = \frac{\sqrt{3}V}{R} = \left| \frac{V_a - V_b}{R} \right| \)

and \( I_b = -I_a \)

Therefore losses in the feeder \( = \left( \frac{\sqrt{3}V}{R} \right)^2 \times r + \left( \frac{\sqrt{3}V}{R} \right)^2 \times r \)

\[= 2 \left( \frac{r}{R} \right) \left( \frac{3V^2}{R} \right) \]

Now, consider another example of a 3 phase system supplying 3-phase load, consisting of three resistors (R) in star as shown in the Fig. 2.9. Let us find out above parameters.

\[
\text{Power supplied to load } = 3 \left( \frac{V}{R} \right)^2 \times R = \frac{3V^2}{R}
\]

\[
\text{Losses in the feeder } = 3 \left( \frac{V}{R} \right)^2 \times r = \left( \frac{r}{R} \right) \left( \frac{3V^2}{R} \right)
\]

Thus, it is interesting to see that power dissipated in the unbalanced system is twice the power loss in balanced circuit. This leads to conclusion that power factor in phases would become less than unity, while for balanced circuit, the power factor is unity. Power analysis of unbalanced circuit shown in Fig. 2.8 is given below.
Fig. 2.9 A three-phase balanced load

The current in phase-\(a\), \(I_a = \frac{V_a - V_b}{R} = \frac{\sqrt{3} V_a}{R} \angle 30^\circ\)

The current in phase-\(b\), \(I_b = -I_a = \frac{\sqrt{3} V}{R} \angle (30 - 180)^\circ = \frac{\sqrt{3} V}{R} \angle -150^\circ\)

The current in phase-\(c\) and neutral are zero, \(I_c = I_n = 0\)

The phase voltages are: \(V_a = V \angle 0^\circ\), \(V_b = V \angle -120^\circ\), \(V_c = V \angle 120^\circ\).

The phase active and reactive and apparent powers are as following.

\[
\begin{align*}
P_a &= V_a I_a \cos \phi_a = V I \cos 30^\circ = \frac{\sqrt{3}}{2} V I \\
Q_a &= V_a I_a \sin \phi_a = V I \sin 30^\circ = \frac{1}{2} V I \\
S_a &= V_a I_a = V I \\
P_b &= V_b I_b \cos \phi_b = V I \cos(-30)^\circ = \frac{\sqrt{3}}{2} V I \\
Q_b &= V_b I_b \sin \phi_b = V I \sin(-30)^\circ = \frac{-1}{2} V I \\
S_b &= V_b I_b = V I \\
P_c &= Q_c = S_c = 0
\end{align*}
\]

Thus total active power \(P = P_a + P_b + P_c = 2 \times \frac{\sqrt{3}}{2} V I = \sqrt{3} V I\)

\[
\begin{align*}
P &= \frac{3 V^2}{R} \\
\text{Total reactive power } Q &= Q_a + Q_b + Q_c = 0
\end{align*}
\]
The vector apparent power, $S_v = \sqrt{P^2 + Q^2} = 3V^2/R = P$

The arithmetic apparent power, $S_A = S_a + S_b + S_c = 2VI = (2/\sqrt{3})P$

From the values of $S_v$ and $S_A$, it implies that,

$$pf_v = \frac{P}{S_v} = \frac{P}{P} = 1$$

$$pf_A = \frac{P}{S_A} = \frac{P}{(2/\sqrt{3})P} = \frac{\sqrt{3}}{2} = 0.866$$

This difference between the arithmetic and vector power factors will be more due to the unbalances in the load.

For balance load $S_A = S_V$, therefore, $pf_A = pf_V = 1.0$. Thus for three-phase electrical circuits, the following holds true.

$$pf_A \leq pf_V \quad (2.72)$$

### 2.3.1 Effective Apparent Power

For unbalanced three-phase circuits, there is one more definition of apparent power, which is known as effective apparent power. The concept assumes that a virtual balanced circuit that has the same power output and losses as the actual unbalanced circuit. This equivalence leads to the definition of effective line current $I_e$ and effective line to neutral voltage $V_e$.

The effective rms current in each phase is given as following.

$$I_e = \sqrt{\frac{(I_a^2 + I_b^2 + I_c^2 + I_n^2)}{3}} \quad (2.73)$$
For the original circuit shown in Fig. 2.8, the effective current $I_e$ is computed using above equation and is given below.

$$I_e = \sqrt{\frac{(I_a^2 + I_b^2)}{3}}$$

since, $I_c = 0$ and $I_n = 0$

$$= \sqrt{\frac{2 I_a^2}{3}} = \sqrt{\frac{2 (\sqrt{3} V/R)^2}{3}}$$

$$= \frac{\sqrt{2} V}{R}$$

To account same power output in circuits shown above, the following identity is used with $R_e = R$ in Fig. 2.10.

$$\frac{V_a^2}{R} + \frac{V_b^2}{R} + \frac{V_c^2}{R} + \frac{V_a^2 + V_b^2 + V_c^2}{3R} = \frac{3V_e^2}{R} + \frac{9V_e^2}{3R}$$

(2.74)

From (2.74), the effective rms value of voltage is expressed as,

$$V_e = \sqrt{\frac{1}{18} \left[ 3(V_a^2 + V_b^2 + V_c^2) + V_{ab}^2 + V_{bc}^2 + V_{ca}^2 \right]}$$

(2.75)

Assuming, $3(V_a^2 + V_b^2 + V_c^2) \approx V_{ab}^2 + V_{bc}^2 + V_{ca}^2$, equation (2.75) can be written as,

$$V_e = \sqrt{\frac{V_a^2 + V_b^2 + V_c^2}{3}} = V$$

(2.76)

Therefore, the effective apparent power ($S_e$), using the values of $V_e$ and $I_e$, is given by,

$$S_e = 3V_e I_e = \frac{3 \sqrt{2} V^2}{R}$$

Thus the effective power factor based on the definition of effective apparent power ($S_e$), for the circuit shown in Fig. 2.8 is given by,

$$pf_e = \frac{P}{S_e} = \frac{3 V^2/R}{3\sqrt{2} V^2/R} = \frac{1}{\sqrt{2}} = 0.707$$

Thus, we observe that,

$$S_V \leq S_A \leq S_e,$$

$$pf_e (0.707) \leq pf_A (0.866) \leq pf_V (1.0).$$

When the system is balanced,

$$V_a = V_b = V_c = V_{en} = V_e,$$

$$I_a = I_b = I_c = I_e,$$

$$I_n = 0,$$

and $S_V = S_A = S_e$. 

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2.3.2 Positive Sequence Powers and Unbalance Power

The unbalance power $S_u$ can be expressed in terms of fundamental positive sequence powers $P^+$, $Q^+$ and $S^+$ as given below.

$$ S_u = \sqrt{S_e^2 - S^2} \quad (2.77) $$

where $S^+ = 3V^+I^+$ and $S^{+2} = P^{+2} + Q^{+2}$.

2.4 Three-phase Non-sinusoidal Balanced System

A three-phase nonsinusoidal system is represented by following set of equations.

$$ v_a(t) = \sqrt{2}V_1 \sin(wt - \alpha_1) + \sqrt{2} \sum_{n=2}^{\infty} V_n \sin(nwt - \alpha_n) $$

$$ v_b(t) = \sqrt{2}V_1 \sin(wt - 120^\circ - \alpha_1) + \sqrt{2} \sum_{n=2}^{\infty} V_n \sin(n(wt - 120^\circ) - \alpha_n) \quad (2.78) $$

$$ v_c(t) = \sqrt{2}V_1 \sin(wt + 120^\circ - \alpha_1) + \sqrt{2} \sum_{n=2}^{\infty} V_n \sin(n(wt + 120^\circ) - \alpha_n) $$

Similarly, the line currents can be expressed as,

$$ i_a(t) = \sqrt{2}I_1 \sin(wt - \beta_1) + \sqrt{2} \sum_{n=2}^{\infty} I_n \sin(nwt - \beta_n) $$

$$ i_b(t) = \sqrt{2}I_1 \sin(wt - 120^\circ - \beta_1) + \sqrt{2} \sum_{n=2}^{\infty} I_n \sin(n(wt - 120^\circ) - \beta_n) \quad (2.79) $$

$$ i_c(t) = \sqrt{2}I_1 \sin(wt + 120^\circ - \beta_1) + \sqrt{2} \sum_{n=2}^{\infty} I_n \sin(n(wt + 120^\circ) - \beta_n) $$

In this case,

$$ S_a = S_b = S_c, $$

$$ P_a = P_b = P_c, $$

$$ Q_a = Q_b = Q_c, $$

$$ D_a = D_b = D_c. \quad (2.80) $$

The above equation suggests that such a system has potential to produce significant additional power loss in neutral wire and ground path.
2.4.1 Neutral Current

The neutral current for three-phase balanced system with harmonics can be given by the following equation.

\[ i_n = i_a + i_b + i_c = \sqrt{2} \left[ I_{a1} \sin (wt - \beta_1) + I_{a2} \sin (2wt - \beta_2) + I_{a3} \sin (3wt - \beta_3) + I_{a1} \sin (wt - 120^\circ - \beta_1) + I_{a2} \sin (2wt - 240^\circ - \beta_2) + I_{a3} \sin (3wt - 360^\circ - \beta_3) + I_{a1} \sin (wt + 120^\circ - \beta_1) + I_{a2} \sin (2wt + 240^\circ - \beta_2) + I_{a3} \sin (3wt + 360^\circ - \beta_3) + I_{a4} \sin (4wt - \beta_4) + I_{a5} \sin (5wt - \beta_5) + I_{a6} \sin (6wt - \beta_6) + I_{a4} \sin (wt + 4 \times 120^\circ - \beta_4) + I_{a5} \sin (5wt - 5 \times 120^\circ - \beta_5) + I_{a6} \sin (6wt - 6 \times 120^\circ - \beta_6) + I_{a4} \sin (wt + 4 \times 120^\circ - \beta_4) + I_{a5} \sin (5wt + 5 \times 120^\circ - \beta_5) + I_{a6} \sin (6wt + 6 \times 120^\circ - \beta_6) + I_{a4} \sin (7wt - \beta_4) + I_{a5} \sin (8wt - \beta_5) + I_{a6} \sin (9wt - \beta_6) + I_{a4} \sin (7wt - 7 \times 120^\circ - \beta_4) + I_{a5} \sin (8wt - 8 \times 120^\circ - \beta_5) + I_{a6} \sin (9wt - 9 \times 120^\circ - \beta_6) + I_{a4} \sin (7wt + 7 \times 120^\circ - \beta_4) + I_{a5} \sin (8wt + 8 \times 120^\circ - \beta_5) + I_{a6} \sin (9wt + 9 \times 120^\circ - \beta_6) \right] \]

\[(2.81)\]

From the above equation, we observe that, the triplen harmonics are added up in the neutral current. All other harmonics except triplen harmonics do not contribute to the neutral current, due to their balanced nature. Therefore the neutral current is given by,

\[ i_n = i_a + i_b + i_c = \sum_{n=3,6,\ldots}^{\infty} 3\sqrt{2}I_n \sin(nwt - \beta_n). \]

\[(2.82)\]

The RMS value of the current in neutral wire is therefore given by,

\[ I_n = 3 \left[ \sum_{n=3,6,\ldots}^{\infty} I_n^2 \right]^{1/2}. \]

\[(2.83)\]

Due to dominant triplen harmonics in electrical loads such as UPS, rectifiers and other power electronic based loads, the current rating of the neutral wire may be comparable to the phase wires.

It is worth to mention here that all harmonics in three-phase balanced systems can be categorized in three groups i.e., \((3n + 1)\), \((3n + 2)\) and \(3n\) (for \(n = 1, 2, 3, \ldots\)) called positive, negative and zero sequence harmonics respectively. This means that balanced fundamental, 4th, 7th 10th,... form positive sequence only. Balanced 2nd, 5th, 8th, 11th,... form negative sequence only and the balanced triplen harmonics i.e. 3rd, 6th, 9th,... form zero sequence only. But in case of unbalanced three-phase systems with harmonics, \((3n + 1)\) harmonics may start forming negative and zero sequence components. Similarly, \((3n + 2)\) may start forming positive and zero sequence components and \(3n\) may start forming positive and negative sequence components.

2.4.2 Line to Line Voltage

For the three-phase balanced system with harmonics, the line-to-line voltages are denoted as \(v_{ab}\), \(v_{bc}\) and \(v_{ca}\). Let us consider, line-to-line voltage between phases \(a\) and \(b\). It is given as following.
\[ v_{ab}(t) = v_a(t) - v_b(t) \]
\[ = \sum_{n=1}^{\infty} \sqrt{2} V_n \sin(n \omega t - \alpha_n) - \sum_{n=1}^{\infty} \sqrt{2} V_n \sin((n \omega t - \alpha_n) - n \times 120^\circ) \]
\[ = \sum_{n=1}^{\infty} \sqrt{2} V_n [\sin(n \omega t - \alpha_n) - \sin(n \omega t - \alpha_n) \cos(n \times 120^\circ) \cos(n \omega t - \alpha_n)] \]
\[ = \sum_{n=1}^{\infty} \sqrt{2} V_n [\sin(n \omega t - \alpha_n) - \sin(n \omega t - \alpha_n) (-1/2) \cos(n \omega t - \alpha_n)] \]
\[ = \sqrt{3} \sum_{n=1}^{\infty} V_n [(\sqrt{3}/2) \sin(n \omega t - \alpha_n) + (\pm \sqrt{3}/2) \cos(n \omega t - \alpha_n)] \]
\[ = \sqrt{3} \sqrt{2} \sum_{n=1}^{\infty} V_n [(\sqrt{3}/2) \sin(n \omega t - \alpha_n) + (\pm 1/2) \cos(n \omega t - \alpha_n)] \tag{2.84} \]

Let \( \sqrt{3}/2 = r_n \cos \phi_n \) and \( \pm 1/2 = r_n \sin \phi_n \). This implies \( r_n = 1 \) and \( \phi_n = \pm 30^\circ \). Using this, equation (2.84) can be written as follows.

\[ v_{ab}(t) = \sqrt{3} \sqrt{2} \sum_{n=1}^{\infty} V_n [(\sqrt{3}/2) \sin(n \omega t - \alpha_n) + (\pm 1/2) \cos(n \omega t - \alpha_n)] \tag{2.85} \]

In equations (2.84) and (2.85), \( v_{ab} = 0 \) for \( n = 3, 6, 9, \ldots \) and for \( n = 1, 2, 4, 5, 7, \ldots \), the ± sign of 1/2 or sign of 30\(^\circ\) changes alternatively. Thus it is observed that triplen harmonics are missing in the line to line voltages, in spite of their presence in phase voltages for balanced three-phase system with harmonics. Thus the following identity hold true for this system,

\[ V_{LL} \leq \sqrt{3} V_{Ln} \tag{2.86} \]

Above equation further implies that,

\[ \sqrt{3} V_{LL} I \leq 3 V_{Ln} I. \tag{2.87} \]

In above equation, \( I \) refers the rms value of the phase current. For above case, \( I_a = I_b = I_c = I \) and \( I_n = 3 \sum_{n=3,6,9,\ldots} V_n \). Therefore, effective rms current, \( I_e \) is given by the following.
\[ I_e = \sqrt{\frac{3 I^2 + 3 \sum_{n=3,6,9\ldots}^\infty I_n^2}{3}} \]
\[ = \sqrt{I^2 + \sum_{n=3,6,9\ldots}^\infty I_n^2} \]  
\[ \geq I \]  

(2.88)

2.4.3 Apparent Power with Budeanu Resolution: Balanced Distortion Case

The apparent power is given as,

\[ S = 3V_{ln}I = \sqrt{P^2 + Q_B^2 + D_B^2} \]
\[ = \sqrt{P^2 + Q^2 + D^2} \]  
\[ \text{(2.89)} \]

where,

\[ P = P_1 + P_H = P_1 + P_2 + P_3 + \ldots \]
\[ = 3V_1I_1 \cos \phi_1 + 3 \sum_{n=1}^\infty V_nI_n \cos \phi_n \]  
\[ \text{(2.90)} \]

\[ Q = Q_B = Q_{B1} + Q_{BH} \]
\[ = Q_1 + Q_H \]  
\[ \text{(2.91)} \]

Where \( Q \) in (2.89) is called as Budeanu’s reactive power (VAr) or simply reactive power which is detailed below.

\[ Q = Q_1 + Q_H = Q_1 + Q_2 + Q_3 + \ldots \]
\[ = 3V_1I_1 \sin \phi_1 + 3 \sum_{n=1}^\infty V_nI_n \sin \phi_n \]  
\[ \text{(2.92)} \]

2.4.4 Effective Apparent Power for Balanced Non-sinusoidal System

The effective apparent power \( S_e \) for the above system is given by,

\[ S_e = 3V_eI_e \]  
\[ \text{(2.93)} \]

For a three-phase, three-wire balanced system, the effective apparent power is found after calculating effective voltage and current as given below.

\[ V_e = \sqrt{(V_{ab}^2 + V_{bc}^2 + V_{ca}^2)/9} \]
\[ = V_{ll}/\sqrt{3} \]  
\[ \text{(2.94)} \]
\[ I_e = \sqrt{\frac{I_a^2 + I_b^2 + I_c^2}{3}} = I \]  

(2.95)

Therefore

\[ S_e = S = \sqrt{3}V_\text{ll}I \]  

(2.96)

**For a four-wire system**, \( V_e \) is same is given (2.94) and \( I_e \) is given by (2.88). Therefore, the effective apparent power is given below.

\[ \sqrt{3}V_\text{ll}I \leq 3V_\text{ln}I_e \]  

(2.97)

The above implies that,

\[ S_e \geq S_A. \]  

(2.98)

Therefore, it can be further concluded that,

\[ pf_e (= P/S_e) \geq pf_A (= P/S_A). \]  

(2.99)

### 2.5 Unbalanced and Non-sinusoidal Three-phase System

In this system, we shall consider most general case i.e., three-phase system with voltage and current quantities which are unbalanced and non-sinusoidal. These voltages and currents are expressed as following.

\[ v_a(t) = \sum_{n=1}^{\infty} \sqrt{2}V_{an} \sin(n \omega t - \alpha_{an}) \]

\[ v_b(t) = \sum_{n=1}^{\infty} \sqrt{2}V_{bn} \sin\{n (\omega t - 120^o) - \alpha_{bn}\} \]  

(2.100)

\[ v_c(t) = \sum_{n=1}^{\infty} \sqrt{2}V_{cn} \sin\{n (\omega t + 120^o) - \alpha_{cn}\} \]

Similarly, currents can be expressed as,

\[ i_a(t) = \sum_{n=1}^{\infty} \sqrt{2}I_{an} \sin(n \omega t - \beta_{an}) \]

\[ i_b(t) = \sum_{n=1}^{\infty} \sqrt{2}I_{bn} \sin\{n (\omega t - 120^o) - \beta_{bn}\} \]  

(2.101)

\[ i_c(t) = \sum_{n=1}^{\infty} \sqrt{2}I_{cn} \sin\{n (\omega t + 120^o) - \beta_{cn}\} \]
For the above voltages and currents in three-phase system, instantaneous power is given as following.

\[ p(t) = v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) \]

\[ = p_a(t) + p_b(t) + p_c(t) \]

\[ = \left( \sum_{n=1}^{\infty} \sqrt{2}V_{an} \sin(n\omega t - \alpha_{an}) \right) \left( \sum_{n=1}^{\infty} \sqrt{2}I_{an} \sin(n\omega t - \beta_{an}) \right) \]

\[ + \left( \sum_{n=1}^{\infty} \sqrt{2}V_{bn} \sin \{n(\omega t - 120^\circ) - \alpha_{bn}\} \right) \left( \sum_{n=1}^{\infty} \sqrt{2}I_{bn} \sin \{n(\omega t - 120^\circ) - \beta_{bn}\} \right) \]

\[ + \left( \sum_{n=1}^{\infty} \sqrt{2}V_{cn} \sin \{n(\omega t + 120^\circ) - \alpha_{cn}\} \right) \left( \sum_{n=1}^{\infty} \sqrt{2}I_{cn} \sin \{n(\omega t + 120^\circ) - \beta_{cn}\} \right) \]

In (2.102), each phase power can be found using expressions derived in Section 1.4 of Unit 1. The direct result is written as following.

\[ p_a(t) = \sum_{n=1}^{\infty} V_{an}I_{an} \cos \phi_{an} \{1 - \cos(2n\omega t - 2\alpha_{an})\} - \sum_{n=1}^{\infty} V_{an}I_{an} \sin \phi_{an} \cos(2n\omega t - 2\alpha_{an}) \]

\[ + \left( \sum_{n=1}^{\infty} \sqrt{2}V_{an} \sin(n\omega t - \alpha_{an}) \right) \left( \sum_{m=1, m \neq n}^{\infty} \sqrt{2}I_{am} \sin(m\omega t - \beta_{am}) \right) \]

\[ = \sum_{n=1}^{\infty} P_{an} \{1 - \cos(2n\omega t - 2\alpha_{an})\} - \sum_{n=1}^{\infty} Q_{an} \cos(2n\omega t - 2\alpha_{an}) \]

\[ + \left( \sum_{n=1}^{\infty} \sqrt{2}V_{an} \sin(n\omega t - \alpha_{an}) \right) \left( \sum_{m=1, m \neq n}^{\infty} \sqrt{2}I_{am} \sin(m\omega t - \beta_{am}) \right) \]

(2.103)

In the above equation, \( \phi_{an} = (\beta_{an} - \alpha_{an}) \). Similarly, for phases b and c, the instantaneous power is expressed as below.

\[ p_b(t) = \sum_{n=1}^{\infty} P_{bn} \{1 - \cos \{2n(\omega t - 120^\circ) - 2\alpha_{bn}\}\} - \sum_{n=1}^{\infty} Q_{bn} \cos \{2n(\omega t - 120^\circ) - 2\alpha_{bn}\} \]

\[ + \left( \sum_{n=1}^{\infty} \sqrt{2}V_{bn} \sin \{n(\omega t - 120^\circ) - \alpha_{bn}\} \right) \left( \sum_{m=1, m \neq n}^{\infty} \sqrt{2}I_{bm} \sin \{m(\omega t - 120^\circ) - \beta_{bm}\} \right) \]

(2.104)
and

\[
p_c(t) = \sum_{n=1}^{\infty} P_{cn} \left[ 1 - \cos \left\{ 2n(\omega t + 120^\circ) - 2\alpha_{cn} \right\} \right] - \sum_{n=1}^{\infty} Q_{cn} \cos \left\{ 2n(\omega t + 120^\circ) - 2\alpha_{cn} \right\} \\
+ \left( \sum_{n=1}^{\infty} \sqrt{2} V_{cn} \sin \left\{ n(\omega t + 120^\circ) - \alpha_{cn} \right\} \right) \left( \sum_{m=1, m \neq n}^{\infty} \sqrt{2} I_{cm} \sin \left\{ m(\omega t + 120^\circ) - \beta_{cm} \right\} \right)
\]

(2.105)

From equations (2.103), (2.104) and (2.105), the real powers in three phases are given as follows.

\[
P_a = \sum_{n=1}^{\infty} V_{an} I_{an} \cos \phi_{an}
\]

(2.106)

\[
P_b = \sum_{n=1}^{\infty} V_{bn} I_{bn} \cos \phi_{bn}
\]

\[
P_c = \sum_{n=1}^{\infty} V_{cn} I_{cn} \cos \phi_{cn}
\]

Similarly, the reactive powers in three phases are given as following.

\[
Q_a = \sum_{n=1}^{\infty} V_{an} I_{an} \sin \phi_{an}
\]

(2.107)

\[
Q_b = \sum_{n=1}^{\infty} V_{bn} I_{bn} \sin \phi_{bn}
\]

\[
Q_c = \sum_{n=1}^{\infty} V_{cn} I_{cn} \sin \phi_{cn}
\]

Therefore, the total active and reactive powers are computed by summing the phase powers using equations (2.106) and (2.107), which are given below.

\[
P = P_a + P_b + P_c = \sum_{n=1}^{\infty} (V_{an} I_{an} \cos \phi_{an} + V_{bn} I_{bn} \cos \phi_{bn} + V_{cn} I_{cn} \cos \phi_{cn})
\]

\[
= V_{a1} I_{a1} \cos \phi_{a1} + V_{b1} I_{b1} \cos \phi_{b1} + V_{c1} I_{c1} \cos \phi_{c1}
\]

\[
+ \sum_{n=2}^{\infty} (V_{an} I_{an} \cos \phi_{an} + V_{bn} I_{bn} \cos \phi_{bn} + V_{cn} I_{cn} \cos \phi_{cn})
\]

\[
= P_{a1} + P_{b1} + P_{c1} + \sum_{n=2}^{\infty} (P_{an} + P_{bn} + P_{cn})
\]

\[
= P_1 + P_H
\]

(2.108)
and,

\[
Q = Q_a + Q_b + Q_c = \sum_{n=1}^{\infty} (V_{an} I_{an} \sin \phi_{an} + V_{bn} I_{bn} \sin \phi_{bn} + V_{cn} I_{cn} \sin \phi_{cn})
\]

\[
= V_{a1} I_{a1} \sin \phi_{a1} + V_{b1} I_{b1} \sin \phi_{b1} + V_{c1} I_{c1} \sin \phi_{c1}
\]

\[
+ \sum_{n=2}^{\infty} (V_{an} I_{an} \sin \phi_{an} + V_{bn} I_{bn} \sin \phi_{bn} + V_{cn} I_{cn} \sin \phi_{cn})
\]

\[
= Q_{a1} + Q_{b1} + Q_{c1} + \sum_{n=2}^{\infty} (Q_{an} + Q_{bn} + Q_{cn})
\]

\[
= Q_1 + Q_H
\]  

(2.109)

### 2.5.1 Arithmetic and Vector Apparent Power with Budeanu’s Resolution

Using Budeanu’s resolution, the arithmetic apparent power for phase-\(a\), \(b\) and \(c\) are expressed as following.

\[
S_a = \sqrt{P_a^2 + Q_a^2 + D_a^2}
\]

\[
S_b = \sqrt{P_b^2 + Q_b^2 + D_b^2}
\]

\[
S_c = \sqrt{P_c^2 + Q_c^2 + D_c^2}
\]

(2.110)

The three-phase arithmetic apparent power is arithmetic sum of \(S_a\), \(S_b\) and \(S_c\) in the above equation. This is given below.

\[
S_A = S_a + S_b + S_c
\]

(2.111)

The three-phase vector apparent power is given as following.

\[
S_v = \sqrt{P^2 + Q^2 + D^2}
\]

(2.112)

Where \(P\) and \(Q\) are given in (2.108) and (2.109) respectively. The total distortion power \(D\) is given as following.

\[
D = D_a + D_b + D_c
\]

(2.113)

Based on above definitions of the apparent powers, the arithmetic and vector power factors are given below.

\[
pf_A = \frac{P}{S_A}
\]

\[
pf_v = \frac{P}{S_v}
\]

(2.114)

From equations (2.111), (2.112) and (2.114), it can be inferred that

\[
S_A \geq S_v
\]

\[
pf_A \leq pf_v
\]

(2.115)
2.5.2 Effective Apparent Power

Effective apparent power \( S_e = 3V_e I_e \) for the three-phase unbalanced systems with harmonics can be found by computing \( V_e \) and \( I_e \) as following. The effective rms current \( (I_e) \) can be resolved into two parts i.e., effective fundamental and effective harmonic components as given below.

\[
I_e = \sqrt{I_{e1}^2 + I_{eH}^2} \tag{2.116}
\]

Similarly,

\[
V_e = \sqrt{V_{e1}^2 + V_{eH}^2} \tag{2.117}
\]

For three-phase four-wire system,

\[
I_e = \sqrt{\frac{I_a^2 + I_b^2 + I_c^2 + I_n^2}{3}} = \sqrt{\frac{I_{a1}^2 + I_{b1}^2 + I_{c1}^2 + \ldots + I_{a2}^2 + I_{b2}^2 + I_{c2}^2 + \ldots + I_{n1}^2 + I_{n2}^2 + \ldots}{3}} \tag{2.118}
\]

\[
I_e = \sqrt{\frac{I_{e1}^2 + I_{eH}^2}{3}} \tag{2.119}
\]

In the above equation,

\[
I_{e1} = \sqrt{\frac{I_{a1}^2 + I_{b1}^2 + I_{c1}^2}{3}} \tag{2.120}
\]

Similarly, the effective rms voltage \( V_e \) is given as following.

\[
V_e = \sqrt{\frac{1}{18}[3(V_a^2 + V_b^2 + V_c^2) + (V_{ab}^2 + V_{bc}^2 + V_{ca}^2)]} = \sqrt{\frac{V_{e1}^2 + V_{eH}^2}{3}} \tag{2.121}
\]

Where

\[
V_{e1} = \sqrt{\frac{1}{18}[3(V_{a1}^2 + V_{b1}^2 + V_{c1}^2) + (V_{ab1}^2 + V_{bc1}^2 + V_{ca1}^2)]} \tag{2.122}
\]

\[
V_{eH} = \sqrt{\frac{1}{18}[3(V_{aH}^2 + V_{bH}^2 + V_{cH}^2) + (V_{abH}^2 + V_{bcH}^2 + V_{caH}^2)]} \tag{2.123}
\]

For three-phase three-wire system, \( I_n = 0 = I_{n1} = I_{nH} \).
\[ I_{e1} = \sqrt{\frac{I_{a1}^2 + I_{b1}^2 + I_{c1}^2}{3}} \]
\[ I_{eH} = \sqrt{\frac{I_{aH}^2 + I_{bH}^2 + I_{cH}^2}{3}} \]  
(2.122)
Similarly
\[ V_{e1} = \sqrt{\frac{V_{ab1}^2 + V_{bc1}^2 + V_{ca1}^2}{9}} \]
\[ V_{eH} = \sqrt{\frac{V_{abH}^2 + V_{bcH}^2 + V_{caH}^2}{9}} \]  
(2.123)
The expression for effective apparent power \( S_e \) is given as following.
\[
S_e = 3V_e I_e \\
= 3\sqrt{V_{e1}^2 + V_{eH}^2} \sqrt{I_{e1}^2 + I_{eH}^2} \\
= \sqrt{9V_{e1}^2 I_{e1}^2 + (9V_{e1}^2 I_{eH}^2 + 9V_{eH}^2 I_{e1}^2 + 9V_{eH}^2 I_{eH}^2)} \\
= \sqrt{S_{e1}^2 + S_{eN}^2} \]  
(2.124)
In the above equation,
\[
S_{e1} = 3V_{e1} I_{e1} \quad (2.125) \\
S_{eN} = \sqrt{S_e^2 - S_{e1}^2} \\
= \sqrt{D_{eV}^2 + D_{eI}^2 + S_{eH}^2} \\
= \sqrt{3(I_{e1} V_{eH}^2) + 3(V_{e1} I_{eH}^2) + 3(V_{eH} I_{eH}^2)} \]  
(2.126)
In equation (2.126), distortion powers \( D_{eI} \), \( D_{eV} \) and harmonic apparent power \( S_{eH} \) are given as following.
\[
D_{eI} = 3V_{eH} I_{eH} \\
D_{eV} = 3V_{eH} I_{e1} \quad (2.127) \\
S_{eH} = 3V_{eH} I_{eH} \]
By defining above effective voltage and current quantities, the effective total harmonic distortion \( (THD_e) \) are expressed below.
\[
THD_{eV} = \frac{V_{eH}}{V_{e1}} \\
THD_{eI} = \frac{I_{eH}}{I_{e1}} \]  
(2.128)
Substituting \( V_{eH} \) and \( I_{eH} \) in (2.126),
\[
S_{eN} = S_{e1} \sqrt{THD_{e1}^2 + THD_{eV}^2 + THD_{eI}^2 THD_{eV}^2}. \]  
(2.129)
In above equation,
\[
\begin{align*}
D_{el} &= S_e_1 THD_I \\
D_{eV} &= S_e_1 THD_V \\
S_{eH} &= S_e_1 (THD_I)(THD_V).
\end{align*}
\] (2.130)

Using (2.124) and (2.129), the effective apparent power is given as below.
\[
S_e = \sqrt{S_{e1}^2 + S_{eN}^2} = S_{e1} \sqrt{1 + THD_{eV}^2 + THD_{eI}^2 + THD_{eV}^2 THD_{eI}^2}
\] (2.131)

Based on above equation, the effective power factor is therefore given as,
\[
pfe = \frac{P}{S_e} = \frac{P_1 + P_H}{S_{e1} \sqrt{1 + THD_{eV}^2 + THD_{eI}^2 + THD_{eV}^2 THD_{eI}^2}}
\] (2.132)

Practically, the THDs in voltage are far less than those of currents THDs, therefore $THD_{eV} << THD_{eI}$. Using this practical constraint and assuming $P_H << P_1$, the above equation can be simplified to,
\[
pfe \approx \frac{pf_{e1}}{\sqrt{1 + THD_{eI}^2}}
\] (2.133)

In the above context, there is another useful term to denote unbalance of the system. This is defined as fundamental unbalanced power and is given below.
\[
S_{U1} = \sqrt{S_{e1}^2 - (S_{1}^+)^2}
\] (2.134)

Where, $S_{1}^+$ is fundamental positive sequence apparent power, which is given below.
\[
S_{1}^+ = \sqrt{(P_{1}^+)^2 + (Q_{1}^+)^2}
\] (2.135)

In above, $P_{1}^+ = 3V_{1}^+ I_{1}^+ \cos \phi_{1}^+$ and $Q_{1}^+ = 3V_{1}^+ I_{1}^+ \sin \phi_{1}^+$. Fundamental positive sequence power factor can thus be expressed as a ratio of $P_{1}^+$ and $S_{1}^+$ as given below.
\[
P_{f1}^+ = \frac{P_{1}^+}{S_{1}^+}
\] (2.136)

**Example 2.3** Consider the following three-phase system. It is given that voltages $V_a, V_b$ and $V_c$ are balanced sinusoids with rms value of 220 V. The feeder impedance is $r_f + jx_f = 0.02 + j0.1 \Omega$. The unbalanced load parameters are: $R_L = 12 \Omega$ and $X_L = 13 \Omega$. Compute the following.

a. The currents in each phase, i.e., $I_a$, $I_b$ and $I_c$ and neutral current, $I_n$. 

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b. Losses in the system.

c. The active and reactive powers in each phase and total three-phase active and reactive powers.

d. Arithmetic, vector and effective apparent powers and power factors based on them.

Solution:

a. Computation of currents

\[ v_a(t) = 220\sqrt{2}\sin(\omega t) \]
\[ v_b(t) = 220\sqrt{2}\sin(\omega t - 120^\circ) \]
\[ v_c(t) = 220\sqrt{2}\sin(\omega t + 120^\circ) \]
\[ v_{ab}(t) = 220\sqrt{6}\sin(\omega t + 30^\circ) \]

Therefore,

\[ I_a = \frac{220\sqrt{3}\angle 30^\circ}{13\angle 90^\circ} = 29.31\angle -60^\circ \text{ A} \]
\[ I_b = -I_a = -29.31\angle -60^\circ = 29.31\angle 120^\circ \text{ A} \]
\[ I_c = \frac{220\angle 120^\circ}{12} = 18.33\angle 120^\circ \text{ A.} \]

Thus, the instantaneous expressions of phase currents can be given as following.

\[ i_a(t) = 41.45\sin(\omega t - 60^\circ) \]
\[ i_b(t) = -i_a(t) = -41.45\sin(\omega t - 60^\circ) = 41.45\sin(\omega t + 120^\circ) \]
\[ i_c(t) = 25.93\sin(\omega t + 120^\circ) \]

b. Computation of losses
The losses occur due to resistance of the feeder impedance. These are computed as below.

\[
\text{Losses} = r_f (I_a^2 + I_b^2 + I_c^2 + I_n^2)
\]
\[
= 0.02 (29.31^2 + 29.31^2 + 18.33^2 + 18.33^2) = 47.80 \text{ W}
\]

\c. Computation of various powers\n
\text{Phase-}\text{a active and reactive power:}

\[
\overline{S}_a = \overline{V}_a \overline{I}_a^* = 220 \angle 0^\circ \times 29.31 \angle 60^\circ = 3224.21 + j5584.49
\]
implies that, \( P_a = 3224.1 \text{ W}, \ Q_a = 5584.30 \text{ VAr} \)

Similarly,

\[
\overline{S}_b = \overline{V}_b \overline{I}_b^* = 220 \angle -120^\circ \times 29.31 \angle 60^\circ = -3224.21 + j5584.49
\]
implies that, \( P_b = -3224.1 \text{ W}, \ Q_b = 5584.30 \text{ VAr} \)

For phase-\text{c},

\[
\overline{S}_c = \overline{V}_c \overline{I}_c^* = 220 \angle 120^\circ \times 18.33 \angle -120^\circ = 4032.6 + j0
\]
implies that, \( P_c = 4032.6 \text{ W}, \ Q_c = 0 \text{ VAr} \)

Total three-phase active and reactive powers are given by,

\[
P_{3-\text{phase}} = P_a + P_b + P_c = 3224.1 - 3224.1 + 4032.6 = 4032.6 \text{ W}
\]
\[
Q_{3-\text{phase}} = Q_a + Q_b + Q_c = 5584.30 + 5584.30 + 0 = 11168.60 \text{ VAr.}
\]

\d. Various apparent powers and power factors\n
The arithmetic, vector and effective apparent powers are computed as below.

\[
S_A = |\overline{S}_a| + |\overline{S}_b| + |\overline{S}_c|
\]
\[
= 6448.12 + 6448.12 + 4032.6 = 16928.84 \text{ VA}
\]

\[
S_v = |\overline{S}_a + \overline{S}_b + \overline{S}_c|
\]
\[
= |4032.6 + j11168.6| = |11874.32 \angle 70.14| = 11874.32 \text{ VA}
\]

\[
S_e = 3V_e I_e = 3 \times 220 \times \sqrt{\frac{I_a^2 + I_b^2 + I_c^2 + I_n^2}{3}}
\]
\[
= 3 \times 220 \times \sqrt{\frac{29.31^2 + 29.31^2 + 18.33^2 + 18.33^2}{3}} = 3 \times 220 \times 28.22
\]
\[= 18629.19 \text{ VA}\]
Based on the above apparent powers, the arithmetic, vector and effective apparent power factors are computed as below.

\[ pf_A = \frac{P_{3-phase}}{S_A} = \frac{4032.6}{16928.84} = 0.2382 \]
\[ pf_v = \frac{P_{3-phase}}{S_v} = \frac{4032.6}{11874.32} = 0.3396 \]
\[ pf_e = \frac{P_{3-phase}}{S_e} = \frac{4032.6}{18629.19} = 0.2165 \]

In the above computation, the effective voltage and current are found as given in the following.

\[ V_e = \sqrt{\frac{V_a^2 + V_b^2 + V_c^2}{3}} = 220 \text{ V} \]
\[ I_e = \sqrt{\frac{I_a^2 + I_b^2 + I_c^2 + I_n^2}{3}} = 28.226 \text{ A} \]

**Example 2.4** A 3-phase, 3-wire system is shown in Fig. 2.12. The 3-phase voltages are balanced sinusoids with RMS value of 230 V. The 3-phase loads connected in star are given as following.

\[ Z_a = 5 + j12 \Omega, \quad Z_b = 6 + j8 \Omega, \quad \text{and} \quad Z_c = 12 - j5 \Omega. \]

Compute the following.

a. Line currents, i.e., \( I_{la}, I_{lb} \) and \( I_{lc} \) and their instantaneous expressions.

b. Load active and reactive powers and power factor of each phase.

c. Compute various apparent powers and power factors based on them.

![Fig. 2.12 A star connected three-phase unbalanced load](image)

**Solution:**

a. Computation of currents
Given that $Z_a = 5 + j12 \, \Omega$, $Z_b = 6 + j8 \, \Omega$, $Z_c = 12 - j5 \, \Omega$.

$\overline{V}_{sa} = 230 \angle 0^\circ \, \text{V}$

$\overline{V}_{sb} = 230 \angle -120^\circ \, \text{V}$

$\overline{V}_{sc} = 230 \angle 120^\circ \, \text{V}$

$\overline{V}_{nN} = \frac{1}{Z_a + \frac{1}{Z_b} + \frac{1}{Z_c}} \left( \frac{\overline{V}_{sa}}{Z_a} + \frac{\overline{V}_{sb}}{Z_b} + \frac{\overline{V}_{sc}}{Z_c} \right)$

$= \frac{1}{\frac{1}{5+j12} + \frac{1}{6+j8} + \frac{1}{12-j5}} \left( \frac{230 \angle 0^\circ}{5+j12} + \frac{230 \angle -120^\circ}{6+j8} + \frac{230 \angle 120^\circ}{12-j5} \right)$

$= \frac{1}{0.2013 \angle -37.09^\circ} \times 31.23 \angle -164.50^\circ$

$= -94.22 - j123.18 = 155.09 \angle -127.41^\circ \, \text{V}$

Now the line currents are computed as below.

$I_{al} = \frac{\overline{V}_{sa} - \overline{V}_{nN}}{Z_a} = \frac{230 \angle 0^\circ - 155.09 \angle -127.41^\circ}{5+j12} = 26.67 \angle -46.56^\circ \, \text{A}$

$I_{bl} = \frac{\overline{V}_{sb} - \overline{V}_{nN}}{Z_b} = \frac{230 \angle -120^\circ - 155.09 \angle -127.41^\circ}{6+j8} = 7.88 \angle -158.43^\circ \, \text{A}$

$I_{cl} = \frac{\overline{V}_{sc} - \overline{V}_{nN}}{Z_c} = \frac{230 \angle 120^\circ - 155.09 \angle -127.41^\circ}{12-j5} = 24.85 \angle 116.3^\circ \, \text{A}$

Thus, the instantaneous expressions of line currents can be given as following.

$i_{al} (t) = 37.72 \sin (\omega t - 46.56^\circ)$

$i_{bl} (t) = 11.14 \sin (\omega t - 158.43^\circ)$

$i_{cl} (t) = 35.14 \sin (\omega t + 116.3^\circ)$

b. Computation of load active and reactive powers

$S_a = \overline{V}_a \overline{I}_a^* = 230 \angle 0^\circ \times 26.67 \angle 46.56^\circ = 4218.03 + j4456.8$

$S_b = \overline{V}_b \overline{I}_b^* = 230 \angle -120^\circ \times 7.88 \angle 158.43^\circ = 1419.82 + j1126.06$

$S_c = \overline{V}_c \overline{I}_c^* = 230 \angle 120^\circ \times 24.85 \angle -116.3^\circ = 5703.43 + j368.11$

implies that,

$P_a = 4218.03 \, \text{W}, \quad Q_a = 4456.8 \, \text{VAR}$

$P_b = 1419.82 \, \text{W}, \quad Q_b = 1126.06 \, \text{VAR}$

$P_c = 5703.43 \, \text{W}, \quad Q_c = 368.11 \, \text{VAR}$
Total three-phase active and reactive powers are given by,

\[
P_{3\text{-phase}} = P_a + P_b + P_c = 4218.03 + 1419.82 + 5703.43 = 11341.29 \text{ W}
\]

\[
Q_{3\text{-phase}} = Q_a + Q_b + Q_c = 4456.8 + 1126.06 + 368.11 = 5950.99 \text{ VAR}
\]

The power factors for phases \(a\), \(b\) and \(c\) are given as follows.

\[
pf_a = \frac{P_a}{|S_a|} = \frac{4218.03}{\sqrt{4218.03^2 + 4456.82^2}} = \frac{4218.03}{6136.3} = 0.6873 \text{ (lag)}
\]

\[
pf_b = \frac{P_b}{|S_b|} = \frac{1419.82}{1419.82^2 + 1126.06^2} = \frac{1419.82}{1812.16} = 0.7835 \text{ (lag)}
\]

\[
pf_c = \frac{P_c}{|S_c|} = \frac{5703.43}{5703.43^2 + 368.11^2} = \frac{5703.43}{5715.30} = 0.9979 \text{ (lag)}
\]

c. Computation of various apparent powers and power factors

The arithmetic, vector and effective apparent powers are computed as below.

\[
S_A = |S_a| + |S_b| + |S_c|
\]

\[
= 6136.3 + 1812.16 + 5715.30 = 13663.82 \text{ VA}
\]

\[
S_v = |S_a + S_b + S_c|
\]

\[
= |11341.29 + j5909.92| = 12807.78 \text{ VA}
\]

\[
S_e = 3V_eI_e = 3 \times 230 \times \sqrt{\frac{I_{a}^2 + I_{b}^2 + I_{c}^2 + I_{ln}^{2}}{3}}
\]

\[
= 3 \times 220 \times \sqrt{\frac{26.67^2 + 7.88^2 + 24.85^2 + 0^2}{3}} = 3 \times 230 \times 21.53
\]

\[
= 14859.7 \text{ VA}
\]

The arithmetic, vector and effective apparent power factors are computed as below.

\[
pf_A = \frac{P_{3\text{-phase}}}{S_A} = \frac{11341.29}{13663.82} = 0.8300
\]

\[
pf_v = \frac{P_{3\text{-phase}}}{S_v} = \frac{11341.29}{12807.78} = 0.8855
\]

\[
pf_e = \frac{P_{3\text{-phase}}}{S_e} = \frac{11341.29}{14859.7} = 0.7632
\]

References


Chapter 3

FUNDAMENTAL THEORY OF LOAD COMPENSATION
(Lectures 19-24)

3.1 Introduction

In general, the loads which cause fluctuations in the supply voltage due to poor power factor, unbalanced and harmonics, d.c components require compensation. Typical loads requiring compensation are arc furnaces, induction furnaces, arc welders, steel rolling mills, winders, very large motors, which start and stop frequently, high energy physics experiments, which require pulse high power supplies. All these loads can be classified into three basic categories.

1. Unbalanced ac load
2. Unbalanced ac + non linear load
3. Unbalanced ac + nonlinear ac + dc component load.

The dc component is generally caused by the usage of have wave rectifiers. These loads, particularly nonlinear loads generate harmonics as well as fundamental frequency voltage variations. For example arc furnaces generate significant amount of harmonics at the load bus. Other serious loads which degrade power quality are adjustable speed drives which include power electronic circuitry, all power electronics based converters such as thyristor controlled drives, rectifiers, cyclo converters etc.. In general, following aspects are important, while we do provide the load compensation in order to improve the power quality [1].

1. Type of Load (unbalance, harmonics and dc component)
2. Real and Reactive power requirements (maximum, minimum and concurrence of maximum real and reactive power requirements in multiple loads)
3. Rate of change of real and reactive power etc.

In this unit, we however, discuss fundamental load compensation techniques for unbalanced linear loads such as combination of resistance, inductance and capacitance and their combinations. The
3.2 Fundamental Theory of Load Compensation

We shall find some fundamental relationship between supply system the load and the compensator. We shall start with the principle of power factor correction, which in its simplest form, and can be studied without reference to supply system [2]–[5].

The supply system, the load and the compensator can be modeled in various ways. The supply system can be modeled as a Thevenin’s equivalent circuit with an open circuit voltage and a series impedance, (its current or power and reactive power) requirements. The compensator can be modeled as variable impedance or as a variable source (or sink) of reactive current. The choice of model varied according to the requirements. The modeling and analysis done here is on the basis of steady state and phasor quantities are used to note the various parameters in system.

3.2.1 Power Factor and its Correction

Consider a single phase system shown in 3.1(a) shown below. The load admittance is represented

by 

$$Y_l = G_l + jB_l$$

supplied from a load bus at voltage $V = V \angle 0$. The load current is $I_l$ is given as,

$$I_l = \frac{V(G_l + jB_l)}{\bar{V}} = V G_l + jV B_l$$

(3.1)

$$= I_R + j I_X$$

According to the above equation, the load current has a two components, i.e. the resistive or in phase component and reactive component or phase quadrature component and are represented by $I_R$ and $I_X$ respectively. The current, $I_X$ will lag 90° for inductive load and it will lead 90° from the reference voltage. This is shown in 3.1(b). The load apparent power can be expressed in terms
of bus voltage $V$ and load current $I_l$ as given below.

$$
\overline{S_l} = \overline{V (I_l)^*} \\
= V (I_R + j I_X)^* \\
= V (I_R - j I_X) \\
= V (I_l \cos \phi_l - j I_l \sin \phi_l) \\
= V I_l \cos \phi_l - j V I_l \sin \phi_l \\
= S_l \cos \phi_l - j S_l \sin \phi_l \quad (3.2)
$$

From (3.1), $\overline{I_l} = \overline{V (G_l + j B_l)} = V G_l + j V B_l$, equation (3.2) can also be written as following.

$$
\overline{S_l} = \overline{V (I_l)^*} \\
= V (V G_l + j V B_l)^* \\
= V (V G_l - j V B_l) \\
= V^2 G_l - j V^2 B_l \\
= P_l + j Q_l \quad (3.3)
$$

From equation (3.129), load active ($P_l$) and reactive power ($Q_l$) are given as,

$$
P_l = V^2 G_l \\
Q_l = -V^2 B_l \quad (3.4)
$$

Now suppose a compensator is connected across the load such that the compensator current, $I_\gamma$ is equal to $-I_X$, thus,

$$
\overline{I_\gamma} = \overline{V Y_\gamma} = V (G_\gamma + j B_\gamma) = -I_X \\
= -j V B_l \quad (3.5)
$$

The above condition implies that $G_\gamma = 0$ and $B_\gamma = -B_l$. The source current $I_s$, can therefore given by,

$$
\overline{I_s} = \overline{I_l + I_\gamma} = I_R \quad (3.6)
$$

Therefore due to compensator action, the source supplies only in phase component of the load current. The source power factor is unity. This reduces the rating of the power conductor and losses due to the feeder impedance. The rating of the compensator is given by the following expression.

$$
\overline{S_\gamma} = P_\gamma + j Q_\gamma = \overline{V (I_\gamma)^*} \\
= \overline{V (-j V B_l)^*} \\
= j V^2 B_l \quad (3.7)
$$

Using (3.4), the above equation indicates the $P_\gamma = 0$ and $Q_\gamma = -Q_l$. This is an interesting inference that the compensator generates the reactive power which is equal and opposite to the load reactive and it has no effect on active power of the load. This is shown in Fig. 3.2.
Using (3.2) and (3.7), the compensator rating can further be expressed as,

\[ Q_\gamma = -Q_l = -S_l \sin \phi_l = -S_l \sqrt{1 - \cos^2 \phi_l} \text{ VAr} \]  

(3.8)

From (3.8),

\[ |Q_\gamma| = \sqrt{1 - \cos^2 \phi_l} \]  

(3.9)

If \( |Q_\gamma| < |Q_l| \) or \( |B_\gamma| < |B_l| \), then load is partially compensated. The compensator of fixed admittance is incapable of following variations in the reactive power requirement of the load. In practical however a compensator such as a bank of capacitors can be divided into parallel sections, each of switched separately, so that discrete changes in the reactive power compensation can be made according to the load. Some sophisticated compensators can be used to provide smooth and dynamic control of reactive power.

Here voltage of supply is being assumed to be constant. In general if supply voltage varies, the \( Q_\gamma \) will not vary separately with the load and compensator error will be there. In the following discussion, voltage variations are examined and some additional features of the ideal compensator will be studied.

### 3.2.2 Voltage Regulation

Voltage regulation can be defined as the proportional change in voltage magnitude at the load bus due to change in load current (say from no load to full load). The voltage drop is caused due to feeder impedance carrying the load current as illustrated in Fig. 3.3(a). If the supply voltage is represented by Thevenin’s equivalent, then the voltage regulation (VR) is given by,

\[ VR = \frac{|E| - |V|}{|V|} = \frac{|E| - |V|}{|V|} \]  

(3.10)

for \( V \) being a reference phasor.

In absence of compensator, the source and load currents are same and the voltage drop due to the
feeder is given by,

$$\Delta V = E - V = Z_s I_l$$  \hspace{1cm} (3.11)

The feeder impedance, $Z_s = R_s + jX_s$. The relationship between the load powers and its voltage and current is expressed below.

$$S_l = V (I_l)^* = P_l + jQ_l$$  \hspace{1cm} (3.12)

Since $V = V$, the load current is expressed as following.

$$I_l = \frac{P_l - jQ_l}{V}$$  \hspace{1cm} (3.13)

Substituting, $I_l$ from above equation into (3.11), we get

$$\Delta V = E - V = (R_s + jX_s) \left( \frac{P_l - jQ_l}{V} \right)$$

$$= \frac{R_s P_l + X_s Q_l}{V} + j \frac{X_s P_l - R_s Q_l}{V}$$

$$= \Delta V_R + j \Delta V_X$$  \hspace{1cm} (3.14)

Thus, the voltage drop across the feeder has two components, one in phase ($\Delta V_R$) and another is in phase quadrature ($\Delta V_X$). This is illustrated in Fig. 3.3(b).

![Fig. 3.3 (a) Single phase system with feeder impedance (b) Phasor diagram](image)

From the above it is evident that load bus voltage ($\bar{V}$) is dependent on the value of the feeder impedance, magnitude and phase angle of the load current. In other words, voltage change ($\Delta \bar{V}$) depends upon the real and reactive power flow of the load and the value of the feeder impedance.

Now let us add compensator in parallel with the load as shown in Fig. 3.4(a). The question is: whether it is possible to make $|E| = |\bar{V}|$, in order to achieve zero voltage regulation irrespective of change in the load. The answer is yes, if the compensator consisting of purely reactive components, has enough capacity to supply to required amount of the reactive power. This situation is shown using phasor diagram in Fig. 3.4(b).

The net reactive at the load bus is now $Q_s = Q_\gamma + Q_l$. The compensator reactive power ($Q_\gamma$) has to be adjusted in such a way as to rotate the phasor $\Delta \bar{V}$ until $|E| = |\bar{V}|$. 

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From (3.14) and Fig. 3.3(b),

\[ E \angle \delta = \left( V + \frac{R_s P_l + X_s Q_s}{V} \right) + j \left( \frac{X_s P_l - R_s Q_s}{V} \right) \]  

(3.15)

The above equation implies that,

\[ E^2 = \left( V + \frac{R_s P_l + X_s Q_s}{V} \right)^2 + \left( \frac{X_s P_l - R_s Q_s}{V} \right)^2 \]  

(3.16)

The above equation can be simplified to,

\[ E^2 V^2 = (V^2 + R_s P_l)^2 + X_s^2 Q_s^2 + 2(V^2 + R_s P_l) X_s Q_s + X_s^2 P_l^2 + R_s^2 Q_s^2 - 2X_s P_l R_s Q_s \]  

(3.17)

Above equation, rearranged in the powers of \( Q_s \), is written as following.

\[ (R_s^2 + X_s^2) Q_s^2 + 2V^2 X_s Q_s + (V^2 + R_s P_l)^2 + (X_s P_l)^2 - E^2 V^2 = 0 \]  

(3.18)

Thus the above equation is quadratic in \( Q_s \) and can be represented using coefficients of \( Q_s \) as given below.

\[ a Q_s^2 + b Q_s + c = 0 \]  

(3.19)

Where \( a = R_s^2 + X_s^2, b = 2V^2 X_s \) and \( c = (V^2 + R_s P_l)^2 + X_s^2 P_l^2 - E^2 V^2 \).

Thus the solution of above equation is as following.

\[ Q_s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]  

(3.20)

In the actual compensator, this value would be determined automatically by control loop. The equation also indicates that, we can find the value of \( Q_s \) by subjecting a condition such as \( E = V \).
irrespective of the requirement of the load powers ($P_l$, $Q_l$). This leads to the following conclusion that a purely reactive compensator can eliminate supply voltage variation caused by changes in both the real and reactive power of the load, provided that there is sufficient range and rate of $Q_s$ both in lagging and leading pf. This compensator therefore acts as an ideal voltage regulator. It is mentioned here that we are regulating magnitude of voltage and not its phase angle. In fact its phase angle is continuously varying depending upon the load current.

It is instructive to consider this principle from different point of view. We have seen that compensator can be made to supply all load reactive power and it acts as power factor correction device. If the compensator is designed to compensate power factor, then $Q_s = Q_l + Q_\gamma = 0$. This implies that $Q_\gamma = -Q_l$. Substituting $Q_s = 0$ for $Q_l$ in (3.14) to achieve this condition, we get the following.

$$\Delta V = \frac{(R_s + jX_s)}{V} P_l$$ (3.21)

From above equation, it is observed that $\Delta V$ is independent of $Q_l$. Thus we conclude that a purely reactive compensator cannot maintain both constant voltage and unity power factor simultaneously. Of course the exception to this rule is a trivial case when $P_l = 0$.

### 3.2.3 An Approximation Expression for the Voltage Regulation

Consider a supply system with short circuit capacity ($S_{sc}$) at the load bus. This short circuit capacity can be expressed in terms of short circuit active and reactive powers as given below.

$$S_{sc} = P_{sc} + jQ_{sc} = \bar{E}\bar{T}_{sc}^* = \bar{E} \left( \frac{\bar{E}}{Z_{sc}} \right)^* = \frac{E^2}{Z_{sc}^*}$$ (3.22)

Where $Z_{sc} = R_s + jX_s$ and $\bar{T}_{sc}$ is the short circuit current. From the above equation

$$|Z_{sc}| = \frac{E^2}{S_{sc}}$$

Therefore, $R_s = \frac{E^2}{S_{sc}} \cos \phi_{sc}$

$X_s = \frac{E^2}{S_{sc}} \sin \phi_{sc}$

$$\tan \phi_{sc} = \frac{X_s}{R_s}$$ (3.23)

Substituting above values of $R_s$ and $X_s$, (3.14) can be written in the following form.

$$\Delta V = \left( \frac{P_l \cos \phi_{sc} + Q_l \sin \phi_{sc}}{V^2} + j \frac{P_l \sin \phi_{sc} - Q_l \cos \phi_{sc}}{V^2} \right) \frac{E^2}{S_{sc}}$$

$$\Delta V = \frac{\Delta V_R}{V} + j \frac{\Delta V_X}{V}$$ (3.24)
Using an approximation that \( E \approx V \), the above equation reduces to the following.

\[
\frac{\Delta V}{V} = \left( \frac{P_l \cos \phi_{sc} + Q_l \sin \phi_{sc}}{S_{sc}} + j \frac{P_l \sin \phi_{sc} - Q_l \cos \phi_{sc}}{S_{sc}} \right)
\] (3.25)

The above implies that,

\[
\frac{\Delta V_R}{V} \approx \frac{P_l \cos \phi_{sc} + Q_l \sin \phi_{sc}}{S_{sc}}
\]

\[
\frac{\Delta V_X}{V} \approx \frac{P_l \sin \phi_{sc} - Q_l \cos \phi_{sc}}{S_{sc}}
\]

Often \( \frac{\Delta V_X}{V} \) is ignored on the ground that the phase quadrature component contributes negligible to the magnitude of overall phasor. It mainly contributes to the phase angle. Therefore the equation (3.25) is simplified to the following.

\[
\frac{\Delta V}{V} = \frac{\Delta V_R}{V} = \frac{P_l \cos \phi_{sc} + Q_l \sin \phi_{sc}}{S_{sc}}
\] (3.26)

Implying that the major change in voltage regulation occurs due to in phase component, \( \Delta V_R \). Although approximate, the above expression is quite useful in terms of short circuit level \( (S_{sc}) \), \( (X_s/R_s) \), active and reactive power of the load.

On the basis of incremental changes in active and reactive powers of the load, i.e., \( 0 \rightarrow P_l \) and \( 0 \rightarrow Q_l \), the above equation can further be written as,

\[
\frac{\Delta V}{V} = \frac{\Delta V_R}{V} = \frac{\Delta P_l \cos \phi_{sc} + \Delta Q_l \sin \phi_{sc}}{S_{sc}}.
\] (3.27)

Further, feeder reactance \( (X_s) \) is far greater than feeder resistance \( (R_s) \), i.e., \( X_s \gg R_s \). This implies that, \( \phi_{sc} \rightarrow 90^\circ \), \( \sin \phi_{sc} \rightarrow 1 \) and \( \cos \phi_{sc} \rightarrow 0 \). Using this approximation the voltage regulation is given as following.

\[
\frac{\Delta V}{V} \approx \frac{\Delta V_R}{V} \approx \frac{\Delta Q_l}{S_{sc}} \sin \phi_{sc} \approx \frac{\Delta Q_l}{S_{sc}}.
\] (3.28)

That is, per unit voltage change is equal to the ratio of the reactive power swing to the short circuit level of the supply system. Representing \( \Delta V \) approximately by \( E - V \), the equation (3.28) can be written as,

\[
\frac{E - V}{V} \approx \frac{Q_l}{S_{sc}}.
\] (3.29)

The above leads to the following expression,

\[
V \approx \frac{E}{(1 + \frac{\Delta Q_l}{S_{sc}})} \approx E(1 - \frac{Q_l}{S_{sc}})
\] (3.30)

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with the assumption that, $Q_l/S_{sc} \ll 1$. Although above relationship is obtained with approximations, however it is very useful in visualizing the action of compensator on the voltage. The above equation is graphically represented as Fig. 3.5. The nature of voltage variation is drooping with increase in inductive reactive power of the load. This is shown by negative slope $-E/S_{sc}$ as indicated in the figure.

The above characteristics also explain that when load is capacitive, $Q_l$ is negative. This makes $V > E$. This is similar to Ferranti effect due to lightly loaded electric lines.

Example 3.1 Consider a supply at 10 kV line to neutral voltage with short circuit level of 250 MVA and $X_s/R_s$ ratio of 5, supplying a star connected load inductive load whose mean power is 25 MW and whose reactive power varies from 0 to 50 MVAr, all quantities per phase.

(a) Find the load bus voltage ($V_l$) and the voltage drop ($\Delta V$) in the supply feeder. Thus determine load current ($I_l$), power factor and system voltage ($E$).

(b) It is required to maintain the load bus voltage to be same as supply bus voltage i.e. $V=10$ kV. Calculate reactive power supplies by the compensator.

(c) What should be the load bus voltage and compensator current if it is required to maintain the unity power factor at the supply?

Solution: The feeder resistance and reactance are computed as following.

$Z_s = E_s^2/S_{sc} = (10 \text{ kV})^2/250 = 0.4 \Omega$/phase

It is given that, $X_s/R_s = \tan \phi_{sc} = 5$, therefore $\phi_{sc} = \tan^{-1} 5 = 78.69^\circ$. From this,

$R_s = Z_s \cos \phi_{sc} = 0.4 \cos(78.69^\circ) = 0.0784 \Omega$

$X_s = Z_s \sin \phi_{sc} = 0.4 \sin(78.69^\circ) = 0.3922 \Omega$

(a) Without compensation $Q_s = Q_l$, $Q_{\gamma} = 0$

To know $\Delta V$, first the voltage at the load bus has to be computed. This is done by rearranging

Fig. 3.5 Voltage variation with reactive power of the load
(3.18) in powers of voltage $V$. This is given below.

$$
\begin{align*}
(R_s^2 + X_s^2) Q_l^2 + 2 V^2 X_s Q_l + (V^2 + R_s P_l) Q_l + X_s P_l^2 - E^2 V^2 &= 0 \\
(R_s^2 + X_s^2) Q_l^2 + 2 V^2 X_s Q_l + V^4 &+ R_s^2 P_l^2 + 2 V^2 R_s P_l + X_s P_l^2 - E^2 V^2 &= 0
\end{align*}
$$

Combining the I, II and III terms in the above equation, we get the following.

$$
V^4 + \{2(R_s P_l + X_s Q_l) - E^2\} V^2 + (R_s^2 + X_s^2)(Q_l^2 + P_l^2) = 0 \tag{3.31}
$$

Now substituting values of $R_s$, $X_s$, $P_l$, $Q_l$ and $E$ in above equation, we get,

$$
V^4 + \{2 [0.0784 \times 25 + 0.3922 \times 50] - 10^2\} V^2 + (0.0784^2 + 0.3922^2)(25^2 + 50^2) = 0
$$

After simplifying the above, we have the following equation.

$$
V^4 - 56.86 V^2 + 500 = 0
$$

Therefore

$$
V^2 = \frac{56.86 \pm \sqrt{56.86 - 4 \times 500}}{2} = 45.985, 10.875
$$

and $V = \pm 6.78 \text{kV}, \pm 3.297 \text{kV}$

Since rms value cannot be negative and maximum rms value must be a feasible solution, therefore $V = 6.78 \text{kV}$.

Now we can compute $\Delta V$ using (3.14), as it is given below.

$$
\Delta V = \frac{R_s P_l + X_s Q_l}{V} + j \frac{X_s P_l - R_s Q_l}{V} = \frac{0.0784 25 + 0.392 50}{6.78} + j \frac{0.392225 - 0.078450}{6.78} = 3.1814 + j 0.8677 \text{kV} = 3.2976 \angle 15.25^\circ \text{kV}
$$

Now the line current can be found out as following.

$$
I_l = \frac{P_l - Q_l}{V} = \frac{25 - j 50}{6.782} = 3.86 - j 7.3746 \text{kA}
$$

$$
= 8.242 \angle -63.44^\circ \text{kA}
$$

The power factor of load is $\cos (\tan^{-1}(Q_l/P_l)) = 0.4472$ lagging. The phasor diagram for this case is similar to what is shown in Fig. 3.3(b).

(b) Compensator as a voltage regulator

Now it is required to maintain $V = E = 10.0 \text{kV}$ at the load bus. For this let their be reactive power $Q_\gamma$ supplied by the compensator at the load bus. Therefore the net reactive power at the load bus is equal to $Q_s$, which is given below.

$$
Q_s = Q_l + Q_\gamma
$$
Thus from (3.18), we get,
\[(R_s^2 + X_s^2)Q_s^2 + 2V^2X_sQ_s + (V^2 + R_sP_l)^2 + X_s^2P_l^2 - E^2V^2 = 0\]
\[(0.784^2 + 0.3922^2)Q_s^2 + 2 \times 10^2 \times 0.3922 \times Q_s + \{(10^2 + 0.784 \times 25)^2 + 0.3922^2 \times 25^2 - 10^4\} = 0\]

From the above we have,
\[0.16Q_s^2 + 78.44Q_s + 491.98 = 0.\]

Solving the above equation we get,
\[Q_s = \frac{-78.44 \pm \sqrt{78.44^2 - 4 \times 0.16 \times 491.98}}{2 \times 0.16} = -6.35 \text{ or } -484 \text{ MVAr.}\]

The feasible solution is \(Q_s = -6.35\) MVAr because it requires less rating of the compensator. Therefore the reactive power of the compensator (\(Q_\gamma\)) is,
\[Q_\gamma = Q_s - Q_l = -6.35 - 50 = -56.35\text{ MVAr.}\]

With \(Q_s = -6.35\) MVAr, the \(\Delta V\) is computed by replacing \(Q_s\) for \(Q_l\) in (3.14) as given below.
\[\Delta V = R_sP_l + X_sQ_s + jX_sP_l - R_sQ_s\]
\[= 0.0784 \times 25 + 0.39225 \times -6.35 + j0.39225 \times 25 - 0.0784 \times (-6.35)\]
\[= \frac{1.96 - 2.4 \times 10}{10} + j\frac{9.805 + 0.4978}{10}\]
\[= -0.0532 + j1.030\text{kV} = 1.03137 \angle 92.95^\circ \text{kV}\]

Now, we can find supply voltage \(E\) as given below.
\[E = \Delta V + V\]
\[= 10 - 0.0532 + j1.030\]
\[= 9.9468 + j1.030 = 10 \angle 5.91^\circ \text{kV}\]

The supply current is,
\[I_s = \frac{P_l - jQ_s}{V}\]
\[= \frac{25 - j(-6.35)}{10}\]
\[= 2.5 + j0.635 \text{kA} = 2.579 \angle 14.25^\circ \text{kA.}\]

This indicates that power factor is not unity for perfect voltage regulation i.e., \(E = V\). For this case the compensator current is given below.
\[I_\gamma = \frac{-jQ_\gamma}{V} = \frac{-j(-56.35)}{10}\]
\[I_\gamma = j5.635 \text{kA}\]
The load current is computed as below.

\[ I_l = \frac{P_l - jQ_l}{V} = \frac{25 - j50}{10} = 2.5 - j5.0 = I_{IR} + jI_{IX} = 5.59\angle63.44^\circ \text{kA} \]

The phasor diagram is similar to the one shown in Fig. 3.4(b). The phasor diagram shown has interesting features. The voltage at the load bus is maintained to 1.0 pu. It is observed that the reactive power of the compensator \( Q_\gamma \) is not equal to load reactive power \( (Q_l) \). It exceeds by 6.35 MVAR. As a result of this compensation, the voltage regulation is perfect, however power factor is not unity. The phase angle between \( V \) and \( I_s \) is \( \cos^{-1} 0.969 = 14.25^\circ \) as computed above. Therefore the angle between \( E \) and \( I_s \) is \( (14.25^\circ - 5.91^\circ = 8.34^\circ) \). Thus, source power factor \( (\phi_s) \) is \( \cos(8.34^\circ) = 0.9956 \) leading.

(c) Compensation for unity power factor

To achieve unity power factor at the load bus, the condition \( Q_\gamma = -Q_l \) must be satisfied, which further implies that the net reactive power at the load bus is zero. Therefore substituting \( Q_l = 0 \) in (3.31), we get the following.

\[ V^4 + \left\{ 2(R_s P_l - E^2) \right\} V^2 + (R_s^2 + X_s^2)(P_l^2 + Q_l^2) = 0 \]
\[ V^4 + (2 \times 0.0784 \times 25 - 10^2)V^2 + (0.0784^2 + 0.3922^2)25^2 = 0 \]

From the above,

\[ V^4 + 96.08V^2 + 99.79 = 0 \]

The solution of the above equation is,

\[ V^2 = \frac{96.08 \pm 93.97}{2} = 95.02, 1.052 \]
\[ V = \pm 9.747 \text{kV}, \pm 1.0256 \text{kV}. \]

Since rms value cannot be negative and maximum rms value must be a feasible solution, therefore \( V = 9.747 \text{kV} \). Thus it is seen that for obtaining unity power factor at the load bus does not ensure desired voltage regulation. Now the other quantities are computed as given below.

\[ I_l = \frac{P_l - jQ_l}{V} = \frac{25 - j50}{9.747} = 2.5648 - j5.129 = 5.7345\angle-63.43^\circ \text{kA} \]

Since \( Q_\gamma = -Q_l \), this implies that \( I_\gamma = -jQ_\gamma/V = jQ_l/V = j5.129 \text{kA} \). The voltage drop across the feeder is given as following.

\[ \Delta V = \frac{R_s P_l + X_s Q_l}{V} + j \frac{X_s P_l - R_s Q_l}{V} \]
\[ = \frac{(0.784 \times 25 + j0.3922 \times 25)}{9.747} \]
\[ = 0.201 + j1.005 = 1.0249\angle5.01^\circ \text{kV} \]
The phasor diagram for the above case is shown in Fig. 3.6.

![Phasor diagram for system with compensator in voltage regulation mode](image)

The percentage voltage change \( = \frac{(10 - 9.748)}{10} \times 100 = 2.5 \). Thus we see that power factor improves voltage regulation enormously compared with uncompensated case. In many cases, degree of improvement is adequate and the compensator can be designed to provide reactive power requirement of load rather than as a ideal voltage regulator.

### 3.3 Some Practical Aspects of Compensator used as Voltage Regulator

In this section, some practical aspects of the compensator in voltage regulation mode will be discussed. The important parameters of the compensator which play significant role in obtaining desired voltage regulation are: Knee point \( (V_k) \), maximum or rated reactive power \( Q_{\gamma max} \) and the compensator gain \( K_{\gamma} \).

The compensator gain \( K_{\gamma} \) is defined as the rate of change of compensator reactive power \( Q_{\gamma} \) with change in the voltage \( V \), as given below.

\[
K_{\gamma} = \frac{dQ_{\gamma}}{dV} \tag{3.32}
\]

For linear relationship between \( Q_{\gamma} \) and \( V \) with incremental change, the above equation be written as the following.

\[
\Delta Q_{\gamma} = \Delta V \cdot K_{\gamma} \tag{3.33}
\]
Assuming compensator characteristics to be linear with $Q_{\gamma} \leq Q_{\gamma \text{max}}$ limit, the voltage can be represented as,

$$V = V_k + \frac{Q_{\gamma}}{K_{\gamma}}$$  \hspace{1cm} (3.34)

This is re-written as,

$$Q_{\gamma} = K_{\gamma}(V - V_k)$$  \hspace{1cm} (3.35)

Flat V-Q characteristics imply that $K_{\gamma} \to \infty$. That means the compensator which can absorb or generate exactly right amount of reactive power to maintain supply voltage constant as the load varies without any constraint. We shall now see the regulating properties of the compensator, when compensator has finite gain $K_r$ operating on supply system with a finite short circuit level, $S_{sc}$. The further which are made in the following study are: high $X_s/R_s$ ratio and negligible load power fluctuations. The net reactive power at the load bus is sum of the load and the compensator reactive power as given below.

$$Q_l + Q_{\gamma} = Q_s$$  \hspace{1cm} (3.36)

Using earlier voltage and reactive power relationship from equation (3.30), it can be written as the following.

$$V \approx E(1 - \frac{Q_s}{S_{sc}})$$  \hspace{1cm} (3.37)

The compensator voltage represented by (3.34) and system voltage represented by (3.37) are shown in Fig. 3.7(a) and (b) respectively.

![Fig. 3.7 (a) Voltage characteristics of compensator (b) System voltage characteristics](image)

Differentiating $V$ with respect to $Q_s$, we get, intrinsic sensitivity of the supply voltage with variation in $Q_s$ as given below.

$$\frac{dV}{dQ_s} = -\frac{E}{S_{sc}}$$  \hspace{1cm} (3.38)
It is seen from the above equation that high value of short circuit level $S_{sc}$ reduces the voltage sensitivity, making voltage variation flat irrespective of $Q_l$. With compensator replacing $Q_s = Q_{\gamma} + Q_l$ in (3.37), we have the following.

$$V \simeq E \left(1 - \frac{Q_l + Q_{\gamma}}{S_{sc}}\right)$$ \hspace{1cm} (3.39)

Substituting $Q_{\gamma}$ from (3.35), we get the following equation.

$$V \simeq E \left[\frac{1 + K_{\gamma} V_k/S_{sc}}{1 + E K_{\gamma}/S_{sc}} - \frac{Q_l/S_{sc}}{1 + E K_{\gamma}/S_{sc}}\right]$$ \hspace{1cm} (3.40)

Although approximate, above equation gives the effects of all the major parameters such as load reactive power, the compensator characteristics $V_{\gamma}$ and $K_{\gamma}$ and the system characteristics $E$ and $S_{sc}$. As we discussed, V-Q characteristics is flat for high or infinite value $K_{\gamma}$. However the higher value of the gain $K_{\gamma}$ means large rating and quick rate of change of the reactive power with variation in the system voltage. This makes cost of the compensator high.

The compensator has two effects as seen from (3.40), i.e., it alters the no load supply voltage ($E$) and it modifies the sensitivity of supply point voltage to the variation in the load reactive power. Differentiating (3.40) with respect to $Q_l$, we get,

$$\frac{dV}{dQ_l} = -\frac{E/S_{sc}}{1 + K_{\gamma} E/S_{sc}}$$ \hspace{1cm} (3.41)

which is voltage sensitivity of supply point voltage to the load reactive power. It can be seen that the voltage sensitivity is reduced as compared to the voltage sensitivity without compensator as indicated in (3.38).

It is useful to express the slope $(-E/S_{sc})$ by a term in a form similar to $K_{\gamma} = dQ_{\gamma}/dV$, as given below.

$$K_s = -\frac{S_{sc}}{E}$$

Thus,

$$\frac{1}{K_s} = -\frac{E}{S_{sc}}$$ \hspace{1cm} (3.42)

Substituting $V$ from (3.39) into (3.35), the following is obtained.

$$Q_{\gamma} = K_{\gamma} \left[ E \left(1 - \frac{Q_l + Q_{\gamma}}{S_{sc}}\right) - V_k\right]$$ \hspace{1cm} (3.43)

Collecting the coefficients of $Q_{\gamma}$ from both sides of the above equation, we get

$$Q_{\gamma} = \frac{K_{\gamma}}{1 + K_{\gamma}(E/S_{sc})} \left[ E \left(1 - \frac{Q_l}{S_{sc}}\right) - V_k\right]$$ \hspace{1cm} (3.44)
Setting knee voltage $V_k$ of the compensator equal to system voltage $E$ i.e., $V_k = E$, the above equation is simplified to,

$$Q_\gamma = -\frac{K_\gamma (E/S_{sc})}{1 + K_\gamma (E/S_{sc})} Q_l = -\left(\frac{K_\gamma / K_s}{1 + K_\gamma / K_s}\right) Q_l. \quad (3.45)$$

From the above equation it is observed that, when compensator gain $K_\gamma \to \infty$, $Q_\gamma \to -Q_l$. This indicates perfect compensation of the load reactive power in order to regulate the load bus voltage.

**Example 3.2** Consider a three-phase system with line-line voltage 11 kV and short circuit capacity of 480 MVA. With compensator gain of 100 pu determine voltage sensitivity with and without compensator. For each case, if a load reactive power changes by 10 MVAr, find out the change in load bus voltage assuming linear relationship between V-Q characteristics. Also find relationship between compensator and load reactive powers.

**Solution:** The voltage sensitivity can be computed using the following equation.

$$\frac{dV}{dQ_l} = -\frac{E/S_{sc}}{1 + K_\gamma E/S_{sc}}$$

**Without compensator** $K_\gamma = 0, E = (11/\sqrt{3}) = 6.35$ kV and $S_{sc} = 480/3 = 160$ MVA.

Substituting these values in the above equation, the voltage sensitivity is given below.

$$\frac{dV}{dQ_l} = -\frac{6.35/160}{1 + 0 \times 6.35/160} = -0.039$$

The change in voltage due to variation of reactive power by 10 MVAr, $\Delta V = -0.039 \times 10 = -0.39$ kV.

**With compensator, $K_\gamma = 100$**

$$\frac{dV}{dQ_l} = -\frac{6.35/160}{1 + 100 \times 6.35/160} = -0.0078$$

The change in voltage due to variation of reactive power by 10 MVAr, $\Delta V = -0.0078 \times 10 = -0.078$ kV.

Thus it is seen that, with finite compensator gain their is quite reduction in the voltage sensitivity, which means that the load bus is fairly constant for considerable change in the load reactive power. The compensator reactive power $Q_\gamma$ and load reactive power $Q_l$ are related by equation (3.45) and is given below.

$$Q_\gamma = -\frac{K_\gamma (E/S_{sc})}{1 + K_\gamma (E/S_{sc})} Q_l = -\frac{100 \times (6.35/160)}{1 + 100 \times (6.35/160)} Q_l \approx -0.79 Q_l$$

It can be observed that when compensator gain, $(Q_\gamma)$ is quite large, then compensator reactive power $Q_\gamma$ is equal and opposite to that of load reactive power i.e., $Q_\gamma = -Q_l$. It is further observed
that due to finite compensator gain i.e., $K_\gamma = 100$, reactive power is partially compensated. The compensator reactive power varies from 0 to 7.9 MVAr for 0 to 10 MVAr change in the load reactive power.

### 3.4 Phase Balancing and Power Factor Correction of Unbalanced Loads

So far we have discussed voltage regulation and power factor correction for single phase systems. In this section we will focus on balancing of three-phase unbalanced loads. In considering unbalanced loads, both load and compensator are modeled in terms of their admittances and impedances.

#### 3.4.1 Three-phase Unbalanced Loads

Consider a three-phase three-wire system supplying unbalanced load as shown in Fig. 3.8.

![Fig. 3.8 Three-phase unbalanced load](image)

Applying Kirchoff’s voltage law for the two loops shown in the figure, we have the following equations.

\[
\begin{align*}
-V_{an} + Z_a I_1 + Z_b (I_1 - I_2) + V_{bn} &= 0 \\
-V_{bn} + Z_b I_2 + Z_b (I_2 - I_1) + V_{cn} &= 0
\end{align*}
\]

Rearranging above, we get the following.

\[
\begin{align*}
V_{an} - V_{bn} &= (Z_a + Z_b) I_1 - Z_b I_2 \\
V_{bn} - V_{cn} &= (Z_b + Z_c) I_2 - Z_b I_1
\end{align*}
\]

The above can be represented in matrix form as given below.

\[
\begin{bmatrix}
V_{an} - V_{bn} \\
V_{bn} - V_{cn}
\end{bmatrix} =
\begin{bmatrix}
(Z_a + Z_b) & -Z_b \\
-Z_b & (Z_b + Z_c)
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]
Therefore the currents are given as below.

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
= \begin{bmatrix}
(Z_a + Z_b) & -Z_b \\
-Z_b & (Z_b + Z_c)
\end{bmatrix}^{-1}
\begin{bmatrix}
V_{an} - V_{bn} \\
V_{bn} - V_{cn}
\end{bmatrix}
= \frac{1}{\Delta Z}
\begin{bmatrix}
(Z_b + Z_c) & Z_b \\
Z_b & (Z_a + Z_b)
\end{bmatrix}
\begin{bmatrix}
V_{an} - V_{bn} \\
V_{bn} - V_{cn}
\end{bmatrix}
\]

Therefore,

\[
\begin{bmatrix}
\tilde{I}_1 \\
\tilde{I}_2
\end{bmatrix}
= \frac{1}{\Delta Z}
\begin{bmatrix}
(Z_b + Z_c) & Z_b \\
Z_b & (Z_a + Z_b)
\end{bmatrix}
\begin{bmatrix}
V_{an} - V_{bn} \\
V_{bn} - V_{cn}
\end{bmatrix}
\]

where, \(\Delta Z = (Z_b + Z_c)(Z_a + Z_b) - Z_b^2 = Z_a Z_b + Z_b Z_c + Z_c Z_a\). The current \(\tilde{I}_1\) is given below.

\[
\tilde{I}_1 = \frac{1}{\Delta Z}
[(Z_b + Z_c)(V_{an} - V_{bn}) + Z_b(V_{bn} - V_{cn})]
= \frac{1}{\Delta Z}
[(Z_b + Z_c)V_{an} - Z_c V_{bn} - Z_b V_{cn}]
\]

(3.50)

Similarly,

\[
\tilde{I}_2 = \frac{1}{\Delta Z}
[Z_b(V_{an} - V_{bn}) + (Z_a + Z_b)(V_{bn} - V_{cn})]
= \frac{1}{\Delta Z}
[Z_b V_{an} + Z_a V_{bn} - (Z_a + Z_b) V_{cn}]
\]

(3.51)

Now,

\[
\tilde{I}_a = \tilde{I}_1 = \frac{1}{\Delta Z}
[(Z_b + Z_c)V_{an} - Z_c V_{bn} - Z_b V_{cn}]
\]

\[
\tilde{I}_b = \tilde{I}_2 - \tilde{I}_1
= \frac{1}{\Delta Z}
[Z_b V_{an} + Z_a V_{bn} - (Z_a + Z_b) V_{cn} - (Z_a + Z_c) V_{an} + Z_c V_{bn} + Z_b V_{cn}]
= \frac{[(Z_c + Z_a)V_{bn} - Z_a V_{cn} - Z_c V_{an}]}{\Delta Z}
= \frac{(Z_c + Z_a)V_{bn} - Z_c V_{an} - Z_a V_{cn}}{\Delta Z}
\]

(3.52)

and

\[
\tilde{I}_c = -\tilde{I}_2 = -\tilde{I}_b - \tilde{I}_a = \frac{(Z_a + Z_b)V_{cn} - Z_b V_{an} - Z_a V_{bn}}{\Delta Z}
\]

(3.53)

Alternatively phase currents can be expressed as following.

\[
\begin{align*}
\tilde{I}_a &= \frac{V_{an} - V_{Nn}}{Z_a} \\
\tilde{I}_b &= \frac{V_{bn} - V_{Nn}}{Z_b} \\
\tilde{I}_c &= \frac{V_{cn} - V_{Nn}}{Z_c}
\end{align*}
\]

(3.54)
Applying Kirchoff’s current law at node \( N \), we get \( I_a + I_b + I_c = 0 \). Therefore from the above equation,

\[
\frac{V_{an} - V_{Nn}}{Z_a} + \frac{V_{bn} - V_{Nn}}{Z_b} + \frac{V_{cn} - V_{Nn}}{Z_c} = 0.
\]  \( (3.55) \)

Which implies that,

\[
\frac{V_{an}}{Z_a} + \frac{V_{bn}}{Z_b} + \frac{V_{cn}}{Z_c} = \left[ \frac{1}{Z_a} + \frac{1}{Z_b} + \frac{1}{Z_c} \right] V_{Nn} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_aZ_bZ_c} V_{Nn}
\]  \( (3.56) \)

From the above equation the voltage between the load and system neutral can be found. It is given below.

\[
\frac{V_{Nn}}{\Delta Z} = \frac{Z_aZ_bZ_c}{\Delta Z} \left[ \frac{V_{an}}{Z_a} + \frac{V_{bn}}{Z_b} + \frac{V_{cn}}{Z_c} \right] = \frac{1}{\frac{1}{Z_a} + \frac{1}{Z_b} + \frac{1}{Z_c}} \left[ \frac{V_{an}}{Z_a} + \frac{V_{bn}}{Z_b} + \frac{V_{cn}}{Z_c} \right]
\]  \( (3.57) \)

Some interesting points are observed from the above formulation.

1. If both source voltage and load impedances are balanced i.e., \( Z_a = Z_b = Z_c = Z \), then \( V_{Nn} = \frac{1}{3} (V_{an} + V_{bn} + V_{cn}) = 0 \). Thus their will not be any voltage between two neutrals.

2. If supply voltage are balanced and load impedances are unbalanced, then \( V_{Nn} \neq 0 \) and is given by the above equation.

3. If supply voltages are not balanced but load impedances are identical, then \( V_{Nn} = \frac{1}{3} (V_{an} + V_{bn} + V_{cn}) \).

This equivalent to zero sequence voltage \( V_0 \).

It is interesting to note that if the two neutrals are connected together i.e., \( V_{Nn} = 0 \), then each phase become independent through neutral. Such configuration is called three-phase four-wire system. In general, three-phase four-wire system has following properties.

\[
\begin{align*}
& V_{Nn} = 0 \\
& I_a + I_b + I_c = I_{Nn} \neq 0
\end{align*}
\]  \( (3.58) \)

The current \( I_{Nn} \) is equivalent to zero sequence current (\( I_0 \)) and it will flow in the neutral wire.

For three-phase three-wire system, the zero sequence current is always zero and therefore following properties are satisfied.

\[
\begin{align*}
& V_{Nn} \neq 0 \\
& I_a + I_b + I_c = 0
\end{align*}
\]  \( (3.59) \)

Thus, it is interesting to observe that three-phase three-wire and three-phase four-wire system have dual properties in regard to neutral voltage and current.
3.4.2 Representation of Three-phase Delta Connected Unbalanced Load

A three-phase delta connected unbalanced and its equivalent star connected load are shown in Fig. 3.9(a) and (b) respectively. The three-phase load is represented by line-line admittances as given below.

\[
\begin{align*}
Y_{ab}^l &= G_{ab}^l + jB_{ab}^l \\
Y_{bc}^l &= G_{bc}^l + jB_{bc}^l \\
Y_{ca}^l &= G_{ca}^l + jB_{ca}^l
\end{align*}
\]  

(3.60)

The delta connected load can be equivalently converted to star connected load using following expressions.

\[
\begin{align*}
Z_a^l &= \frac{Z_{ab}^l Z_{ca}^l}{Z_{ab}^l + Z_{bc}^l + Z_{ca}^l} \\
Z_b^l &= \frac{Z_{bc}^l Z_{ab}^l}{Z_{ab}^l + Z_{bc}^l + Z_{ca}^l} \\
Z_c^l &= \frac{Z_{ca}^l Z_{bc}^l}{Z_{ab}^l + Z_{bc}^l + Z_{ca}^l}
\end{align*}
\]  

(3.61)

Where \( Z_{ab}^l = 1/Y_{ab}^l \), \( Z_{bc}^l = 1/Y_{bc}^l \) and \( Z_{ca}^l = 1/Y_{ca}^l \). The above equation can also be written in admittance form

\[
\begin{align*}
Y_{a}^l &= \frac{Y_{ab}^l Y_{bc}^l + Y_{bc}^l Y_{ca}^l + Y_{ca}^l Y_{ab}^l}{Y_{bc}^l} \\
Y_{b}^l &= \frac{Y_{ab}^l Y_{bc}^l + Y_{bc}^l Y_{ca}^l + Y_{ca}^l Y_{ab}^l}{Y_{ca}^l} \\
Y_{c}^l &= \frac{Y_{ab}^l Y_{bc}^l + Y_{bc}^l Y_{ca}^l + Y_{ca}^l Y_{ab}^l}{Y_{ab}^l}
\end{align*}
\]  

(3.62)

**Example 3.3** Consider three-phase system supply a delta connected unbalanced load with \( Z_a^l = R_a = 10 \, \Omega \), \( Z_b^l = R_b = 15 \, \Omega \) and \( Z_c^l = R_c = 30 \, \Omega \) as shown in Fig. 3.8. Determine the voltage...
between neutrals and find the phase currents. Assume a balance supply voltage with rms value of 230 V. Find out the vector and arithmetic power factor. Comment upon the results.

**Solution:** The voltage between neutrals $V_{\text{Nn}}$ is given as following.

$$V_{\text{Nn}} = \frac{R_a R_b R_c}{R_a R_b + R_b R_c + R_c R_a} \left[ \frac{V_{\text{an}}}{R_a} + \frac{V_{\text{bn}}}{R_b} + \frac{V_{\text{cn}}}{R_c} \right]$$

$$= \frac{10 \times 15 \times 30}{10 \times 15 + 15 \times 30 + 30 \times 10} \left[ \frac{V \angle 0^\circ}{10} + \frac{V \angle -120^\circ}{15} + \frac{V \angle 120^\circ}{30} \right]$$

$$= \frac{4500}{900} \left[ \frac{3V \angle 0^\circ + 2V \angle -120^\circ + V \angle 120^\circ}{30} \right]$$

$$= \frac{4500}{900} \frac{1}{30} V \left[ 3 + 2 \left( -\frac{1}{2} - j\frac{\sqrt{3}}{2} \right) + \left( -\frac{1}{2} + j\frac{\sqrt{3}}{2} \right) \right]$$

$$= \frac{V}{6} \left[ 3 - 1 - \frac{1}{2} - j2 \times \frac{\sqrt{3}}{2} + j\frac{\sqrt{3}}{2} \right]$$

$$= \frac{V}{6} \left[ 2 - j\frac{\sqrt{3}}{2} \right] = V \left[ \frac{1}{4} - j\frac{1}{4\sqrt{3}} \right] = V [0.25 - j0.1443]$$

$$V_{\text{Nn}} = \frac{V}{2\sqrt{3}} \angle -30^\circ = 66.39 \angle -30^\circ \text{ Volts}$$

Knowing this voltage, we can find phase currents as following.

$$I_a = \frac{V_{\text{an}} - V_{\text{Nn}}}{R_a} = \frac{V \angle 0^\circ - V/(2\sqrt{3}) \angle -30^\circ}{10}$$

$$= \frac{V [1 - 0.25 + j0.1443]}{10}$$

$$= 230 \times [0.075 + j0.01443]$$

$$= 17.56 \angle 10.89^\circ \text{ Amps}$$

Similarly,

$$I_b = \frac{V_{\text{bn}} - V_{\text{Nn}}}{R_b} = \frac{V \angle -120^\circ - V/(2\sqrt{3}) \angle -30^\circ}{15}$$

$$= 230 \times [-0.05 - j0.04811]$$

$$= 15.94 \angle -136.1^\circ \text{ Amps}$$

and

$$I_c = \frac{V_{\text{cn}} - V_{\text{Nn}}}{Z_c} = \frac{V \angle 120^\circ - V/(2\sqrt{3}) \angle -30^\circ}{30}$$

$$= 230 \times [-0.025 + j0.03367]$$

$$= 9.64 \angle 126.58^\circ \text{ Amps}$$

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It can be seen that $\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0$. The phase powers are computed as below.

\[
\begin{align*}
\mathcal{S}_a &= \mathbf{V}_a (\mathbf{I}_a)^* = P_a + jQ_a = 230 \times 17.56\angle -10.81^\circ = 3976.12 - j757.48 \text{ VA} \\
\mathcal{S}_b &= \mathbf{V}_b (\mathbf{I}_b)^* = P_b + jQ_b = 230 \times 15.94\angle(-120^\circ + 136.1^\circ) = 3522.4 + j1016.69 \text{ VA} \\
\mathcal{S}_c &= \mathbf{V}_c (\mathbf{I}_c)^* = P_c + jQ_c = 230 \times 9.64\angle(120^\circ - 126.58^\circ) = 2202.59 - j254.06 \text{ VA}
\end{align*}
\]

From the above the total apparent power $\mathcal{S}_V = \mathcal{S}_a + \mathcal{S}_b + \mathcal{S}_c = 9692.11 + j0 \text{ VA}$. Therefore, $\mathcal{S}_V = \left|\mathcal{S}_a + \mathcal{S}_b + \mathcal{S}_c\right| = 9692.11 \text{ VA}.$

The total arithmetic apparent power $S_A = |\mathcal{S}_a| + |\mathcal{S}_b| + |\mathcal{S}_c| == 9922.2 \text{ VA}$. Therefore, the arithmetic and vector apparent power factors are given by,

\[
\begin{align*}
p_{fA} &= \frac{P}{S_A} = \frac{9692.11}{9922.2} = 0.9768 \\
p_{fV} &= \frac{P}{S_V} = \frac{9622.11}{9622.11} = 1.00.
\end{align*}
\]

It is interesting to note that although the load in each phase is resistive but each phase has some reactive power. This is due to unbalance of the load currents. This apparently increases the rating of power conductors for given amount of power transfer. It is also to be noted that the net reactive power $Q = Q_a + Q_b + Q_c = 0$ leading to the unity vector apparent power factor. However the arithmetic apparent power factor is less than unity showing the effect of the unbalance loads on the power factor.

### 3.4.3 An Alternate Approach to Determine Phase Currents and Powers

In this section, an alternate approach will be discussed to solve phase currents and powers directly without computing the neutral voltage for the system shown in Fig.3.9(a). First we express three-phase voltage in the following form.

\[
\begin{align*}
\mathbf{V}_a &= V \angle 0^\circ \\
\mathbf{V}_b &= V \angle -120^\circ = \alpha^2 V \\
\mathbf{V}_c &= V \angle 120^\circ = \alpha V
\end{align*}
\]

Where, in above equation, $\alpha$ is known as complex operator and value of $\alpha$ and $\alpha^2$ are given below.

\[
\begin{align*}
\alpha &= e^{j2\pi/3} = 1\angle 120^\circ = -1/2 + j\sqrt{3}/2 \\
\alpha^2 &= e^{j4\pi/3} = 1\angle 240 = 1\angle -120 = -1/2 - j\sqrt{3}/2
\end{align*}
\]

Also note the following property,

\[
1 + \alpha + \alpha^2 = 0.
\]

Using the above, the line to line voltages can be expressed as following.
\[
\begin{align*}
\bar{V}_{ab} &= \bar{V}_a - \bar{V}_b = (1 - \alpha^2)V \\
\bar{V}_{bc} &= \bar{V}_b - \bar{V}_c = (\alpha^2 - \alpha)V \\
\bar{V}_{ca} &= \bar{V}_c - \bar{V}_a = (\alpha - 1)V
\end{align*}
\]

(3.66)

Therefore, currents in line \( ab, bc \) and \( ca \) are given as,

\[
\begin{align*}
\bar{I}_{abl} &= Y_{ab}^{l} \bar{V}_{ab} = Y_{ab}^{l} (1 - \alpha^2)V \\
\bar{I}_{bcl} &= Y_{bc}^{l} \bar{V}_{bc} = Y_{bc}^{l} (\alpha^2 - \alpha)V \\
\bar{I}_{cal} &= Y_{ca}^{l} \bar{V}_{ca} = Y_{ca}^{l} (\alpha - 1)V
\end{align*}
\]

(3.67)

\[
\begin{align*}
\bar{I}_{al} &= \bar{I}_{abl} - \bar{I}_{cal} = [Y_{ab}^{l} (1 - \alpha^2) - Y_{ca}^{l} (\alpha - 1)]V \\
\bar{I}_{bl} &= \bar{I}_{bcl} - \bar{I}_{abl} = [Y_{bc}^{l} (\alpha^2 - \alpha) - Y_{ab}^{l} (1 - \alpha^2)]V \\
\bar{I}_{cl} &= \bar{I}_{cal} - \bar{I}_{bcl} = [Y_{ca}^{l} (\alpha - 1) - Y_{bc}^{l} (\alpha^2 - \alpha)]V
\end{align*}
\]

(3.68)

**Example 3.4** Compute line currents by using above expressions directly for the problem in Example 3.3.

**Solution:** To compute line currents directly from the above expressions, we need to compute \( Y_{ab}^{l} \). These are given below

\[
\begin{align*}
Y_{ab}^{l} &= \frac{1}{Z_{ab}^{l}} = \frac{Z_{c}^{l}}{Z_{a}^{l} + Z_{b}^{l} Z_{c}^{l} + Z_{c}^{l} Z_{a}^{l}} \\
Y_{bc}^{l} &= \frac{1}{Z_{bc}^{l}} = \frac{Z_{a}^{l}}{Z_{a}^{l} Z_{b}^{l} + Z_{b}^{l} Z_{c}^{l} + Z_{c}^{l} Z_{a}^{l}} \\
Y_{ca}^{l} &= \frac{1}{Z_{ca}^{l}} = \frac{Z_{b}^{l}}{Z_{a}^{l} Z_{b}^{l} + Z_{b}^{l} Z_{c}^{l} + Z_{c}^{l} Z_{a}^{l}}
\end{align*}
\]

(3.69)

Substituting, \( Z_{a}^{l} = R_{a} = 10 \Omega, Z_{b}^{l} = R_{b} = 15 \Omega \) and \( Z_{c}^{l} = R_{c} = 30 \Omega \) into above equation, we get the following.

\[
\begin{align*}
Y_{ab}^{l} &= G_{ab}^{l} = \frac{1}{30} \Omega \\
Y_{bc}^{l} &= G_{bc}^{l} = \frac{1}{90} \Omega \\
Y_{ca}^{l} &= G_{ca}^{l} = \frac{1}{60} \Omega
\end{align*}
\]

Substituting above values of the admittances in (3.68), line currents are computed as below.
Thus it is found that the above values are similar to what have been found in previous Example 3.3. The other quantities such as powers and power factors are same.

### 3.4.4 An Example of Balancing an Unbalanced Delta Connected Load

An unbalanced delta connected load is shown in Fig. 3.10(a). As can be seen from the figure that between phase-a and b there is admittance $Y_{ab}^\gamma = G_{ab}^\gamma$ and other two branches are open. This is an example of extreme unbalanced load. Obviously for this load, line currents will be extremely unbalanced. Now we aim to make these line currents to be balanced and in phase with their phase voltages. So, let us assume that we add admittances $Y_{\gamma}^{ab}$, $Y_{\gamma}^{bc}$ and $Y_{\gamma}^{ca}$ between phases ab, bc and ca respectively as shown in Fig. 3.10(b) and (c). Let values of compensator susceptances are given by,

\[
Y_{\gamma}^{ab} = 0 \\
Y_{\gamma}^{bc} = j \frac{G_{ab}^\gamma}{\sqrt{3}} \\
Y_{\gamma}^{ca} = -j \frac{G_{ab}^\gamma}{\sqrt{3}}
\]
Thus total admittances between lines are given by,

\[
Y_{ab} = Y_{l}^{ab} + Y_{\gamma}^{ab} = G_{l}^{ab} + 0 = G_{l}^{ab} \\
Y_{bc} = Y_{l}^{bc} + Y_{\gamma}^{bc} = 0 + jG_{l}^{ab}/\sqrt{3} = jG_{l}^{ab}/\sqrt{3} \\
Y_{ca} = Y_{l}^{ca} + Y_{\gamma}^{ca} = 0 - jG_{l}^{ab}/\sqrt{3} = -jG_{l}^{ab}/\sqrt{3}.
\]

Therefore the impedances between load lines are given by,

\[
Z_{ab} = \frac{1}{Y_{ab}} = \frac{1}{G_{l}^{ab}} \\
Z_{bc} = \frac{1}{Y_{bc}} = \frac{-j\sqrt{3}}{G_{l}^{ab}} \\
Z_{ca} = \frac{1}{Y_{ca}} = \frac{j\sqrt{3}}{G_{l}^{ab}}
\]

Note that \(Z_{ab} + Z_{bc} + Z_{ca} = 1/G_{l}^{ab} - j\sqrt{3}/G_{l}^{ab} + j\sqrt{3}/G_{l}^{ab} = 1/G_{l}^{ab}\).

The impedances, \(Z_a\), \(Z_b\) and \(Z_c\) of equivalent star connected load are given as follows.

\[
Z_a = \frac{Z_{ab} \times Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}} \\
    = \left(\frac{1}{G_{l}^{ab}} \times \frac{j\sqrt{3}}{G_{l}^{ab}}\right)/\left(\frac{1}{G_{l}^{ab}}\right) \\
    = \frac{j\sqrt{3}}{G_{l}^{ab}}
\]
\[ Z_b = \frac{Z^{bc} \times Z^{ab}}{Z^{ab} + Z^{bc} + Z^{ca}} \]

\[ = \frac{1}{G_{lb}} \times \frac{-j\sqrt{3}}{G_{lb}} \times \left( \frac{1}{G_{lb}} \right) \]

\[ = \frac{-j\sqrt{3}}{G_{lb}} \]

\[ Z_c = \frac{Z^{ca} \times Z^{bc}}{Z^{ab} + Z^{bc} + Z^{ca}} \]

\[ = \frac{1}{G_{lb}} \times \frac{j\sqrt{3}}{G_{lb}} \times \left( \frac{1}{G_{lb}} \right) \]

\[ = \frac{3}{G_{lb}} \]

The above impedances seen from the load side are shown in Fig. 3.11(a) below. Using (3.57),

![Fig. 3.11 Compensated system (a) Load side (b) Source side](image)

the voltage between load and system neutral of delta equivalent star load as shown in Fig. 3.11, is computed as below.

\[ V_{Nn} = \frac{1}{Z_a + \frac{1}{Z_b} + \frac{1}{Z_c}} \left[ \frac{V_{an}}{Z_a} + \frac{V_{bn}}{Z_b} + \frac{V_{cn}}{Z_c} \right] \]

\[ = \frac{1}{j\sqrt{3}/G_{lb}^b} + \frac{1}{(-j\sqrt{3}/G_{lb}^b) + \frac{3}{G_{lb}^b}} \left[ \frac{V\angle 0^\circ}{j\sqrt{3}/G_{lb}^b} + \frac{V\angle -120^\circ}{-j\sqrt{3}/G_{lb}^b} + \frac{V\angle 120^\circ}{3/G_{lb}^b} \right] \]

\[ = 3 V \left( \frac{G_{lb}^{ab}}{3} - j \frac{G_{lb}^{ab}}{\sqrt{3}} \right) \]

\[ = 2 V \left( \frac{1}{2} - j \frac{\sqrt{3}}{2} \right) \]

\[ = 2 V \angle -60^\circ \]

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Using above value of neutral voltage the line currents are computed as following.

\[ I_a = \frac{V_{an} - V_{Nn}}{Z_a} = \frac{[V \angle 0^\circ - 2V \angle -60^\circ]}{j \frac{\sqrt{3}}{G_{ab}}} = G_{ab}^a V = G_{ab}^a V_a \]

\[ I_b = \frac{V_{bn} - V_{Nn}}{Z_b} = \frac{[V \angle -120^\circ - 2V \angle -60^\circ]}{-j \frac{\sqrt{3}}{G_{ab}}} = G_{ab}^b V \angle 240^\circ = G_{ab}^b V_b \]

\[ I_c = \frac{V_{cn} - V_{Nn}}{Z_c} = \frac{[V \angle 120^\circ - 2V \angle -60^\circ]}{3 \frac{1}{G_{ab}}} = G_{ab}^c V \angle 120^\circ = G_{ab}^c V_c \]

From the above example, it is seen that the currents in each phase are balanced and in phase with their respective voltages. This is equivalently shown in Fig. 3.11(b). It is to be mentioned here that the two neutrals in Fig. 3.11 are not same. In Fig. 3.11(b), the neutral \( N \) is same as the system neutral as shown in Fig. 3.8, whereas in Fig. 3.11(a), \( V_{Nn} = 2V \angle -60^\circ \). However the reader may be curious to know why \( Y_{\gamma}^{ab} = 0 \), \( Y_{\gamma}^{bc} = jG_{ab}^a / \sqrt{3} \) and \( \gamma^{ca} = -jG_{ab}^a / \sqrt{3} \) have been chosen as compensator admittance values. The answer of the question can be found by going following sections.

### 3.5 A Generalized Approach for Load Compensation using Symmetrical Components

In the previous section, we have expressed line currents \( I_a, I_b \) and \( I_c \), in terms load admittances and the voltage \( V \) for a delta connected unbalanced load as shown in Fig 3.12(a). For the sake of completeness, theses are reproduced below.
I_{al} = T_{abl} - T_{cal} = [Y_{l}^{ab}(1 - \alpha^2) - Y_{l}^{ca}(\alpha - 1)]V
I_{bl} = T_{bcl} - T_{lab} = [Y_{l}^{bc}(\alpha^2 - \alpha) - Y_{l}^{ab}(1 - \alpha^2)]V
I_{cl} = T_{cal} - T_{bcl} = [Y_{l}^{ca}(\alpha - 1) - Y_{l}^{bc}(\alpha^2 - \alpha)]V

(3.70)

\begin{align*}
\begin{bmatrix}
T_{0l} \\
T_{1l} \\
T_{2l}
\end{bmatrix} &= \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 1 & 1 \\
1 & a & a^2 \\
1 & a^2 & a
\end{bmatrix}
\begin{bmatrix}
T_{al} \\
T_{bl} \\
T_{cl}
\end{bmatrix}
\end{align*}

From the above equation, zero sequence current is given below.

\[ T_{0l} = (T_{al} + T_{al} + T_{al}) / \sqrt{3} \]

The positive sequence current is as follows.

\[ T_{1l} = \frac{1}{\sqrt{3}} \left[ T_{al} + \alpha T_{bl} + \alpha^2 T_{cl} \right] \]
\[ = \frac{1}{\sqrt{3}} \left[ Y_{l}^{ab}(1 - \alpha^2) - Y_{l}^{ca}(\alpha - 1) + \alpha \left\{ Y_{l}^{bc}(\alpha^2 - \alpha) - Y_{l}^{ab}(1 - \alpha^2) \right\} + \alpha^2 \left\{ Y_{l}^{ca}(\alpha - 1) - Y_{l}^{bc}(\alpha - \alpha^2) \right\} \right] V \]
\[ = \frac{1}{\sqrt{3}} \left[ Y_{l}^{ab} - \alpha^2 Y_{l}^{ab} + Y_{l}^{ca} - \alpha Y_{l}^{ca} + \alpha Y_{l}^{bc} - \alpha^2 Y_{l}^{bc} - \alpha Y_{l}^{ab} - \alpha^3 Y_{l}^{ab} + \alpha^2 Y_{l}^{ca} - \alpha^3 Y_{l}^{ca} - \alpha^2 Y_{l}^{ca} - \alpha^4 Y_{l}^{bc} + \alpha Y_{l}^{bc} \right] V \]
\[ = \left( Y_{l}^{ab} + Y_{l}^{bc} + Y_{l}^{ca} \right) V \sqrt{3} \]

Fig. 3.12 (a) An unbalanced delta connected load (b) Compensated system

Since loads are currents are unbalanced, these will have positive and negative currents. The zero sequence current will be zero as it is three-phase and three-wire system. These symmetrical components of the load currents are expressed as following.

\[ \begin{bmatrix}
T_{0l} \\
T_{1l} \\
T_{2l}
\end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 1 & 1 \\
1 & a & a^2 \\
1 & a^2 & a
\end{bmatrix}
\begin{bmatrix}
T_{al} \\
T_{bl} \\
T_{cl}
\end{bmatrix} \quad (3.71) \]
Similarly negative sequence component of the current is,

\[
I_{2l} = \frac{1}{\sqrt{3}} \left[ I_{al} + \alpha^2 I_{bl} + \alpha I_{cl} \right]
\]

\[
= \frac{1}{\sqrt{3}} \left[ Y_{I}^{ab} (1 - \alpha^2) - Y_{I}^{ca} (\alpha - 1) + \alpha^2 \left\{ Y_{I}^{bc} (\alpha^2 - \alpha) - Y_{I}^{ab} (1 - \alpha^2) \right\} \right]
\]

\[
+ \alpha \left\{ Y_{I}^{ca} (\alpha - 1) - Y_{I}^{bc} (\alpha^2 - \alpha) \right\} \bar{V}
\]

\[
= \frac{1}{\sqrt{3}} \left[ Y_{I}^{ab} - \alpha^2 Y_{I}^{ab} - \alpha Y_{I}^{ca} + Y_{I}^{ca} + \alpha^4 Y_{I}^{bc} - \alpha^3 Y_{I}^{bc} - \alpha^2 Y_{I}^{ab}
\]

\[
+ \alpha^2 Y_{I}^{ab} - \alpha Y_{I}^{ca} - \alpha Y_{I}^{bc} + \alpha^2 Y_{I}^{bc} \right\} V
\]

\[
= \frac{1}{\sqrt{3}} \left[ -3 \alpha^2 Y_{I}^{ab} - 3 Y_{I}^{bc} - 3 \alpha Y_{I}^{ca} \right] V
\]

\[
= - [\alpha^2 Y_{I}^{ab} + Y_{I}^{bc} + \alpha Y_{I}^{ca}] \sqrt{3} V
\]

From the above, it can be written that,

\[
\bar{T}_{0l} = 0
\]

\[
\bar{T}_{1l} = (Y_{I}^{ab} + Y_{I}^{bc} + Y_{I}^{ca}) \sqrt{3} V \quad (3.72)
\]

\[
\bar{T}_{2l} = - (\alpha^2 Y_{I}^{ab} + Y_{I}^{bc} + \alpha Y_{I}^{ca}) \sqrt{3} V
\]

When compensator is used, three delta branches \( Y_{\gamma}^{ab}, Y_{\gamma}^{bc} \) and \( Y_{\gamma}^{ca} \) are added as shown in Fig. 3.12(b). Using above analysis, the sequence components of the compensator currents can be given as below.

\[
\bar{T}_{0\gamma} = 0
\]

\[
\bar{T}_{1\gamma} = (Y_{\gamma}^{ab} + Y_{\gamma}^{bc} + Y_{\gamma}^{ca}) \sqrt{3} V \quad (3.73)
\]

\[
\bar{T}_{2\gamma} = - (\alpha^2 Y_{\gamma}^{ab} + Y_{\gamma}^{bc} + \alpha Y_{\gamma}^{ca}) \sqrt{3} V
\]

Since, compensator currents are purely reactive, i.e., \( G_{\gamma}^{ab} = G_{\gamma}^{bc} = G_{\gamma}^{ca} = 0 \),

\[
Y_{\gamma}^{ab} = G_{\gamma}^{ab} + j B_{\gamma}^{ab} = j B_{\gamma}^{ab}
\]

\[
Y_{\gamma}^{bc} = G_{\gamma}^{bc} + j B_{\gamma}^{bc} = j B_{\gamma}^{bc}
\]

\[
Y_{\gamma}^{ca} = G_{\gamma}^{ca} + j B_{\gamma}^{ca} = j B_{\gamma}^{ca} \quad (3.74)
\]

Using above, the compensated sequence currents can be written as,

\[
\bar{T}_{0\gamma} = 0
\]

\[
\bar{T}_{1\gamma} = j (B_{\gamma}^{ab} + B_{\gamma}^{bc} + B_{\gamma}^{ca}) \sqrt{3} V \quad (3.75)
\]

\[
\bar{T}_{2\gamma} = - j (\alpha^2 B_{\gamma}^{ab} + B_{\gamma}^{bc} + \alpha B_{\gamma}^{ca}) \sqrt{3} V
\]

Knowing nature of compensator and load currents, we can set compensation objectives as following.

1. All negative sequence component of the load current must be supplied from the compensator negative current, i.e.,

\[
\bar{T}_{2l} = - \bar{T}_{2\gamma} \quad (3.76)
\]
The above further implies that,
\[\text{Re}(\bar{I}_2) + j \text{Im}(\bar{I}_2) = -\text{Re}(\bar{I}_2) - j \text{Im}(\bar{I}_2)\] (3.77)

2. The total positive sequence current, which is source current should have desired power factor from the source, i.e.,
\[\frac{\text{Im}(\bar{I}_{ll} + \bar{I}_{1\gamma})}{\text{Re}(\bar{I}_{ll} + \bar{I}_{1\gamma})} = \tan \phi = \beta\] (3.78)

Where, \(\phi\) is the desired phase angle between the line currents and the supply voltages. The above equation thus implies that,
\[\text{Im}(\bar{I}_{ll} + \bar{I}_{1\gamma}) = \beta \text{Re}(\bar{I}_{ll} + \bar{I}_{1\gamma})\] (3.79)

Since \(\text{Re}(\bar{I}_{1\gamma}) = 0\), the above equation is rewritten as following.
\[\text{Im}(\bar{I}_{ll}) - \beta \text{Re}(\bar{I}_{ll}) = -\text{Im}(\bar{I}_{1\gamma})\] (3.80)

The equation (3.77) gives two conditions and equation (3.79) gives one condition. There are three unknown variables, i.e., \(B_{\gamma}^{ab}, B_{\gamma}^{bc}\) and \(B_{\gamma}^{ca}\) and three conditions. Therefore the unknown variables can be solved. This is described in the following section. Using (3.75), the current \(\bar{I}_{2\gamma}\) is expressed as following.

\[\bar{I}_{2\gamma} = -j[\alpha^2 B_{\gamma}^{ab} + B_{\gamma}^{bc} + \alpha B_{\gamma}^{ca}] \sqrt{3} V\]
\[= -j \left[ \left( -\frac{1}{2} - j \frac{\sqrt{3}}{2} \right) B_{\gamma}^{ab} + B_{\gamma}^{bc} + \left( -\frac{1}{2} + j \frac{\sqrt{3}}{2} \right) B_{\gamma}^{ca} \right] \sqrt{3} V\]
\[= \left[ -\frac{\sqrt{3}}{2} B_{\gamma}^{ab} + \frac{\sqrt{3}}{2} B_{\gamma}^{ca} \right] - j \left[ -\frac{1}{2} B_{\gamma}^{ab} + B_{\gamma}^{bc} - \frac{1}{2} B_{\gamma}^{ca} \right] \sqrt{3} V\] (3.81)

Thus the above equation implies that
\[\left( \frac{\sqrt{3}}{2} B_{\gamma}^{ab} - \frac{\sqrt{3}}{2} B_{\gamma}^{ca} \right) = -\frac{1}{\sqrt{3}V} \text{Re}(\bar{I}_{2\gamma}) = \frac{1}{\sqrt{3}V} \text{Re}(\bar{I}_{2l})\] (3.82)

and,
\[\left( -\frac{1}{2} B_{\gamma}^{ab} + B_{\gamma}^{bc} - \frac{1}{2} B_{\gamma}^{ca} \right) = -\frac{1}{\sqrt{3}V} \text{Im}(\bar{I}_{2\gamma}) = \frac{1}{\sqrt{3}V} \text{Im}(\bar{I}_{2l})\] (3.83)

Or
\[(-B_{\gamma}^{ab} + 2 B_{\gamma}^{bc} - B_{\gamma}^{ca}) = \frac{1}{\sqrt{3}V} 2 \text{Im}(\bar{I}_{2l})\] (3.84)
From (3.75), \( \text{Im} (\bar{T}_{1\gamma}) \) can be written as,
\[
\text{Im} (\bar{T}_{1\gamma}) = (B_{\gamma}^{ab} + B_{\gamma}^{bc} + B_{\gamma}^{ca}) \sqrt{3} V. \tag{3.85}
\]
Substituting \( \text{Im} (\bar{T}_{1\gamma}) \) from above equation into (3.85), we get the following.
\[
-(B_{\gamma}^{ab} + B_{\gamma}^{bc} + B_{\gamma}^{ca}) = -\frac{1}{\sqrt{3}V} \text{Im} (\bar{T}_{1\gamma}) = \frac{1}{\sqrt{3}V} \{ \text{Im} (\bar{T}_{1\gamma}) - \beta \text{Re} (\bar{T}_{1\gamma}) \} \tag{3.86}
\]
Subtracting (3.86) from (3.84), the following is obtained,
\[
B_{\gamma}^{bc} = -\frac{1}{3\sqrt{3}V} \{ \text{Im} (\bar{T}_{1\gamma}) - 2 \text{ Im} (\bar{T}_{2\gamma}) - \beta \text{ Re} (\bar{T}_{1\gamma}) \}. \tag{3.87}
\]
Now, from (3.82) we have
\[
-\frac{1}{2} B_{\gamma}^{ab} - \frac{1}{2} B_{\gamma}^{ca} = \frac{1}{\sqrt{3}V} \text{Im} (\bar{T}_{2\gamma}) - B_{\gamma}^{bc}
\]
\[
= \frac{1}{\sqrt{3}V} \text{Im} (\bar{T}_{2\gamma}) - \left[ -\frac{1}{3\sqrt{3}V} \{ \text{Im} (\bar{T}_{1\gamma}) - \beta \text{ Re} (\bar{T}_{1\gamma}) - 2 \text{ Im} (\bar{T}_{2\gamma}) \} \right]
\]
\[
= \frac{1}{3\sqrt{3}V} \{ \text{Im} (\bar{T}_{2\gamma}) + \text{Im} (\bar{T}_{1\gamma}) - \beta \text{ Re} (\bar{T}_{1\gamma}) \} \tag{3.88}
\]
Reconsidering (3.88) and (3.82), we have
\[
-\frac{1}{2} B_{\gamma}^{ab} - \frac{1}{2} B_{\gamma}^{ca} = \frac{1}{3\sqrt{3}V} \{ \text{Im} (\bar{T}_{1\gamma}) + \text{Im} (\bar{T}_{2\gamma}) - \beta \text{ Re} (\bar{T}_{1\gamma}) \}
\]
\[
\frac{1}{2} B_{\gamma}^{ab} - \frac{1}{2} B_{\gamma}^{ca} = \frac{1}{3\sqrt{3}V} [\sqrt{3} \text{Re} (\bar{T}_{2\gamma})]
\]
Adding above equations, we get
\[
B_{\gamma}^{ca} = \frac{-1}{3\sqrt{3}V} [\text{Im} (\bar{T}_{1\gamma}) + \text{Im} (\bar{T}_{2\gamma}) + \sqrt{3} \text{Re} (\bar{T}_{2\gamma}) - \beta \text{Re} (\bar{T}_{1\gamma})]. \tag{3.89}
\]
Therefore,
\[
B_{\gamma}^{ab} = B_{\gamma}^{ca} + \frac{2}{3\sqrt{3}V} [\sqrt{3} \text{Re} (\bar{T}_{2\gamma})] \tag{3.90}
\]
\[
= \frac{-1}{3\sqrt{3}V} [\text{Im} (\bar{T}_{1\gamma}) + \text{Im} (\bar{T}_{2\gamma}) + \sqrt{3} \text{Re} (\bar{T}_{2\gamma}) - \beta \text{Re} (\bar{T}_{1\gamma}) - 2\sqrt{3} \text{Re} (\bar{T}_{2\gamma})]
\]
\[
= \frac{-1}{3\sqrt{3}V} [\text{Im} (\bar{T}_{1\gamma}) + \text{Im} (\bar{T}_{2\gamma}) - \sqrt{3} \text{Re} (\bar{T}_{2\gamma}) - \beta \text{Re} (\bar{T}_{1\gamma})] \tag{3.91}
\]
From the above, the compensator susceptances in terms of real and imaginary parts of the load
current can be written as following.

\[
B_{\gamma}^{ab} = \frac{-1}{3\sqrt{3}V} [\text{Im} (\bar{T}_{1l}) + \text{Im} (\bar{T}_{2l}) - \sqrt{3} \text{Re} (\bar{T}_{2l}) - \beta \text{Re} (\bar{T}_{1l})] \\
B_{\gamma}^{bc} = \frac{-1}{3\sqrt{3}V} [\text{Im} (\bar{T}_{1l}) - \beta \text{Re} (\bar{T}_{1l}) - 2\text{Im} (\bar{T}_{2l} - \beta \text{Re} (\bar{T}_{1l}))] \\
B_{\gamma}^{ca} = \frac{-1}{3\sqrt{3}V} [\text{Im} (\bar{T}_{1l}) + \text{Im} (\bar{T}_{2l}) + \sqrt{3} \text{Re} (\bar{T}_{2l}) - \beta \text{Re} (\bar{T}_{1l})]
\] (3.92)

In the above equation, the susceptances of the compensator are expressed in terms of real and imaginary parts of symmetrical components of load currents. It is however advantageous to express these susceptances in terms of instantaneous values of voltages and currents from implementation point of view. The first step to achieve this is to express these susceptances in terms of load currents, i.e., \( I_{al}, I_{bl} \) and \( I_{cl} \), which is described below. Using equation (3.71), the sequence components of the load currents are expressed as,

\[
\bar{I}_{0l} = \frac{1}{\sqrt{3}} [I_{al} + I_{bl} + I_{cl}] \\
\bar{I}_{1l} = \frac{1}{\sqrt{3}} [I_{al} + \alpha I_{bl} + \alpha^2 I_{cl}] \\
\bar{I}_{2l} = \frac{1}{\sqrt{3}} [I_{al} + \alpha^2 I_{1l} + \alpha I_{cl}].
\] (3.93)

Substituting these values of sequence components of load currents, in (3.92), we can obtain compensator susceptances in terms of real and imaginary components of the load currents. Let us start from the \( B_{\gamma}^{bc} \), as obtained following.

\[
B_{\gamma}^{bc} = -\frac{1}{3\sqrt{3}V} [\text{Im} (\bar{T}_{1l}) - 2\text{Im} (\bar{T}_{2l}) - \beta \text{Re} (\bar{T}_{1l})] \\
= -\frac{1}{3\sqrt{3}V} \left[ \text{Im} \left( \frac{I_{al} + \alpha I_{bl} + \alpha^2 I_{cl}}{\sqrt{3}} \right) - 2 \text{Im} \left( \frac{I_{al} + \alpha^2 I_{bl} + \alpha I_{cl}}{\sqrt{3}} \right) - \beta \text{Re} \left( \frac{I_{al} + \alpha I_{bl} + \alpha^2 I_{cl}}{\sqrt{3}} \right) \right] \\
= -\frac{1}{9V} \left[ \text{Im} \left\{ \left( -I_{al} + (2 + 3\alpha) I_{bl} + (2 + 3\alpha^2) I_{cl} \right) - \beta \text{Re} \left( I_{al} + \alpha I_{bl} + \alpha^2 I_{cl} \right) \right\} \right] \\
= -\frac{1}{9V} \left[ \text{Im} \left\{ -I_{al} + 2I_{bl} + 2I_{cl} + 3\alpha I_{bl} + 3\alpha^2 I_{cl} \right\} - \beta \text{Re} \left( I_{al} + \alpha I_{bl} + \alpha^2 I_{cl} \right) \right]
\]

By adding and subtracting \( I_{bl} \) and \( I_{cl} \) in the above equation we get,

\[
B_{\gamma}^{bc} = -\frac{1}{9V} \left[ \text{Im} \left\{ \left( -I_{al} + I_{bl} - I_{cl} \right) + 3I_{bl} + 3I_{cl} + 3\alpha I_{bl} + \alpha^2 I_{cl} \right\} - \beta \text{Re} \left( I_{al} + \alpha I_{bl} + \alpha^2 I_{cl} \right) \right]
\]

We know that \( I_{al} + I_{bl} + I_{cl} = 0 \), therefore \( I_{al} + I_{bl} = -I_{cl} \).

\[
B_{\gamma}^{bc} = -\frac{1}{3V} \left[ -\text{Im} \left( I_{al} \right) + \text{Im} \left( \alpha I_{bl} \right) + \text{Im} \left( \alpha^2 I_{cl} \right) - \frac{\beta}{3} \text{Re} \left( I_{al} + \alpha I_{bl} + \alpha^2 I_{cl} \right) \right]
\] (3.94)
Similarly, it can be proved that,

\[ B_{ca}^\gamma = -\frac{1}{3V} \left[ \text{Im}(\bar{T}_{al}) - \text{Im}(\alpha \bar{T}_{bl}) + \text{Im}(\alpha^2 \bar{T}_{cl}) - \frac{\beta}{3} \text{Re}(\bar{T}_{al} + \alpha \bar{T}_{bl} + \alpha^2 \bar{T}_{cl}) \right] \]  

\( (3.95) \)

\[ B_{ab}^\gamma = -\frac{1}{3V} \left[ \text{Im}(\bar{T}_{al}) + \text{Im}(\alpha \bar{T}_{bl}) - \text{Im}(\alpha^2 \bar{T}_{cl}) - \frac{\beta}{3} \text{Re}(\bar{T}_{al} + \alpha \bar{T}_{bl} + \alpha^2 \bar{T}_{cl}) \right] \]  

\( (3.96) \)

The above expressions for \( B_{ca}^\gamma \) and \( B_{ab}^\gamma \) are proved below. For convenience, the last term associated with \( \beta \) is not considered. For the sake simplicity in equations (3.95) and (3.96) are proved to those given in equations (3.92).

\[ B_{ca}^\gamma = -\frac{1}{3V} \left[ \text{Im}(\bar{T}_{al}) - \text{Im}(\alpha \bar{T}_{bl}) + \text{Im}(\alpha^2 \bar{T}_{cl}) \right] \]

\[ = -\frac{1}{3V} \left[ \text{Im} \left( \frac{\bar{T}_{al} + \bar{T}_{1l} + \bar{T}_{2l}}{\sqrt{3}} \right) - \alpha \frac{\bar{T}_{0l} + \alpha \bar{T}_{1l} + \alpha^2 \bar{T}_{2l}}{\sqrt{3}} + \alpha^2 \frac{\bar{T}_{0l} + \alpha \bar{T}_{1l} + \alpha^2 \bar{T}_{2l}}{\sqrt{3}} \right] \]

Since \( T_{0l} = 0 \)

\[ B_{ca}^\gamma = -\frac{1}{3\sqrt{3}V} \text{Im}(\bar{T}_{1l} - 2\alpha^2 \bar{T}_{2l}) \]

\[ = -\frac{1}{3\sqrt{3}V} \text{Im} \left[ \bar{T}_{1l} - 2 \left( -\frac{1}{2} - j\frac{\sqrt{3}}{2} \right) \bar{T}_{2l} \right] \]

\[ = -\frac{1}{3\sqrt{3}V} \text{Im} \left[ \bar{T}_{1l} + \bar{T}_{2l} + j\sqrt{3} \bar{T}_{2l} \right] \]

\[ = -\frac{1}{3\sqrt{3}V} \left[ \text{Im}(\bar{T}_{1l}) + \text{Im}(\bar{T}_{2l}) + \sqrt{3} \text{Re}(\bar{T}_{2l}) \right] \]

Note that \( \text{Im}(j\bar{T}_{2l}) = \text{Re}(\bar{T}_{2l}) \). Adding \( \beta \) term, we get the following.

\[ B_{ca}^\gamma = -\frac{1}{3\sqrt{3}V} \left[ \text{Im}(\bar{T}_{1l}) + \text{Im}(\bar{T}_{2l}) + \sqrt{3} \text{Re}(\bar{T}_{2l}) - \beta \text{Re}(\bar{T}_{1l}) \right] \]

Similarly,

\[ B_{ab}^\gamma = -\frac{1}{3V} \left[ \text{Im}(\bar{T}_{al}) + \text{Im}(\alpha \bar{T}_{bl}) - \text{Im}(\alpha^2 \bar{T}_{cl}) \right] \]

\[ = -\frac{1}{3V} \left[ \text{Im} \left( \frac{\bar{T}_{al} + \bar{T}_{1l} + \bar{T}_{2l}}{\sqrt{3}} \right) + \alpha \frac{\bar{T}_{0l} + \alpha \bar{T}_{1l} + \alpha^2 \bar{T}_{2l}}{\sqrt{3}} - \alpha^2 \frac{\bar{T}_{0l} + \alpha \bar{T}_{1l} + \alpha^2 \bar{T}_{2l}}{\sqrt{3}} \right] \]
\[ B_{ab}^{\gamma} = -\frac{1}{3\sqrt{3}V} \text{Im} \left[ I_{1l} - 2\alpha I_{2l} \right] \]
\[ = -\frac{1}{3\sqrt{3}V} \text{Im} \left[ I_{1l} - 2 \left( -\frac{1}{2} + j\frac{\sqrt{3}}{2} \right) I_{2l} \right] \]
\[ = -\frac{1}{3\sqrt{3}V} \text{Im} \left[ I_{1l} + I_{2l} - j\sqrt{3}I_{2l} \right] \]
\[ = -\frac{1}{3\sqrt{3}V} \left[ \text{Im} \left( I_{1l} \right) + \text{Im} \left( I_{2l} \right) - \sqrt{3}\text{Re} \left( I_{2l} \right) \right] \]

Thus, Compensator susceptances are expressed as following.

\[ B_{ab}^{\gamma} = -\frac{1}{3V} \left[ \text{Im} \left( I_{al} \right) + \text{Im} \left( \alpha I_{bl} \right) - \text{Im} \left( \alpha^2 I_{cl} \right) - \frac{\beta}{3} \text{Re} \left( I_{al} + \alpha I_{bl} + \alpha^2 I_{cl} \right) \right] \]
\[ B_{bc}^{\gamma} = -\frac{1}{3V} \left[ -\text{Im} \left( I_{al} \right) + \text{Im} \left( \alpha I_{bl} \right) + \text{Im} \left( \alpha^2 I_{cl} \right) - \frac{\beta}{3} \text{Re} \left( I_{al} + \alpha I_{bl} + \alpha^2 I_{cl} \right) \right] \]
\[ B_{ca}^{\gamma} = -\frac{1}{3V} \left[ \text{Im} \left( I_{al} \right) - \text{Im} \left( \alpha I_{bl} \right) + \text{Im} \left( \alpha^2 I_{cl} \right) - \frac{\beta}{3} \text{Re} \left( I_{al} + \alpha I_{bl} + \alpha^2 I_{cl} \right) \right] \]

(3.97)

An unity power factor is desired from the source. For this \( \cos \phi_l = 1 \), implying \( \tan \phi_l = 0 \) hence \( \beta = 0 \). Thus we have,

\[ B_{ab}^{\gamma} = -\frac{1}{3V} \left[ \text{Im} \left( I_{al} \right) + \text{Im} \left( \alpha I_{bl} \right) - \text{Im} \left( \alpha^2 I_{cl} \right) \right] \]
\[ B_{bc}^{\gamma} = -\frac{1}{3V} \left[ -\text{Im} \left( I_{al} \right) + \text{Im} \left( \alpha I_{bl} \right) + \text{Im} \left( \alpha^2 I_{cl} \right) \right] \]
\[ B_{ca}^{\gamma} = -\frac{1}{3V} \left[ \text{Im} \left( I_{al} \right) - \text{Im} \left( \alpha I_{bl} \right) + \text{Im} \left( \alpha^2 I_{cl} \right) \right] \]

(3.98)

The above equations are easy to realize in order to find compensator susceptances. As mentioned above, sampling and averaging techniques will be used to convert above equation into their time equivalents. These are described below.

### 3.5.1 Sampling Method

Each current phasor in above equation can be expressed as,

\[ \bar{I}_{al} = \text{Re} \left( I_{al} \right) + j\text{Im} \left( I_{al} \right) \]
\[ = I_{al,R} + jI_{al,X} \]

(3.99)
An instantaneous phase current is written as follows.

\[ i_{al}(t) = \sqrt{2} I_{al} \text{Im} (e^{j\omega t}) \]

\[ = \sqrt{2} \text{Im} (I_{al} e^{j\omega t}) \]

\[ = \sqrt{2} \text{Im} [(I_{al,R} + jI_{al,X}) e^{j\omega t}] \]

\[ = \sqrt{2} \text{Im} [(I_{al,R} + jI_{al,X})(\cos \omega t + j \sin \omega t)] \]

\[ = \sqrt{2} \text{Im} [(I_{al,R} \cos \omega t - I_{al,X} \sin \omega t) + j(I_{al,R} \sin \omega t + I_{al,X} \cos \omega t)] \]

\[ = \sqrt{2} [(I_{al,R} \sin \omega t + I_{al,X} \cos \omega t)] \quad (3.100) \]

\[ \text{Im} (I_{al}) = \sqrt{2} I_{al} \quad \text{at} \sin \omega t = 0, \cos \omega t = 1 \quad (3.101) \]

From equation (3.63), the phase voltages can be expressed as below.

\[ v_{a}(t) = \sqrt{2} V \sin \omega t \]

\[ v_{b}(t) = \sqrt{2} V \sin(\omega t - 120^\circ) \quad (3.102) \]

\[ v_{c}(t) = \sqrt{2} V \sin(\omega t + 120^\circ) \]

From above voltage expressions, it is to be noted that, \( \sin \omega t = 0, \cos \omega t = 1 \) implies that the phase-\( a \) voltage, \( v_{a}(t) \) is going through a positive zero crossing, hence, \( v_{a}(t) = 0 \) and \( \frac{dv_{a}}{dt} = 0 \). Therefore, equation (3.101), can be expressed as following.

\[ I_{a,l} = \frac{i_{al}(t)}{\sqrt{2}} \text{ when, } v_{a}(t) = 0, \frac{dv_{a}}{dt} > 0 \quad (3.103) \]

Similarly,

\[ I_{b,l} = I_{bl,R} + jI_{bl,X} \quad (3.104) \]

Therefore,

\[ \alpha (I_{b,l}) = \alpha (I_{bl,R} + jI_{bl,X}) \]

\[ = \left( -\frac{1}{2} + j \frac{\sqrt{3}}{2} \right) (I_{bl,R} + jI_{bl,X}) \]

\[ = \left( -\frac{1}{2} I_{bl,R} - \frac{\sqrt{3}}{2} I_{bl,X} \right) + j \left( \frac{\sqrt{3}}{2} I_{bl,R} - \frac{1}{2} I_{bl,X} \right) \quad (3.105) \]

From the above,

\[ \text{Im} \{ \alpha (I_{b}) \} = \frac{\sqrt{3}}{2} I_{bl,R} - \frac{1}{2} I_{bl,X} \quad (3.106) \]
Similar to equation (3.100), we can express phase-b current in terms $\text{Im}(\alpha I_{bl})$, as given below.

\[
i_{bl}(t) = \sqrt{2} \text{Im}(I_{bl} e^{j\omega t})
= \sqrt{2} \text{Im}(\alpha I_{bl} e^{j\omega t} \alpha^{-1})
= \sqrt{2} \text{Im}(\alpha I_{bl} e^{j(\omega t - 120^\circ)})
= \sqrt{2} \text{Im} \left[ \left( -\frac{1}{2} + j\frac{\sqrt{3}}{2} \right) (I_{bl,R} + jI_{bl,X}) e^{j(\omega t - 120^\circ)} \right]
= \sqrt{2} \text{Im} \left[ \left( -\frac{1}{2} I_{bl,R} - \frac{\sqrt{3}}{2} I_{bl,X} \right) + j \left( \frac{\sqrt{3}}{2} I_{bl,R} - \frac{1}{2} I_{bl,X} \right) \right]
\{(\cos(\omega t - 120^\circ) + j \sin(\omega t - 120^\circ))\} \right]
= \sqrt{2} \left[ -\frac{1}{2} I_{bl,R} - \frac{\sqrt{3}}{2} I_{bl,X} \right] \sin(\omega t - 120^\circ) + \left( \frac{\sqrt{3}}{2} I_{bl,R} - \frac{1}{2} I_{bl,X} \right) \cos(\omega t - 120^\circ) \right]
\]

(3.107)

From the above equation, we get the following.

\[\text{Im}(\alpha I_{bl}) = \frac{i_{bl}(t)}{\sqrt{2}} \text{ when, } v_b(t) = 0, dv_b/dt > 0 \quad (3.108)\]

Similarly for Phase-c, it can proved that,

\[\text{Im}(\alpha^2 I_{cl}) = \frac{i_{cl}(t)}{\sqrt{2}} \text{ when, } v_c(t) = 0, dv_c/dt > 0 \quad (3.109)\]

Substituting $\text{Im}(I_{al})$, $\text{Im}(\alpha I_{bl})$ and $\text{Im}(\alpha^2 I_{cl})$ from (3.103), (3.108) and (3.109) respectively, in (3.98), we get the following.

\[
B_{ar}^{ab} = -\frac{1}{3\sqrt{2}V} \left[ i_a \mid (v_a=0, \frac{dv_a}{dt} > 0) + i_b \mid (v_b=0, \frac{dv_b}{dt} > 0) - i_c \mid (v_c=0, \frac{dv_c}{dt} > 0) \right]
B_{ar}^{bc} = -\frac{1}{3\sqrt{2}V} \left[ -i_a \mid (v_a=0, \frac{dv_a}{dt} > 0) + i_b \mid (v_b=0, \frac{dv_b}{dt} > 0) + i_c \mid (v_c=0, \frac{dv_c}{dt} > 0) \right]
B_{ar}^{ca} = -\frac{1}{3\sqrt{2}V} \left[ -i_a \mid (v_a=0, \frac{dv_a}{dt} > 0) - i_b \mid (v_b=0, \frac{dv_b}{dt} > 0) + i_c \mid (v_c=0, \frac{dv_c}{dt} > 0) \right]
\]

(3.110)

Thus the desired compensating susceptances are expressed in terms of the three line currents sampled at instants defined by positive-going zero crossings of the line-neutral voltages $v_a, v_b, v_c$. An artificial neutral at ground potential may be created measuring voltages $v_a, v_b$ and $v_c$ to implement above algorithm.

### 3.5.2 Averaging Method

In this method, we express the compensator susceptances in terms of real and reactive power terms and finally expressed them in time domain through averaging process. The method is described
From equation (3.98), susceptance, $B_{ab}$, can be re-written as following.

$$B_{ab} = -\frac{1}{3V} \left[ \text{Im} (I_{al}) + \text{Im} (\alpha I_{bl}) - \text{Im} (\alpha^2 I_{cl}) \right]$$

$$= -\frac{1}{3V^2} \left[ V \text{Im} (I_{al}) + V \text{Im} (\alpha I_{bl}) - V \text{Im} (\alpha^2 I_{cl}) \right]$$

$$= -\frac{1}{3V^2} \left[ \text{Im} (V I_{al}) + \text{Im} (\alpha V I_{bl}) - \text{Im} (\alpha^2 V I_{cl}) \right]$$

(3.111)

Note the following property of phasors and applying it for the simplification of the above expression.

$$\text{Im} (V I) = -\text{Im} (V^* I^*)$$

(3.112)

Using above equation (3.111) can be written as,

$$B_{ab} = \frac{1}{3V^2} \left[ \text{Im} (V^* I_{al})^* + \text{Im} (\alpha V^* I_{bl})^* - \text{Im} (\alpha^2 V^* I_{cl})^* \right]$$

$$= \frac{1}{3V^2} \left[ \text{Im} (V^* I_{al})^* + \text{Im} (\alpha V^* I_{bl})^* - \text{Im} (\alpha^2 V^* I_{cl})^* \right]$$

$$= \frac{1}{3V^2} \left[ \text{Im} (V^* I_{al})^* + \text{Im} (\alpha V^* I_{bl})^* - \text{Im} (\alpha V^* I_{cl})^* \right]$$

(3.113)

Since $V_a = V \angle 0^\circ$ is a reference phasor, therefore $V_a = V_b^* = V$, $\alpha^2 V_a = V_b^*$ and $\alpha V_a = V_c^*$. Using this, the above equation can be written as following.

$$B_{ab} = \frac{1}{3V^2} \left[ \text{Im} (V_a I_{al})^* + \text{Im} (V_b I_{bl})^* - \text{Im} (V_c I_{cl})^* \right]$$

Similarly, $B_{bc} = \frac{1}{3V^2} \left[ -\text{Im} (V_a I_{al})^* + \text{Im} (V_b I_{bl})^* + \text{Im} (V_c I_{cl})^* \right]$ (3.114)

$$B_{ca} = \frac{1}{3V^2} \left[ \text{Im} (V_a I_{al})^* - \text{Im} (V_b I_{bl}) + \text{Im} (V_c I_{cl})^* \right]$$

It can be further proved that,

$$\text{Im} (V_a I_{al})^* = \frac{1}{T} \int_0^T v_a(t) \angle (-\pi/2) i_{al}(t) \, dt$$

$$\text{Im} (V_b I_{bl})^* = \frac{1}{T} \int_0^T v_b(t) \angle (-\pi/2) i_{bl}(t) \, dt$$

$$\text{Im} (V_c I_{cl})^* = \frac{1}{T} \int_0^T v_c(t) \angle (-\pi/2) i_{cl}(t) \, dt$$
Since,
\[ v_a(t)\angle(-\pi/2) = \frac{v_{bc}(t)}{\sqrt{3}} \]
\[ v_b(t)\angle(-\pi/2) = \frac{v_{ca}(t)}{\sqrt{3}} \]
\[ v_c(t)\angle(-\pi/2) = \frac{v_{ca}(t)}{\sqrt{3}} \]

Equation (3.115) can be written as,
\[
\begin{align*}
\text{Im}(V_a T_a^*) &= \frac{1}{\sqrt{3}T} \int_0^T v_{bc}(t) i_{al}(t) dt \\
\text{Im}(V_b T_b^*) &= \frac{1}{\sqrt{3}T} \int_0^T v_{ca}(t) i_{bl}(t) dt \\
\text{Im}(V_c T_c^*) &= \frac{1}{\sqrt{3}T} \int_0^T v_{ab}(t) i_{cl}(t) dt
\end{align*}
\] (3.115)

Substituting above values of \( \text{Im}(V_a T_a^*) \), \( \text{Im}(V_b T_b^*) \) and \( \text{Im}(V_c T_c^*) \) into (3.113), we get the following.

\[
\begin{align*}
B_{\gamma}^{ab} &= \frac{1}{(3\sqrt{3}V^2)} \frac{1}{T} \int_0^T (v_{bc} i_{al} + v_{ca} i_{bl} - v_{ab} i_{cl}) dt \\
B_{\gamma}^{bc} &= \frac{1}{(3\sqrt{3}V^2)} \frac{1}{T} \int_0^T (-v_{bc} i_{al} + v_{ca} i_{bl} + v_{ab} i_{cl}) dt \\
B_{\gamma}^{ca} &= \frac{1}{(3\sqrt{3}V^2)} \frac{1}{T} \int_0^T (v_{bc} i_{al} + v_{ca} i_{bl} + v_{ab} i_{cl}) dt
\end{align*}
\] (3.116)

The above equations can directly be used to know the compensator susceptances by performing the averaging on line the product of the line to line voltages and phase load currents. The term \( \int_0^T = \int_{t_1}^{t_1+T} \) can be implemented using moving average of one cycle. This improves transient response by computing average value at each instant. But in this case the controller response which changes the susceptance value, should match to that of the above computing algorithm.

### 3.6 Compensator Admittance Represented as Positive and Negative Sequence Admittance Network

Recalling the following relations from equation (3.92) for unity power factor operation i.e. \( \beta = 0 \), we get the following.

\[
\begin{align*}
B_{\gamma}^{ab} &= \frac{-1}{3\sqrt{3}V} [\text{Im}(\bar{T}_{1l}) + \text{Im}(\bar{T}_{2l}) - \sqrt{3} \text{Re}(\bar{T}_{2l})] \\
B_{\gamma}^{bc} &= \frac{-1}{3\sqrt{3}V} [\text{Im}(\bar{T}_{1l}) - \beta \text{Re}(\bar{T}_{1l}) - 2\text{Im}(\bar{T}_{2l})] \\
B_{\gamma}^{ca} &= \frac{-1}{3\sqrt{3}V} [\text{Im}(\bar{T}_{1l}) + \text{Im}(\bar{T}_{2l}) + \sqrt{3} \text{Re}(\bar{T}_{2l})]
\end{align*}
\] (3.117)
From these equations, it is evident the the first terms form the positive sequence susceptance as they involve \( I_{1l} \) terms. Similarly, the second and third terms in above equation form negative sequence susceptance of the compensator, as these involve \( I_{2l} \) terms. Thus, we can write,

\[
B_{\gamma}^{ab} = B_{\gamma 1}^{ab} + B_{\gamma 2}^{ab} \\
B_{\gamma}^{ab} = B_{\gamma 1}^{ab} + B_{\gamma 2}^{ab} \\
B_{\gamma}^{ab} = B_{\gamma 1}^{ab} + B_{\gamma 2}^{ab}
\]

(3.118)

Therefore,

\[
B_{\gamma 1}^{ab} = B_{\gamma 1}^{bc} = B_{\gamma 1}^{ca} = -\frac{1}{3\sqrt{3}V} \left[ \text{Im}(I_{1l}) \right]
\]

(3.119)

And,

\[
B_{\gamma 2}^{ab} = -\frac{1}{3\sqrt{3}V} \left( \text{Im}(I_{2l}) - \sqrt{3} \text{Re}(I_{2l}) \right) \\
B_{\gamma 2}^{bc} = -\frac{1}{3\sqrt{3}V} \left( -2 \text{Im}(I_{2l}) \right) \\
B_{\gamma 2}^{ca} = -\frac{1}{3\sqrt{3}V} \left( \text{Im}(I_{2l}) + \sqrt{3} \text{Re}(I_{2l}) \right)
\]

(3.120)

Earlier in equation, (3.121), it was established that,

\[
\begin{align*}
I_{0l} &= 0 \\
I_{1l} &= (Y_{l}^{ab} + Y_{l}^{bc} + Y_{l}^{ca}) \sqrt{3}V \\
I_{2l} &= -\left( \alpha^2 Y_{l}^{ab} + Y_{l}^{bc} + \alpha Y_{l}^{ca} \right) \sqrt{3}V
\end{align*}
\]

Noting that,

\[
\begin{align*}
Y_{l}^{ab} &= G_{l}^{ab} + jB_{l}^{ab} \\
Y_{l}^{bc} &= G_{l}^{bc} + jB_{l}^{bc} \\
Y_{l}^{ca} &= G_{l}^{ca} + jB_{l}^{ca}
\end{align*}
\]

Therefore,

\[
\text{Im}(I_{1l}) = \text{Im} \left( (Y_{l}^{ab} + Y_{l}^{bc} + Y_{l}^{ca}) \sqrt{3}V \right) = \left( B_{l}^{ab} + B_{l}^{bc} + B_{l}^{ca} \right) \sqrt{3}V
\]

(3.121)

Thus equation (3.119) is re-written as following.

\[
B_{\gamma 1}^{ab} = B_{\gamma 1}^{bc} = B_{\gamma 1}^{ca} = -\frac{1}{3} \left( B_{l}^{ab} + B_{l}^{bc} + B_{l}^{ca} \right)
\]

(3.122)

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Now we shall compute $B^{ab}_{\gamma_2}$, $B^{bc}_{\gamma_2}$ and $B^{ca}_{\gamma_2}$ using equations (3.120) as following. Knowing that,

$$I_{2l} = -\left(\alpha^2 Y^{ab}_l + Y^{bc}_l + \alpha Y^{ca}_l\right) \sqrt{3} V$$

$$= -\left[\left(-\frac{1}{2} - \frac{j\sqrt{3}}{2}\right) (G^{ab}_l + jB^{ab}_l) + (G^{bc}_l + jB^{bc}_l) + \left(-\frac{1}{2} + \frac{j\sqrt{3}}{2}\right) (G^{ca}_l + jB^{ca}_l)\right] \sqrt{3} V$$

$$= -\left[-\frac{G^{ab}_l}{2} + \frac{\sqrt{3}}{2} B^{ab}_l + G^{bc}_l - \frac{\sqrt{3}}{2} B^{ca}_l - j \left(\frac{\sqrt{3}}{2} G^{ab}_l + \frac{B^{ab}_l}{2} - B^{bc}_l - \frac{\sqrt{3}}{2} G^{ca}_l + \frac{B^{ca}_l}{2}\right)\right] \sqrt{3} V$$

$$= \left[\frac{G^{ab}_l}{2} - \frac{\sqrt{3}}{2} B^{ab}_l - G^{bc}_l + \frac{G^{ca}_l}{2} + \frac{\sqrt{3}}{2} B^{ca}_l + j \left(\frac{\sqrt{3}}{2} G^{ab}_l + \frac{B^{ab}_l}{2} - B^{bc}_l - \frac{\sqrt{3}}{2} G^{ca}_l + \frac{B^{ca}_l}{2}\right)\right] \sqrt{3} V$$

(3.123)

The above implies that,

$$\text{Im}(I_{2l}) = \left(\frac{\sqrt{3}}{2} G^{ab}_l + \frac{B^{ab}_l}{2} - B^{bc}_l - \frac{\sqrt{3}}{2} G^{ca}_l + \frac{B^{ca}_l}{2}\right) \sqrt{3} V$$

$$- \sqrt{3} \text{Re}(I_{2l}) = \left(-\frac{\sqrt{3}}{2} G^{ab}_l + \frac{3}{2} B^{ab}_l + \sqrt{3} G^{bc}_l - \frac{\sqrt{3}}{2} G^{ca}_l - \frac{3}{2} B^{ca}_l\right) \sqrt{3} V$$

Thus, $B^{ab}_{\gamma_2}$ can be given as,

$$B^{ab}_{\gamma_2} = -\frac{1}{3\sqrt{3} V} \left[2B^{ab}_l - B^{bc}_l - B^{ca}_l + \sqrt{3} G^{bc}_l - \sqrt{3} G^{ca}_l\right] \sqrt{3} V$$

$$= -\frac{1}{3} \left[2B^{ab}_l - B^{bc}_l - B^{ca}_l + \sqrt{3} (G^{bc}_l - G^{ca}_l)\right]$$

$$= \frac{1}{\sqrt{3}} (G^{bc}_l - G^{ca}_l) + \frac{1}{3} (B^{bc}_l + B^{ca}_l - 2B^{ab}_l)$$

(3.124)

Similarly,

$$B^{ca}_{\gamma_2} = -\frac{1}{3\sqrt{3} V} \left[\text{Im}(I_{2l}) + \sqrt{3} \text{Re}(I_{2l})\right]$$

$$= -\frac{1}{3\sqrt{3} V} \left[\frac{\sqrt{3}}{2} G^{ab}_l + \frac{B^{ab}_l}{2} - B^{bc}_l - \frac{\sqrt{3}}{2} G^{ca}_l + \frac{B^{ca}_l}{2} + \sqrt{3} G^{ab}_l - \frac{3}{2} B^{ab}_l - \sqrt{3} G^{bc}_l + \frac{3}{2} G^{ca}_l + \frac{3}{2} B^{ca}_l\right] \sqrt{3} V$$

$$= -\frac{1}{3} \left[\sqrt{3} (G^{ab}_l - G^{ca}_l) - \sqrt{3} G^{bc}_l - B^{ab}_l - B^{ca}_l + 2B^{ca}_l\right]$$

$$= \frac{1}{\sqrt{3}} (G^{bc}_l - G^{ab}_l) + \frac{1}{3} (B^{bc}_l + B^{ca}_l - 2B^{ab}_l)$$

(3.125)

And, $B^{bc}_{\gamma_2}$ is computed as below.
\[ B_{\gamma 2}^{bc} = -\frac{1}{3\sqrt{3}V} \left[ -2\text{Im}(\hat{I}_{2l}) \right] \]
\[ = \frac{2}{3\sqrt{3}V} \left( \frac{\sqrt{3}}{2} G_{i}^{ab} + \frac{B_{i}^{ab} - \sqrt{3}}{2} G_{i}^{ca} + \frac{B_{i}^{ca}}{2} \right) V \sqrt{3} \]
\[ = \frac{1}{\sqrt{3}} (G_{i}^{ab} - G_{i}^{ca}) + \frac{1}{3} (B_{i}^{ab} + B_{i}^{ca} - 2B_{i}^{bc}) \]  

(3.126)

Using (3.119), (3.124)-(3.126), We therefore can find the overall compensator susceptances as following.

\[ B_{\gamma}^{ab} = B_{\gamma 1}^{ab} + B_{\gamma 2}^{ab} \]
\[ = -\frac{1}{3} (B_{\gamma}^{ab} + B_{\gamma}^{bc} + B_{\gamma}^{ca}) + \frac{1}{\sqrt{3}} (G_{i}^{ab} - G_{i}^{bc}) + \frac{1}{3} (B_{i}^{ab} + B_{i}^{ca} - 2B_{i}^{bc}) \]
\[ = -B_{i}^{ab} + \frac{1}{\sqrt{3}} (G_{i}^{ac} - G_{i}^{bc}) \]

Similarly,

\[ B_{\gamma}^{bc} = B_{\gamma 1}^{bc} + B_{\gamma 2}^{bc} \]
\[ = -\frac{1}{3} (B_{\gamma}^{ab} + B_{\gamma}^{bc} + B_{\gamma}^{ca}) + \frac{1}{\sqrt{3}} (G_{i}^{bc} - G_{i}^{ab}) + \frac{1}{3} (B_{i}^{ab} + B_{i}^{ca} - 2B_{i}^{bc}) \]
\[ = -B_{i}^{bc} + \frac{1}{\sqrt{3}} (G_{i}^{ab} - G_{i}^{bc}) \]

And,

\[ B_{\gamma}^{ca} = B_{\gamma 1}^{ca} + B_{\gamma 2}^{ca} \]
\[ = -\frac{1}{3} (B_{\gamma}^{ab} + B_{\gamma}^{bc} + B_{\gamma}^{ca}) + \frac{1}{\sqrt{3}} (G_{i}^{bc} - G_{i}^{ab}) + \frac{1}{3} (B_{i}^{bc} + B_{i}^{ab} - 2B_{i}^{ca}) \]
\[ = -B_{i}^{ca} + \frac{1}{\sqrt{3}} (G_{i}^{bc} - G_{i}^{ab}) \]

Thus, the compensator susceptances in terms of load parameters are given as in the following.

\[ B_{\gamma}^{ab} = -B_{i}^{ab} + \frac{1}{\sqrt{3}} (G_{i}^{ca} - G_{i}^{bc}) \]
\[ B_{\gamma}^{bc} = -B_{i}^{bc} + \frac{1}{\sqrt{3}} (G_{i}^{ab} - G_{i}^{ca}) \]
\[ B_{\gamma}^{ca} = -B_{i}^{ca} + \frac{1}{\sqrt{3}} (G_{i}^{bc} - G_{i}^{ab}) \]  

(3.127)
It is interesting to observe above equations. The first parts of the equation nullifies the effect of the load susceptances and the second parts of the equations correspond to the unbalance in resistive load. The two terms together make source current balanced and in phase with the supply voltages. The compensator’s positive and negative sequence networks are shown in Fig. 3.13.

What happens if we just use the following values of the compensator susceptances as given below?

\[
\begin{align*}
B_{\gamma}^{ab} &= -B_{l}^{ab} \\
B_{\gamma}^{be} &= -B_{l}^{be} \\
B_{\gamma}^{ca} &= -B_{l}^{ca}
\end{align*}
\] (3.128)

In above case, load susceptance parts of the admittance are fully compensated. However the source currents after compensation remain unbalanced due to unbalance conductance parts of the load.

Example 3.5 For a delta connected load shown in 3.14, the load admittances are given as following,

\[
\begin{align*}
Y_{l}^{ab} &= G_{l}^{ab} + jB_{l}^{ab} \\
Y_{l}^{be} &= G_{l}^{be} + jB_{l}^{be} \\
Y_{l}^{ca} &= G_{l}^{ca} + jB_{l}^{ca}
\end{align*}
\]

Given the load parameters:

\[
\begin{align*}
Z_{l}^{ab} &= 1/Y_{l}^{ab} = 5 + j12 \Omega \\
Z_{l}^{be} &= 1/Y_{l}^{be} = 3 + j4 \Omega \\
Z_{l}^{ca} &= 1/Y_{l}^{ca} = 9 - j12 \Omega
\end{align*}
\]

Determine compensator susceptances \((B_{\gamma}^{ab}, B_{\gamma}^{be}, B_{\gamma}^{ca})\) so that the supply sees the load as balanced and unity power factor. Also find the line currents and source active and reactive powers before and after compensation.
Fig. 3.14 A delta connected load

Solution:

\[ Z_{ab}^l = 5 + j12 \, \Omega \Rightarrow Y_{ab}^l = 0.03 - j0.0710 \, \Omega \]
\[ Z_{bc}^l = 3 + j4 \, \Omega \Rightarrow Y_{bc}^l = 0.12 - j0.16 \, \Omega \]
\[ Z_{ca}^l = 9 - j13 \, \Omega \Rightarrow Y_{ca}^l = 0.04 + j0.0533 \, \Omega \]

Once we know the admittances we know,

\[ G_{ab}^l = 0.03, \quad B_{ab}^l = -0.0710 \]
\[ G_{bc}^l = 0.12, \quad B_{bc}^l = -0.16 \]
\[ G_{ca}^l = 0.04, \quad B_{ca}^l = 0.0533 \]

\[ B_{\gamma}^{ab} = -B_{\gamma}^{ab} + \frac{1}{\sqrt{3}}(G_{\gamma}^{ca} - G_{\gamma}^{bc}) = 0.0248 \, \Omega \]
\[ B_{\gamma}^{bc} = -B_{\gamma}^{bc} + \frac{1}{\sqrt{3}}(G_{\gamma}^{ab} - G_{\gamma}^{ca}) = 0.1540 \, \Omega \]
\[ B_{\gamma}^{ca} = -B_{\gamma}^{ca} + \frac{1}{\sqrt{3}}(G_{\gamma}^{bc} - G_{\gamma}^{ab}) = -0.0011 \, \Omega \]

Total admittances are:

\[ Y_{ab}^\gamma = Y_{ab}^l + Y_{ab}^\gamma = 0.03 - j0.0462 \, \Omega \]
\[ Y_{bc}^\gamma = Y_{bc}^l + Y_{bc}^\gamma = 0.12 - j0.006 \, \Omega \]
\[ Y_{ca}^\gamma = Y_{ca}^l + Y_{ca}^\gamma = 0.04 + j0.0522 \, \Omega \]

Knowing these total admittances, we can find line currents using following expressions.

Current Before Compensation
\[ T_a = T_{ab} - T_{ca} = [(1 - \alpha^2)Y^{ab} - (\alpha - 1)Y^{ca}] \quad V = 0.2150 V \angle -9.51^\circ A \]
\[ T_b = T_{bc} - T_{ab} = [(\alpha^2 - \alpha)Y^{bc} - (1 - \alpha^2)Y^{ab}] \quad V = 0.4035 V \angle -161.66^\circ A \]
\[ T_c = T_{ca} - T_{bc} = [(\alpha - 1)Y^{ca} - (\alpha^2 - \alpha)Y^{bc}] \quad V = 0.2358 V \angle 43.54^\circ A \]

**Powers Before Compensation**

\[
\begin{align*}
\bar{S}_a &= \overline{V}_a (\overline{T}_{al})^* = P_a + jQ_a = V (0.2121 + j0.0355) \\
\bar{S}_b &= \overline{V}_b (\overline{T}_{bl})^* = P_b + jQ_b = V (0.3014 + j0.2682) \\
\bar{S}_c &= \overline{V}_c (\overline{T}_{cl})^* = P_c + jQ_c = V (0.0552 + j0.2293) \\
\end{align*}
\]

Total real power, \( P \) = \( P_a + P_b + P_c = V \times 0.5688 \) W

Total reactive power, \( Q \) = \( Q_a + Q_b + Q_c = V \times 0.5330 \) VAr

Power factor in phase-a, \( p_{fa} \) = \( \cos \phi_a = \cos(9.51^\circ) = 0.9863 \) lag

Power factor in phase-b, \( p_{fb} \) = \( \cos \phi_b = \cos(41.63^\circ) = 0.7471 \) lag

Power factor in phase-c, \( p_{fc} \) = \( \cos \phi_c = \cos(76.45^\circ) = 0.2334 \) lag

Thus we observe that the phases draw reactive power from the lines and currents are unbalanced in magnitude and phase angles.

**After Compensation**

\[ T_a = T_{ab} - T_{ca} = [(1 - \alpha^2)Y^{ab} - (\alpha - 1)Y^{ca}] \quad V = 0.1896 V \angle 0^\circ A \]
\[ T_b = T_{bc} - T_{ab} = [(\alpha^2 - \alpha)Y^{bc} - (1 - \alpha^2)Y^{ab}] \quad V = 0.1896 V \angle -120^\circ A \]
\[ T_a = T_{ca} - T_{bc} = [(\alpha - 1)Y^{ca} - (\alpha^2 - \alpha)Y^{bc}] \quad V = 0.1896 V \angle 120^\circ A \]

**Powers After Compensation**

\[
\begin{align*}
\bar{S}_a &= \overline{V}_a (\overline{T}_{al})^* = P_a + jQ_a = V (0.1986 + j0.0) \\
\bar{S}_b &= \overline{V}_b (\overline{T}_{bl})^* = P_b + jQ_b = V (0.1986 + j0.0) \\
\bar{S}_c &= \overline{V}_c (\overline{T}_{cl})^* = P_c + jQ_c = V (0.1986 + j0.0) \\
\end{align*}
\]

Total real power, \( P \) = \( P_a + P_b + P_c = (V \times 0.5688) \) W

Total reactive power, \( Q \) = \( Q_a + Q_b + Q_c = 0 \) VAr

Power factor in phase-a, \( p_{fa} \) = \( \cos \phi_a = \cos(0^\circ) = 1.0 \)

Power factor in phase-b, \( p_{fb} \) = \( \cos \phi_b = \cos(0^\circ) = 1.0 \)

Power factor in phase-c, \( p_{fc} \) = \( \cos \phi_c = \cos(0^\circ) = 1.0 \)

From above results we observe that after placing compensator of suitable values as calculated above, the line currents become balanced and have unity power factor relationship with their voltages.
Example 3.6 Consider the following 3-phase, 3-wire system. The 3-phase voltages are balanced sinusoids with RMS value of 230 V at 50 Hz. The load impedances are $Z_a = 3 + j 4 \, \Omega$, $Z_b = 5 + j 12 \, \Omega$, $Z_c = 12 - j 5 \, \Omega$. Compute the following.

1. The line currents $I_{la}$, $I_{lb}$, $I_{lc}$.

2. The active ($P$) and reactive ($Q$) powers of each phase.

3. The compensator susceptance ($B_{ab} \gamma$, $B_{bc} \gamma$, $B_{ca} \gamma$), so that the supply sees the load balanced and unity power factor.

4. For case (3), compute the source, load, compensator active and reactive powers (after compensation).

\[ \begin{align*}
\mathcal{P} & = \frac{V_{la} V_{lb} \cos \theta_{ab} + V_{lb} V_{lc} \cos \theta_{bc} + V_{lc} V_{la} \cos \theta_{ca}}{3} \\
\mathcal{Q} & = \frac{V_{la} V_{lb} \sin \theta_{ab} + V_{lb} V_{lc} \sin \theta_{bc} + V_{lc} V_{la} \sin \theta_{ca}}{3} \\
\mathcal{S} & = \sqrt{\mathcal{P}^2 + \mathcal{Q}^2}
\end{align*} \]

\[ \mathcal{P} = 113.4 \, \text{kW}, \quad \mathcal{Q} = 80.75 \, \text{kVAR} \]

Solution:

Given that $Z_a = 3 + j 4 \, \Omega$, $Z_b = 5 + j 12 \, \Omega$, $Z_c = 12 - j 5 \, \Omega$.

1. **Line currents** $I_{la}$, $I_{lb}$, and $I_{lc}$ are found by first computing neutral voltage as given below.

\[
V_{nN} = \frac{1}{\left(\frac{1}{Z_a} + \frac{1}{Z_b} + \frac{1}{Z_c}\right)} \left(\frac{V_a}{Z_a} + \frac{V_b}{Z_b} + \frac{V_c}{Z_c}\right) \\
= \left(\frac{Z_a Z_b Z_c}{Z_a Z_b + Z_b Z_c + Z_c Z_a}\right) \left(\frac{V_a}{Z_a} + \frac{V_b}{Z_b} + \frac{V_c}{Z_c}\right) \\
= Z_{abc} \left(\frac{V_a}{Z_a} + \frac{V_b}{Z_b} + \frac{V_c}{Z_c}\right) \\
= \frac{50 \angle 97.82^\circ}{252.41 \angle 55.5^\circ} \left(24.12 \angle -99.55^\circ\right) \\
= 43.79 - j 67.04 \, \text{V} \\
= 80.75 \angle -57.15^\circ \, \text{V}
\]

Now the line currents are computed as below.
\[ I_{la} = \frac{V_a - V_{nN}}{Z_a} \]
\[ = \frac{230\angle0^\circ - 80.75\angle-57.15^\circ}{3 + j4} \]
\[ = 33.2 - j21.65 \]
\[ = 39.63\angle-33.11^\circ A \]

\[ I_{lb} = \frac{V_b - V_{nN}}{Z_b} \]
\[ = \frac{230\angle-120^\circ - 80.75\angle-57.15^\circ}{5 + j12} \]
\[ = 15.85\angle152.21^\circ A \]

\[ I_{lc} = \frac{V_c - V_{nN}}{Z_c} \]
\[ = \frac{230\angle120^\circ - 80.75\angle-57.15^\circ}{12 - j5} \]
\[ = 23.89\angle143.35^\circ A \]

2. **Active and Reactive Powers**

For phase a

\[ P_a = V_a I_a \cos \phi_a = 230 \times 39.63 \times \cos(33.11^\circ) = 7635.9 \text{ W} \]

\[ Q_a = V_a I_a \sin \phi_a = 230 \times 39.63 \times \sin(33.11^\circ) = 4980 \text{ VAr} \]

For phase b

\[ P_b = V_b I_b \cos \phi_b = 230 \times 15.85 \times \cos(-152.21^\circ - 120^\circ) = 140.92 \text{ W} \]

\[ Q_b = V_b I_b \sin \phi_b = 230 \times 15.85 \times \sin(-152.21^\circ - 120^\circ) = 3643.2 \text{ VAr} \]

For phase c

\[ P_c = V_c I_c \cos \phi_c = 230 \times 23.89 \times \cos(-143.35^\circ + 120^\circ) = 5046.7 \text{ W} \]
\[ Q_c = V_c I_c \sin \phi_c = 230 \times 23.89 \times \sin(-143.35^o + 120^o) = -2179.3 \text{ VAr} \]

**Total three phase powers**

\[ P = P_a + P_b + P_c = 12823 \text{ W} \]

\[ Q = Q_a + Q_b + Q_c = 6443.8 \text{ VAr} \]

**3. Compensator Susceptance**

First we convert star connected load to a delta load as given below.

\[ Z_{ab}^l = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_c} = 4 + j 19 = 19.42 \angle 77.11^o \Omega \]

\[ Z_{bc}^l = \frac{\Delta Z}{Z_a} = 50.44 + j 2.08 = 50.42 \angle 2.36^o \Omega \]

\[ Z_{ca}^l = \frac{\Delta Z}{Z_b} = 19.0 - j 4.0 = 19.42 \angle -11.89^o \Omega \]

The above implies that,

\[ Y_{ab}^l = 1/Z_{ab}^l = G_{ab}^l + j B_{ab}^l = 0.0106 - j0.050 \Omega \]

\[ Y_{bc}^l = 1/Z_{bc}^l = G_{bc}^l + j B_{bc}^l = 0.0198 - j0.0008 \Omega \]

\[ Y_{ca}^l = 1/Z_{ca}^l = G_{ca}^l + j B_{ca}^l = 0.0504 + j0.0106 \Omega \]

From the above, the compensator susceptances are computed as following.

\[ B_{ab}^\gamma = -B_{ab}^l + \frac{(G_{ca}^l - G_{bc}^l)}{\sqrt{3}} = 0.0681 \Omega \]

\[ B_{bc}^\gamma = -B_{bc}^l + \frac{(G_{ab}^l - G_{ca}^l)}{\sqrt{3}} = -0.0222 \Omega \]

\[ B_{ca}^\gamma = -B_{ca}^l + \frac{(G_{bc}^l - G_{ab}^l)}{\sqrt{3}} = -0.0053 \Omega \]

\[ Z_{ab}^\gamma = -j 14.69 \Omega \text{ (capacitance)} \]

\[ Z_{bc}^\gamma = j 45.13 \Omega \text{ (inductance)} \]

\[ Z_{ca}^\gamma = j 188.36 \Omega \text{ (inductance)} \]
4. After Compensation

\[ Z_{i}^{bl} = Z_{i}^{bl} || Z_{y}^{ab} = 24.97 - j 41.59 = 48.52\angle - 59.01^\circ \Omega \]

\[ Z_{i}^{bc} = Z_{i}^{bc} || Z_{y}^{bc} = 21.52 - j 24.98 = 32.97\angle 49.25^\circ \Omega \]

\[ Z_{i}^{ca} = Z_{i}^{ca} || Z_{y}^{ca} = 24.97 - j 41.59 = 19.7332\angle - 6.0^\circ \Omega \]

Let us convert delta connected impedances to star connected.

\[ Z_a = \frac{Z_{ab'} \times Z_{ca'}}{Z_{ab'} + Z_{bc'} + Z_{ca'}} = \frac{9.0947 - j 10.55}{13.93\angle - 49.25^\circ \Omega} \]

\[ Z_b = \frac{Z_{bc'} \times Z_{ab'}}{Z_{ab'} + Z_{bc'} + Z_{ca'}} = \frac{23.15 + j 2.43}{23.28\angle 6.06^\circ \Omega} \]

\[ Z_c = \frac{Z_{ca'} \times Z_{bc'}}{Z_{ab'} + Z_{bc'} + Z_{ca'}} = \frac{4.8755 + j 8.12}{9.47\angle 59.01^\circ \Omega} \]

The new voltage between the load and system neutral after compensation is given by,

\[ V_{nN}' = \frac{1}{\left(\frac{1}{Z_a} + \frac{1}{Z_b} + \frac{1}{Z_c}\right)} \left(\frac{V_a}{Z_a} + \frac{V_b}{Z_b} + \frac{V_c}{Z_c}\right) = 205.42\angle -57.15^\circ V \]

Based on the above, the line currents are computed as following.

\[ I_a = \frac{V_a - V_{nN}'}{Z_a} = 18.58\angle 0^\circ A \]

\[ I_b = \frac{V_b - V_{nN}'}{Z_b} = 18.58\angle -120^\circ A \]

\[ I_c = \frac{V_c - V_{nN}'}{Z_c} = 18.58\angle 120^\circ A \]
Thus, it is seen that after compensation, the source currents are balanced and have unity power factor with respective supply voltages.

Source powers after compensation

\[ P_a = P_b = P_c = 230 \times 18.58 = 4272.14 \text{ W} \]

\[ P = 3P_a = 12820.2 \text{ W} \]

\[ Q_a = Q_b = Q_c = 0 \]

\[ Q = 0 \text{ VAr} \]

Compensator powers

\[
\overline{S}^{ab}_\gamma = \bar{V}_{ab}(\bar{T}^{ab*}_\gamma) \\
= \bar{V}_{ab}(\bar{V}^{*}_{ab} \cdot Y^{ab*}_{\gamma}) \\
= \bar{V}_{ab}^2 Y_{\gamma}^{ab*} \\
= (230 \times \sqrt{3})^2 \times (-j \ 0.0068) \\
= -j \ 10802 \text{ VA}
\]

\[
\overline{S}^{bc}_\gamma = \bar{V}_{bc}^2 Y_{\gamma}^{bc*} \\
= (230 \times \sqrt{3})^2 \times (j \ 0.0222) \\
= j \ 3516 \text{ VA}
\]

\[
\overline{S}^{ca}_\gamma = \bar{V}_{ca}^2 Y_{\gamma}^{ca*} \\
= (230 \times \sqrt{3})^2 \times (j \ 0.0053) \\
= j \ 842 \text{ VA}
\]

References


Chapter 4

CONTROL THEORIES FOR LOAD COMPENSATION
(Lectures 25-35)

4.1 Introduction

In the previous chapter, we studied the methods of load compensation. These methods can eliminate only the fundamental reactive power and unbalance in the steady state. These kinds of compensators can be realized using passive LC filters and thyristor controlled devices. However, when harmonics are present in the system, these methods fail to provide correct compensation. To correct load with unbalance and harmonics, instantaneous load compensation methods are used. The two important theories in this context are Instantaneous Theory of load compensation often known as \(pq\) theory [1] and Instantaneous Symmetrical Component Theory for load compensation [2]. These theories will be discussed in this chapter. Their merits and demerits and applications will be explored in detail.

To begin with \(pq\) theory, we shall first recall the \(a\beta0\) transformation, which was discussed in Chapter 2. For three-phase system shown in Fig. 4.1, the \(a\beta0\) transformations for voltages and currents are given below.

![Fig. 4.1 A three phase system](image-url)
\[
\begin{bmatrix}
v_0 \\
v_{\alpha} \\
v_{\beta}
\end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
1 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
\] (4.1)

\[
\begin{bmatrix}
i_0 \\
i_{\alpha} \\
i_{\beta}
\end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
1 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{bmatrix}
\begin{bmatrix}
i_a \\
i_b \\
i_c
\end{bmatrix}
\] (4.2)

The instantaneous active, \( p(t) \) and reactive, \( q(t) \) powers were defined in Chapter 2 through equations (2.14)-(2.15) respectively. For the sake of completeness these are given below.

\[
p_{3\phi}(t) = v_{\alpha}i_{\alpha} + v_{\beta}i_{\beta} + v_0i_0
\]

\[
= v_{\alpha}i_{\alpha} + v_{\beta}v_{\beta} + v_0i_0
\]

\[
= p_{\alpha} + p_{\beta} + p_0
\]

\[
= p_{\alpha\beta} + p_0
\] (4.3)

Where, \( p_{\alpha\beta} = p_{\alpha} + p_{\beta} = v_{\alpha}i_{\alpha} + v_{\beta}i_{\beta} \) and \( p_0 = v_0i_0 \).

In the instantaneous reactive power theory, as discussed in Chapter 2, the instantaneous reactive power, \( q(t) \) was defined as,

\[
q(t) = q_{\alpha\beta} = v_{\alpha} \times i_{\beta} + v_{\beta} \times i_{\alpha}
\]

\[
= v_{\alpha}i_{\beta} - v_{\beta}i_{\alpha}
\]

\[
= -\frac{1}{\sqrt{3}} [v_{\alpha}i_{\alpha} + v_{\beta}i_{\beta} + v_{ab}i_{c}]
\] (4.4)

Therefore, powers \( p_0, p_{\alpha\beta} \) and \( q_{\alpha\beta} \) can be expressed in matrix form as given below.

\[
\begin{bmatrix}
p_0 \\
p_{\alpha\beta} \\
q_{\alpha\beta}
\end{bmatrix} =
\begin{bmatrix}
v_0 & 0 & 0 \\
0 & v_{\alpha} & v_{\beta} \\
0 & -v_{\beta} & v_{\alpha}
\end{bmatrix}
\begin{bmatrix}
i_0 \\
i_{\alpha} \\
i_{\beta}
\end{bmatrix}
\] (4.5)

From the above equation, the currents, \( i_0, i_{\alpha} \) and \( i_{\beta} \) are computed as given below.

\[
\begin{bmatrix}
i_0 \\
i_{\alpha} \\
i_{\beta}
\end{bmatrix} = \begin{bmatrix}
v_0 & 0 & 0 \\
0 & v_{\alpha} & v_{\beta} \\
0 & -v_{\beta} & v_{\alpha}
\end{bmatrix}^{-1}
\begin{bmatrix}
p_0 \\
p_{\alpha\beta} \\
q_{\alpha\beta}
\end{bmatrix} = \frac{1}{v_0(v_{\alpha}^2 + v_{\beta}^2)}
\begin{bmatrix}
v_{\alpha}^2 + v_{\beta}^2 & 0 & 0 \\
v_0v_{\alpha} & 0 & -v_0v_{\beta} \\
v_0v_{\beta} & 0 & v_0v_{\alpha}
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_{\alpha\beta} \\
q_{\alpha\beta}
\end{bmatrix}
\] (4.6)

In the above equation,

\[
i_0 = \frac{p_0(v_{\alpha}^2 + v_{\beta}^2)}{v_0(v_{\alpha}^2 + v_{\beta}^2)} = \frac{p_0v_{\alpha}i_0}{v_0} = \frac{v_0i_0}{v_0} = i_0
\] (4.7)

\[
i_{\alpha} = \frac{v_{\alpha}}{v_{\alpha}^2 + v_{\beta}^2} p_{\alpha\beta} + \frac{-v_{\beta}}{v_{\alpha}^2 + v_{\beta}^2} q_{\alpha\beta}
\]

\[
= i_{\alpha p} + i_{\alpha q}
\] (4.8)
\[ i_\beta = \frac{v_\beta}{v_\alpha^2 + v_\beta^2} p_{\alpha\beta} + \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} q_{\alpha\beta} \]
\[
= i_\beta^p + i_\beta^q \tag{4.9}
\]

where

- \( i_0 \) = zero sequence instantaneous current
- \( i_{\alpha p} \) = \( \alpha \)-phase instantaneous active current \( \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} p \)
- \( i_{\beta p} \) = \( \beta \)-phase instantaneous active current \( \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} p \)
- \( i_{\alpha q} \) = \( \alpha \)-phase instantaneous reactive current \( -\frac{v_\beta}{v_\alpha^2 + v_\beta^2} q \)
- \( i_{\beta q} \) = \( \beta \)-phase instantaneous reactive current \( \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} q \)

Using above definitions of various components of currents, the three phase instantaneous power can be expressed as,

\[
p_{3\phi} = \sum_{i=0}^{\alpha,\beta} \left( v_i i_i \right) \sum_{j=0}^{\alpha,\beta} \left( v_j i_j \right) \left( v_i v_j \right) \tag{4.10}
\]

In the above equation,

\[
p_{\alpha q} + p_{\beta q} = v_\alpha i_{\alpha q} + v_\beta i_{\beta q} = 0 \tag{4.11}
\]

If referred to compensator (or filter), the equation (4.6) can be written as,

\[
\begin{bmatrix}
i_{f0} \\
i_{fa} \\
i_{f\beta}
\end{bmatrix} = \frac{1}{v_0(v_\alpha^2 + v_\beta^2)} \begin{bmatrix}
v_\alpha^2 + v_\beta^2 & 0 & 0 \\
0 & v_0 v_\alpha & -v_0 v_\beta \\
0 & v_0 v_\beta & v_0 v_\alpha
\end{bmatrix} \begin{bmatrix}
p_{f0} \\
p_{f\alpha} \\
p_{f\beta}
\end{bmatrix} \tag{4.12}
\]

Since the compensator does not supply any instantaneous real power, therefore,

\[
p_{f\beta \phi} = p_{f0} + p_{f\alpha \beta} = 0 \tag{4.13}
\]

The instantaneous zero sequence power exchanges between the load and the compensator and compensator reactive power must be equal to load reactive power. Therefore we have,

\[
p_{f0} = p_{l0} = v_0 i_{l0} \tag{4.14}
\]
\[
p_{f\alpha \beta} = -p_{l0} = -v_\alpha i_{l0} \tag{4.15}
\]
\[
q_{f\alpha \beta} = q_i = v_\alpha i_{l\beta} - v_\beta i_{l\alpha} \tag{4.16}
\]
Since over all real power from the compensator is equal to zero, therefore the following should be satisfied.

\[ p_{fo} + p_{f_{a\beta}} = 0 \]  \hspace{1cm} (4.17)

The power flow description is shown in Fig. 4.2.

![Fig. 4.2 Power flow description of three-phase 4-wire compensated system](image)

Also the zero sequence current should be circulated through the compensator, therefore,

\[ i_{fo} = i_{t_{lo}} \]  \hspace{1cm} (4.18)

Using the conditions of compensator powers as given in the above, the \( \alpha \) and \( \beta \) components of compensator currents can be given as following.

\[
 i_{f_\alpha} = \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} \left[ p_{f_{\alpha\beta}} + \frac{-v_\beta}{v_\alpha^2 + v_\beta^2} q_{f_{\alpha\beta}} \right] \\
= \frac{1}{v_\alpha^2 + v_\beta^2} \left[ v_\alpha (-v_\alpha i_{t_{lo}}) - v_\beta (v_\alpha i_{t_{\beta}} - v_\beta i_{t_{\alpha}}) \right] \\
= \frac{1}{v_\alpha^2 + v_\beta^2} \left[ -v_\alpha v_\alpha i_{t_{lo}} - v_\beta v_\alpha i_{t_{\beta}} + v_\beta^2 i_{t_{\alpha}} \right]  \hspace{1cm} (4.19)
\]

Similarly,

\[
 i_{f_\beta} = \frac{1}{v_\alpha^2 + v_\beta^2} \left[ v_\beta \left( p_{f_{\alpha\beta}} + v_\alpha (q_{f_{\alpha\beta}}) \right) \right] \\
= \frac{1}{v_\alpha^2 + v_\beta^2} \left[ v_\beta (-v_\alpha i_{t_{lo}}) + v_\alpha (v_\alpha i_{t_{\beta}} - v_\beta i_{t_{\alpha}}) \right] \\
= \frac{1}{v_\alpha^2 + v_\beta^2} \left[ -v_\beta v_\alpha i_{t_{lo}} - v_\alpha v_\beta i_{t_{\alpha}} + v_\alpha^2 i_{t_{\beta}} \right]  \hspace{1cm} (4.20)
\]
The above equations are derived based on assumption that in general \( v_{lo} \neq 0 \). If \( v_{lo} = 0 \), then

\[
\begin{align*}
if_0 &= i_{lo} \\
if_\alpha &= \frac{1}{v_\alpha^2 + v_\beta^2} \left[ v_\alpha^2 i_{lo} - v_\alpha v_\beta i_{l_\beta} \right] \\
if_\beta &= \frac{1}{v_\alpha^2 + v_\beta^2} \left[ -v_\alpha v_\beta i_{lo} + v_\alpha^2 i_{l_\beta} \right]
\end{align*}
\] (4.21)

Once the compensator currents, \( if_o, if_\alpha \) and \( if_\beta \) are known, they are transformed back to the abc frame in order to implement in real time. This transformation is given below.

\[
\begin{bmatrix}
if_{*a} \\
if_{*b} \\
if_{*c}
\end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix}
\frac{1}{\sqrt{2}} & 1 & 0 \\
\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix} \begin{bmatrix}
if_o \\
if_\alpha \\
if_\beta
\end{bmatrix}
\] (4.22)

These reference currents are shown in Fig. 4.3. Once reference compensator currents are known, these are tracked using voltage source inverter (VSI). The other details of the scheme can be as following.

Thus, compensator powers can be expressed in terms of load powers as following.

\[
\begin{align*}
p_{fo} &= p_{lo} \\
p_{fo\alpha} &= (p_l - \bar{p}_{lavg}) - p_{lo} = \bar{p}_l \\
p_f &= p_{fo} + p_{fo\alpha} = \bar{p}_l
\end{align*}
\]
The power components and various components of currents are related as following.

\[ \begin{bmatrix} p_\alpha \\ p_\beta \end{bmatrix} = \begin{bmatrix} v_\alpha i_\alpha \\ v_\beta i_\beta \end{bmatrix} = \begin{bmatrix} v_\alpha (i_{\alpha p} + i_{\alpha q}) \\ v_\beta (i_{\beta p} + i_{\beta q}) \end{bmatrix} = \begin{bmatrix} v_\alpha i_\alpha \\ v_\beta i_\beta \end{bmatrix} + \begin{bmatrix} v_\alpha i_{\alpha p} \\ v_\beta i_{\beta p} \end{bmatrix} = \begin{bmatrix} [P_{\alpha p}] \\ [P_{\beta p}] \end{bmatrix} + \begin{bmatrix} [P_{\alpha q}] \\ [P_{\beta q}] \end{bmatrix} \] (4.23)

The following quantities are defined.

- \( \alpha \) - axis instantaneous active power = \( p_{\alpha p} = v_\alpha i_{\alpha p} \)
- \( \alpha \) - axis instantaneous reactive power = \( p_{\alpha q} = v_\alpha i_{\alpha q} \)
- \( \beta \) - axis instantaneous active power = \( p_{\beta p} = v_\beta i_{\beta p} \)
- \( \beta \) - axis instantaneous reactive power = \( p_{\beta q} = v_\beta i_{\beta q} \)

It is seen that,

\[ p_{\alpha p} + p_{\beta p} = v_\alpha i_{\alpha p} + v_\beta i_{\beta p} = \left( \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} \right) p + \left( \frac{v_\beta}{v_\alpha^2 + v_\beta^2} \right) p = \left( \frac{v_\alpha^2 + v_\beta^2}{v_\alpha^2 + v_\beta^2} \right) p = p \] (4.24)

and

\[ p_{\alpha q} + p_{\beta q} = v_\alpha i_{\alpha q} + v_\beta i_{\beta q} = \left( -\frac{v_\beta}{v_\alpha^2 + v_\beta^2} \right) p + \left( \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} \right) p = 0 \] (4.25)

Thus, it can be observed that the sum of \( p_{\alpha p} \) and \( p_{\beta p} \) is equal to total instantaneous real power \( p(t) \) and the sum of \( p_{\alpha q} \) and \( p_{\beta q} \) is equal to zero. Therefore,

\[ p_{\beta \phi} = p + p_o = p_\alpha + p_\beta + p_o = p_{\alpha p} + p_{\beta p} + p_o \] (4.26)

For an ideal compensator,

\[ p_{fo} = p_{io} = v_o i_o = p_o \]
\[ p_{fo\beta} = -p_{io} \]
\[ q_{fo\beta} = q_i \] (4.27)

For practical compensator, the switching and ohmic losses should be considered. These losses should be met from the source in order to maintain the dc link voltage constant. Let these losses are denoted by \( P_{loss} \), then the following formulation is used to include this term. Let the average
power that must be supplied to the compensator be $\Delta p$, then $\Delta p$ is given as following.

$$\Delta p = \bar{p}_o + P_{loss} \quad (4.28)$$

Thus, the compensator powers can be expressed as,

$$p_{f0} = \bar{p}_0$$
$$p_{f\alpha\beta} = \bar{p}_l - \Delta p \quad (4.29)$$
$$q_{f\alpha\beta} = q_l$$

Once these compensator powers are obtained, the compensator currents $i_{f0}$, $i_{f\alpha}$ and $i_{f\beta}$ are computed using (4.18), (4.19) and (4.20). Knowing these currents, we can obtain compensator currents in $abc$ frame using equation (4.22). These currents are realized using voltage source inverter (VSI). One of the common VSI topology is illustrated in Fig. 4.4. This VSI topology is known as neutral clamped inverter.

While realizing compensator using voltage source inverter, there are switching and other losses in the inverter circuit. Therefore, a fraction of total power is required to maintain dc capacitor voltage to a reference value by generating $P_{loss}$ term. Once reference filter currents ($i_{f0}$, $i_{f\alpha}$, $i_{f\beta}$) are obtained, the filter currents in $abc$ system are obtained as below.

$$\begin{bmatrix}
  i^*_{fa} \\
  i^*_{fb} \\
  i^*_{fc}
\end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix}
  \frac{1}{\sqrt{2}} & 1 & 0 \\
  \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
  \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix} \begin{bmatrix}
  i_{f0} \\
  i_{f\alpha} \\
  i_{f\beta}
\end{bmatrix}$$

As discussed above, the compensator powers are substituted in equation (4.29), the compensator currents are expressed as below.
Now once we know $i^*_a$, $i^*_b$, $i^*_c$ signals, these have to be synthesized using voltage source inverter. A typical voltage source inverter (VSI) along with a three-phase compensated system as shown in Fig. 4.5.

![Fig. 4.5 Control algorithm for three-phase compensated system](image)

### 4.1.1 State Space Modeling of the Compensator

There are different VSI topologies to realize [3]. The most commonly used is neutral clamped inverter topology as shown in Fig. 4.4. Since this is a three phase four-wire system each phase can be considered independently. Therefore, to analyze above circuit, only one phase is considered, which is shown in Fig. 4.6. The other phases work similarly. In Fig. 4.6(a), for switch $S_1$ is closed and switch $S_4$ is open, the KVL can be written as below.

$$L_f \frac{di_{fa}}{dt} + R_f i_{fa} + v_{sa} - V_{dc1} = 0 \quad (4.30)$$

From the above equation,

$$\frac{di_{fa}}{dt} = -\frac{R_f}{L_f} i_{fa} - \frac{v_{sa}}{L_f} + \frac{V_{dc1}}{L_f} \quad (4.31)$$
Similarly, when $S_1$ is open and $S_4$ is closed as shown in Fig. 4.6 (b),

$$\frac{di_{fa}}{dt} = -\frac{R_f}{L_f} i_{fa} - \frac{v_{sa}}{L_f} - \frac{V_{dc2}}{L_f}. \quad (4.32)$$

The above two equations can be combined into one by using switching signals $S_a$, $\overline{S}_a$, as given below.

$$\frac{di_{fa}}{dt} = -\frac{R_f}{L_f} i_{fa} + S_a \frac{V_{dc1}}{L_f} - \overline{S}_a \frac{V_{dc2}}{L_f} - \frac{v_{sa}}{L_f}. \quad (4.33)$$

Similarly, for phases $b$ and $c$, the first order derivative of filter currents can be written as following.

$$\frac{di_{fb}}{dt} = -\frac{R_f}{L_f} i_{fb} + S_b \frac{V_{dc1}}{L_f} - \overline{S}_b \frac{V_{dc2}}{L_f} - \frac{v_{sb}}{L_f}. \quad (4.34)$$

$$\frac{di_{fc}}{dt} = -\frac{R_f}{L_f} i_{fc} + S_c \frac{V_{dc1}}{L_f} - \overline{S}_c \frac{V_{dc2}}{L_f} - \frac{v_{sc}}{L_f}. \quad (4.35)$$

where, $S_a = 0$ and $\overline{S}_a = 1$ implies that the top switch is open and bottom switch is closed and, $S_a = 1$ and $\overline{S}_a = 0$ implies that the top switch is closed and bottom switch is open. The two logic signals $S_a$ and $\overline{S}_a$ are complementary to each other. This logic also holds for the other two phases. The inverter currents $i_1$ and $i_2$ as shown in Fig. 4.4, can be expressed in terms of filter currents and switching signals. These are given below.

$$i_1 = S_a i_{fa} + S_b i_{fb} + S_c i_{fc}$$

$$i_2 = \overline{S}_a i_{fa} + \overline{S}_b i_{fb} + \overline{S}_c i_{fc}. \quad (4.36)$$

The relationship between DC capacitor voltages $V_{dc1}, V_{dc2}$ and inverter currents $i_1$ and $i_2$ is given as below.

$$C_{dc1} \frac{dV_{dc1}}{dt} = -i_1$$

$$C_{dc2} \frac{dV_{dc2}}{dt} = i_2. \quad (4.37)$$
Considering \(C_{dc1} = C_{dc2} = C_{dc}\) and substituting \(i_1\) and \(i_2\) from (4.36), the above equations can be written as,

\[
\frac{dV_{dc1}}{dt} = -\frac{S_a}{C_{dc}} i_{fa} - \frac{S_b}{C_{dc}} i_{fb} - \frac{S_c}{C_{dc}} i_{fc} \tag{4.38}
\]

\[
\frac{dV_{dc2}}{dt} = \frac{S_a}{C_{dc}} i_{fa} + \frac{S_b}{C_{dc}} i_{fb} + \frac{S_c}{C_{dc}} i_{fc} \tag{4.39}
\]

The equations (4.33), (4.34), (4.35), (4.38) and (4.39) can be represented in state space form as given below.

\[
\frac{d}{dt} \begin{bmatrix} i_{fa} \\ i_{fb} \\ i_{fc} \\ V_{dc1} \\ V_{dc2} \end{bmatrix} = \begin{bmatrix} -\frac{R_f}{L_f} & 0 & 0 & \frac{S_a}{L_f} & -\frac{S_a}{L_f} \\ 0 & -\frac{R_f}{L_f} & 0 & \frac{S_b}{L_f} & -\frac{S_b}{L_f} \\ 0 & 0 & -\frac{R_f}{L_f} & \frac{S_c}{L_f} & -\frac{S_c}{L_f} \\ -\frac{S_a}{C} & -\frac{S_b}{C} & -\frac{S_c}{C} & 0 & 0 \\ \frac{S_a}{C} & \frac{S_b}{C} & \frac{S_c}{C} & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{fa} \\ i_{fb} \\ i_{fc} \\ V_{dc1} \\ V_{dc2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{L_f} & 0 & 0 \\ 0 & -\frac{1}{L_f} & 0 \\ 0 & 0 & -\frac{1}{L_f} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{sa} \\ v_{sb} \\ v_{sc} \end{bmatrix} \tag{4.40}
\]

The above equation is in the form,

\[
\dot{x} = Ax + Bu. \tag{4.41}
\]

Where, \(x\) is a state vector, \(A\) is system matrix, \(B\) is input matrix and \(u\) is input vector. This state space equation can be solved using MATLAB to implement the compensator for simulation study.

4.1.2 Switching Control of the VSI

In the equation (4.40), the switching signals \(S_a, \overline{S}_a, S_b, \overline{S}_b, S_c,\) and \(\overline{S}_c\) are generated using a hysteresis band current control. This is described as following. The upper and lower bands of the reference filter current (say phase-\(a\)) are formed using hysteresis \(h\) i.e., \(i_{fa}^* + h\) and \(i_{fa}^* - h\). Then, following logic is used to generate switching signals.

If \(i_{fa} \geq (i_{fa}^* + h)\)

\(S_a = 0\) and \(\overline{S}_a = 1\)

else if \(i_{fa} \leq (i_{fa}^* - h)\)

\(S_a = 1\) and \(\overline{S}_a = 0\)

else if \((i_{fa}^* - h) < i_{fa} < (i_{fa}^* + h)\)

Retain the current status of the switches.

end

The first order derivative of state variables can be easily solved using ‘c2d’ (continuous to discrete) command in MATLAB. It is given below.

\[
[A_d \ B_d] = c2d(A, \ B, \ t_d) \tag{4.42}
\]
The value of the state vector is updated using the following equation.

\[ x[(k+1)T] = A_d x[kT] + B_d u[kT] \] (4.43)

Where \( x(k+1) \) refers the value of the state vector at \( (k+1)^{th} \) sample. The \( A_d \) and \( B_d \) computed by ‘c2d’ function as described above can be expressed as below.

The solution of state equation given by (4.41) is given as following [Nagrath].

\[ x(t) = e^{(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{(t-\tau)} B \ u(\tau) \ d\tau \] (4.44)

Where \( t_0 \) represents initial time and \( t \) represents final time. The above equation can be re-written as following.

\[ x(t) = \phi(t-t_0) x(t_0) + \int_{t_0}^{t} \phi(t-\tau) B \ u(\tau) \ d\tau \] (4.45)

Writing above equation for small time interval, \( kT \leq t \leq (k+1)T \) with \( t_0 = kT \) and \( t = (k+1)T \),

\[ x[(k+1)T] = e^{AT} x[kT] + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B \ u[kT] \ d\tau \]

\[ = e^{AT} x[kT] + \int_{kT}^{(k+1)T} \{ e^{A((k+1)T-\tau)} B \} \ u[kT] \] (4.46)

Comparing (4.43) and (4.46), the discrete matrices \( A_d \) and \( B_d \) computed by ‘c2d’ MATLAB function can be written as following.

\[ A_d = e^{AT} \]

\[ B_d = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B \ d\tau \] (4.47)

### 4.1.3 Generation of \( P_{loss} \) to maintain dc capacitor voltage

The next step is to determine \( P_{loss} \) in order to maintain the dc link voltage close to its reference value. In compensation, what could be an indication of \( P_{loss} \) to account losses in the inverter. The average voltage variation of dc link may be an indicator of \( P_{loss} \) in the inverter. If losses are more than supplied by the inverter, the dc link voltage, i.e., \( V_{dc} = V_{dc1} + V_{dc2} \), will decline towards zero and vice versa. For proper operation of compensator, we need to maintain dc capacitor voltage to two times of the reference value of each capacitor voltage i.e., \( V_{dc1} + V_{dc2} = V_{dc} = 2V_{dcref} \). Thus, we have to replenish losses in inverter and sustain dc capacitor voltage to \( 2V_{dcref} \) with each
capacitor voltage to $V_{dcref}$. This is achieved with the help of proportional integral (PI) controller described below [4]. Let's define an error signal as following.

$$e_{Vdc} = 2V_{dcref} - (V_{dc1} + V_{dc2}) = 2V_{dcref} - V_{dc}$$

Then, the term $P_{loss}$ is computed as following.

$$P_{loss} = K_p e_{Vdc} + K_i \int_0^{T_d} e_{Vdc} \, dt$$

This control loop need not be very fast and may be updated once in a voltage cycle, preferably at the positive of phase-$a$ voltage and generate $P_{loss}$ term at these points. The above controller can be implemented using digital domain as following.

$$P_{loss}(k) = K_p e_{Vdc}(k) + K_i \sum_{j=0}^{k} e_{Vdc}(j) T_d. \quad (4.48)$$

In the above equation, $k$ represents the $k^{th}$ sample of error, $e_{Vdc}$. For $k = 1$, the above equation can be written as,

$$P_{loss}(1) = K_p e_{Vdc}(1) + K_i \sum_{j=0}^{1} e_{Vdc}(j) T_d = K_p e_{Vdc}(1) + K_i [e_{Vdc(0)} + e_{Vdc(1)}] T_d. \quad (4.49)$$

Similarly for $k = 2$, we can write,

$$P_{loss}(2) = K_p e_{Vdc}(1) + K_i [e_{Vdc(0)} + e_{Vdc(1)} + e_{Vdc(2)}] T_d. \quad (4.50)$$

Replacing $K_i [e_{Vdc(0)} + e_{Vdc(1)}]$ from (4.49), we get,

$$K_i [e_{Vdc(0)} + e_{Vdc(1)}] T_d = P_{loss}(1) - K_p e_{Vdc}(1) \quad (4.51)$$

Substituting above value in (4.50), we obtain the following.

$$P_{loss}(2) = K_p e_{Vdc}(2) + P_{loss}(1) - K_p e_{Vdc}(1) + K_i e_{Vdc}(2) T_d$$

$$= P_{loss}(1) + K_p [e_{Vdc}(2) - e_{Vdc}(1)] + K_i e_{Vdc}(2) T_d. \quad (4.52)$$

In general, for $k^{th}$ sample of $P_{loss}$,

$$P_{loss}(k) = P_{loss}(k-1) + K_p [e_{Vdc}(k) - e_{Vdc}(k-1)] + K_i e_{Vdc}(k) T_d. \quad (4.53)$$

The algorithm can be used to implement PI controller to generate $P_{loss}$. The control action can be updated at every positive zero crossing of phase-$a$ voltage for example.

### 4.1.4 Computation of load average power ($P_{lavg}$)

In reference current expressions, the average load power ($P_{lavg}$) is required to be computed. Although low pass filter can be used to find this, however the dynamic response is quite slow. The
dynamic performance of computation of $P_{avg}$ plays significant role in compensation. For this reason, a moving average algorithm can be used, which is described below.

\[
\langle p_t \rangle = P_{avg} = \langle v_a i_a + v_b i_b + v_c i_c \rangle \\
= \frac{1}{T} \int_0^T (v_a i_a + v_b i_b + v_c i_c) \, dt \quad (4.54)
\]

The above equation can be written with integration from $t_1$ to $t_1 + T$ as given in the following.

\[
P_{avg} = \frac{1}{T} \int_{t_1}^{t_1+T} (v_a i_a + v_b i_b + v_c i_c) \, dt 
\quad (4.55)
\]

This is known as moving average filter (MAF). Any change in variables instantly reflected with settling time of one cycle.

### 4.2 Some Misconception in Reactive Power Theory

The instantaneous reactive power theory, that has evolved from Fortesque, Park and Clarke Transformations of voltage and current specified in phases-$a$, $b$ and $c$ coordinates [5]. In general, for 3-phase, 4-wire system,

\[
\begin{bmatrix}
v_o \\
v_a \\
v_\beta
\end{bmatrix}
= \frac{\sqrt{2}}{3} \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
1 & -\frac{1}{2} & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
\quad (4.56)
\]

Similarly, for currents, the $\alpha\beta0$ components are given as following.

\[
\begin{bmatrix}
i_o \\
i_\alpha \\
i_\beta
\end{bmatrix}
= \frac{\sqrt{2}}{3} \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
1 & -\frac{1}{2} & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix}
\begin{bmatrix}
i_o \\
i_b \\
i_c
\end{bmatrix}
\quad (4.57)
\]

For balanced system $v_0 = (v_a + v_b + v_c)/\sqrt{3} = 0$. For three-phase, three-wire system, $i_a + i_b + i_c = 0$, which implies that $i_0 = 0$. Using these details, the above transformations in equations (4.56) and (4.57) result to the following.

\[
\begin{bmatrix}
v_a \\
v_\beta
\end{bmatrix}
= \frac{\sqrt{2}}{3} \begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
\quad (4.58)
\]

\[
\begin{bmatrix}
i_o \\
i_\alpha \\
i_\beta
\end{bmatrix}
= \frac{\sqrt{2}}{3} \begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix}
\begin{bmatrix}
i_o \\
i_b \\
i_c
\end{bmatrix}
\quad (4.59)
\]
From equation (4.58), and using \( v_a + v_b + v_c = 0 \), we get the following.

\[
v_\alpha = \sqrt{\frac{2}{3}} \left[ v_a - \frac{v_b + v_c}{2} \right]
= \sqrt{\frac{2}{3}} \left[ v_a - \frac{v_b}{2} - \frac{v_c}{2} \right]
= \sqrt{\frac{2}{3}} \left[ v_a - \frac{v_b}{2} - \left( -\frac{v_a}{2} - \frac{v_b}{2} \right) \right]
= \frac{\sqrt{3}}{2} v_a
\]  
(4.60)

\[
v_\beta = \sqrt{\frac{2}{3}} \left[ \sqrt{\frac{3}{2}} v_b - \sqrt{\frac{3}{2}} v_c \right]
= \sqrt{\frac{2}{3}} \left[ \sqrt{\frac{3}{2}} v_b - \sqrt{\frac{3}{2}} (-v_a - v_b) \right]
= \sqrt{\frac{2}{3}} \left[ \sqrt{\frac{3}{2}} v_b + \sqrt{\frac{3}{2}} \left( -\frac{v_a}{2} - \frac{v_b}{2} \right) \right]
= \frac{1}{\sqrt{2}} v_a + \sqrt{2} v_b
\]  
(4.61)

Writing equations (4.60) and (4.61) in matrix form we get,

\[
\begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix}
\]  
(4.62)

Similarly, using \( i_a + i_b + i_c = 0 \) the following can be written.

\[
\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \end{bmatrix}
\]  
(4.63)

According to the \( pq \) theory, the \( abc \) components of voltages and currents are transformed to the \( \alpha \) and \( \beta \) coordinates and the instantaneous powers \( p \) and \( q \) of the load can be expressed as following.

\[
p = v_\alpha i_\alpha + v_\beta i_\beta
\]  
(4.64)

\[
q = v_\alpha i_\beta - v_\beta i_\alpha
\]  
(4.65)

The above equations representing instantaneous active and reactive powers can be expressed in matrix form as following [1].

\[
\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} v_\alpha & v_\beta \\ -v_\beta & v_\alpha \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix}
\]  
(4.66)
Therefore from above equation (4.66), the $\alpha \beta$ components of currents can be expressed as following.

\[
\begin{bmatrix}
    i_{\alpha} \\
    i_{\beta}
\end{bmatrix} = \begin{bmatrix} v_{\alpha} & v_{\beta} \\ -v_{\beta} & v_{\alpha} \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix}
\]

The matrix, \( \begin{bmatrix} v_{\alpha} & v_{\beta} \\ -v_{\beta} & v_{\alpha} \end{bmatrix}^{-1} \) is given as following.

\[
\begin{bmatrix} v_{\alpha} & v_{\beta} \\ -v_{\beta} & v_{\alpha} \end{bmatrix}^{-1} = \frac{1}{v_{\alpha}^2 + v_{\beta}^2} \begin{bmatrix} v_{\alpha} & -v_{\beta} \\ v_{\beta} & v_{\alpha} \end{bmatrix}
\] (4.67)

From the above equation,

\[
i_{\alpha} = \frac{v_{\alpha}}{v_{\alpha}^2 + v_{\beta}^2} p - \frac{v_{\beta}}{v_{\alpha}^2 + v_{\beta}^2} q
\] (4.68)

\[
i_{\beta} = \frac{v_{\beta}}{v_{\alpha}^2 + v_{\beta}^2} p + \frac{v_{\alpha}}{v_{\alpha}^2 + v_{\beta}^2} q
\] (4.69)

Which can further be written as,

\[
i_{\alpha} = i_{\alpha p} + i_{\alpha q}
\] (4.70)

\[
i_{\beta} = i_{\beta p} + i_{\beta q}
\] (4.71)

In the above equation,

\[
i_{\alpha p} = \frac{v_{\alpha}}{v_{\alpha}^2 + v_{\beta}^2} p
\] (4.72)

\[
i_{\alpha q} = -\frac{v_{\beta}}{v_{\alpha}^2 + v_{\beta}^2} q
\] (4.73)

\[
i_{\beta p} = \frac{v_{\beta}}{v_{\alpha}^2 + v_{\beta}^2} p
\] (4.74)

\[
i_{\beta q} = \frac{v_{\alpha}}{v_{\alpha}^2 + v_{\beta}^2} q
\] (4.75)

The instantaneous active and reactive components of currents in supplying line can be calculated from the $\alpha$ and $\beta$ components of the current as given in the following.

\[
\begin{bmatrix}
    i_a \\
    i_b
\end{bmatrix} = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{bmatrix}^{-1} \begin{bmatrix}
    i_{\alpha} \\
    i_{\beta}
\end{bmatrix} = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i_{\alpha p} + i_{\alpha q} \\ i_{\beta p} + i_{\beta q} \end{bmatrix}
\]

\[
= \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix}
    i_{\alpha p} \\
    i_{\beta p}
\end{bmatrix} + \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix}
    i_{\alpha q} \\
    i_{\beta q}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    i_{\alpha p} \\
    i_{\beta p}
\end{bmatrix} = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix}
    i_{\alpha p} \\
    i_{\beta p}
\end{bmatrix}
\] (4.76)
and,

\[
\begin{bmatrix}
i_{aq} \\
i_{bq}
\end{bmatrix} = \begin{bmatrix}
\sqrt{\frac{2}{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}
\end{bmatrix} \begin{bmatrix}
i_{aq} \\
i_{bq}
\end{bmatrix}
\]  

(4.77)

The active and reactive components of the line currents must be consistent to the basic definitions. However, these components of currents have little in common with the reactive power of the load as defined in [5]. This is shown in the following illustrations.

**Example 4.1** Assume a resistive load connected as shown in the 4.7 below. It is supplied from a symmetrical source of a sinusoidal balanced voltage with \( v_a = \sqrt{2}V \sin\omega t \), with \( V = 230 \) Volts. Express the voltage and currents for primary and secondary side of the transformer. Express the active and reactive component of the currents, powers and discuss them.

![Fig. 4.7 An unbalanced resistive load supplied by three-phase delta-star connected transformer](image)

**Solution:** With the above given values, the primary side phase voltages with respect to virtual ground could be expressed as the following.

\[
\begin{align*}
v_A &= \sqrt{2}V \sin \omega t = 230\sqrt{2} \sin \omega t \\
v_B &= \sqrt{2}V \sin(\omega t - 120^\circ) = 230\sqrt{2}\sin(\omega t - 120^\circ) \\
v_C &= \sqrt{2}V \sin(\omega t + 120^\circ) = 230\sqrt{2}\sin(\omega t + 120^\circ)
\end{align*}
\]

(4.78)

In phasor form,

\[
\begin{align*}
\bar{V}_A &= 230\angle 0^\circ \text{ V} \\
\bar{V}_B &= 230\angle -120^\circ \text{ V} \\
\bar{V}_C &= 230\angle 120^\circ \text{ V}
\end{align*}
\]

Therefore, the primary side line-to-line voltage are expressed as following.

\[
\begin{align*}
v_{AB} &= \sqrt{2}\sqrt{3}V \sin(\omega t + 30^\circ) \\
v_{BC} &= \sqrt{2}\sqrt{3}V \sin(\omega t - 90^\circ) \\
v_{CA} &= \sqrt{2}\sqrt{3}V \sin(\omega t + 150^\circ)
\end{align*}
\]

(4.79)
In phasor form,

\[
\begin{align*}
\mathbf{V}_{AB} &= 398.37\angle30^\circ \text{ V} \\
\mathbf{V}_{BC} &= 398.37\angle-90^\circ \text{ V} \\
\mathbf{V}_{CA} &= 398.37\angle150^\circ \text{ V}
\end{align*}
\]

These voltages are transformed to the secondaries and are expressed below.

\[
\begin{align*}
v_a &= \sqrt{2}\sqrt{3}V \sin \omega t = 398.37\sqrt{2}\sin(\omega t + 30^\circ) \\
v_b &= \sqrt{2}\sqrt{3}V \sin(\omega t - 120^\circ) = 398.37\sqrt{2}\sin(\omega t - 90^\circ) \quad (4.80) \\
v_c &= \sqrt{2}\sqrt{3}V \sin(\omega t + 120^\circ) = 398.37\sqrt{2}\sin(\omega t + 150^\circ)
\end{align*}
\]

In phasor form,

\[
\begin{align*}
\mathbf{V}_a &= 398.37\angle30^\circ \text{ V} \\
\mathbf{V}_b &= 398.37\angle-90^\circ \text{ V} \\
\mathbf{V}_c &= 398.37\angle150^\circ \text{ V}
\end{align*}
\]

Therefore the currents on the secondary side are given below.

\[
\begin{align*}
i_a &= \frac{\sqrt{2}\sqrt{3}V}{R} \cos \omega t \\
i_b &= 0 \\
i_c &= 0 \quad (4.81)
\end{align*}
\]

Taking \(V = 230 \text{ V and } R = 4 \Omega\), the currents on the secondary side of the transformer are given as following.

\[
\begin{align*}
i_a &= \frac{v_a}{R} = \frac{\sqrt{2}\sqrt{3}V}{4} \sin \omega t = 99.56\sqrt{2}\sin(\omega t + 30^\circ) = \sqrt{2}I \sin(\omega t + 30^\circ) \quad (4.82) \\
i_b &= 0 \\
i_c &= 0
\end{align*}
\]

In phasor form, the above can be expressed as,

\[
\begin{align*}
T_a &= 99.59\angle30^\circ \text{ A} \quad (4.83) \\
T_b &= 0 \text{ A} \quad (4.84) \\
T_c &= 0 \text{ A} \quad (4.85)
\end{align*}
\]

This phase-\(a\) current \(i_a\) in the secondary side of transformer is transformed to the primary of the delta connected winding, therefore the currents on the secondary side of transformer are given as following.

\[
\begin{align*}
i_A &= \sqrt{2}I \sin(\omega t + 30^\circ) \quad (4.86) \\
i_B &= -i_A = -\sqrt{2}I \sin(\omega t + 30^\circ) \\
i_C &= 0
\end{align*}
\]
The above can be written in phasor form as given below.

\[ I_A = 99.59 \angle 30^\circ = I \angle 30^\circ \]
\[ I_B = -I_A = 1 \angle -180^\circ \times 99.59 \angle 30^\circ \]
\[ = 99.59 \angle -150^\circ = I \angle -150^\circ \]
\[ I_C = 0 \]

(4.87)

After, knowing the voltages and currents of the primary side of the transformer, their \( \alpha \) and \( \beta \) components are expressed as below.

\[ \begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix} = \begin{bmatrix} \sqrt{3}^2 & 0 \\ \frac{1}{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} v_A \\ v_B \end{bmatrix} \]  

(4.88)

Substituting \( v_A \) and \( v_B \) from (4.102) in the above equation, we get the following.

\[ \begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix} = \begin{bmatrix} \sqrt{3}^2 & 0 \\ \frac{1}{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} V \sin(\omega t) \\ \sqrt{3} V \sin(\omega t - 120^\circ) \end{bmatrix} = \begin{bmatrix} \sqrt{3} V \sin(\omega t) \\ -\sqrt{3} V \cos(\omega t) \end{bmatrix} \]  

(4.89)

And,

\[ \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \begin{bmatrix} \sqrt{3}^2 & 0 \\ \frac{1}{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} i_A \\ i_B \end{bmatrix} = \begin{bmatrix} \sqrt{3} I \sin(\omega t + 30^\circ) \\ \sqrt{3} I \sin(\omega t + 30^\circ) \end{bmatrix} \]  

(4.90)

Based on the above the active and reactive powers are computed as below.

\[ p(t) = v_\alpha i_\alpha + v_\beta i_\beta \]
\[ = \sqrt{3} V \sin(\omega t) \sqrt{3} I \sin(\omega t + 30^\circ) - \sqrt{3} V \cos(\omega t) (-I) \sin(\omega t + 30^\circ) \]
\[ = 2\sqrt{3} V I \sin(\omega t + 30^\circ) \left[ \frac{\sqrt{3}}{2} \sin(\omega t) + \frac{1}{2} \cos(\omega t) \right] \]
\[ = \sqrt{3} V I \left[ 2 \sin(\omega t + 30^\circ) \sin(\omega t) \right] \]
\[ = \sqrt{3} V I \left[ 2 \sin^2(\omega t + 30^\circ) \right] \]
\[ = \sqrt{3} V I \left[ 1 - \cos^2(\omega t + 30^\circ) \right] \]  

(4.91)

\[ q(t) = v_\alpha i_\beta - v_\beta i_\alpha \]
\[ = \sqrt{3} V \sin(\omega t) \{-I \sin(\omega t + 30^\circ) \} - (-\sqrt{3} V \cos(\omega t)) \sqrt{3} I \sin(\omega t + 30^\circ) \]
\[ = -\sqrt{3} V I 2 \sin(\omega t + 30^\circ) \left[ \frac{1}{2} \sin(\omega t) - \frac{\sqrt{3}}{2} \cos(\omega t) \right] \]
\[ = -\sqrt{3} V I 2 \sin(\omega t + 30^\circ) (-\cos(\omega t + 30^\circ)) \]
\[ = \sqrt{3} V I \sin(2(\omega t + 30^\circ)) \]  

(4.92)
Based on above values of $p$ and $q$ powers, the $\alpha$ and $\beta$ components of active and reactive components are given below.

$$\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \begin{bmatrix} i_{\alpha p} + i_{\alpha q} \\ i_{\beta p} + i_{\beta q} \end{bmatrix}$$

Where,

$$i_{\alpha p} = \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} p = \frac{\sqrt{3} V \sin(\omega t)}{(\sqrt{3} V \sin(\omega t))^2 + (\sqrt{3} V \cos(\omega t))^2} p = \frac{1}{\sqrt{3} V} \sin(\omega t) \sqrt{3} V I (1 - \cos 2(\omega t + 30^\circ)) = I \sin \omega t \{1 - \cos 2(\omega t + 30^\circ)\} \tag{4.93}$$

Similarly,

$$i_{\beta p} = \frac{v_\beta}{v_\alpha^2 + v_\beta^2} p = \frac{-\sqrt{3} V \cos(\omega t)}{(\sqrt{3} V I \sin(\omega t))^2 + (\sqrt{3} V \cos(\omega t))^2} p = \frac{-\sqrt{3} V \cos(\omega t)}{3V^2} \sqrt{3} V I (1 - \cos 2(\omega t + 30^\circ)) = -I \cos \omega t \{1 - \cos 2(\omega t + 30^\circ)\} \tag{4.94}$$

$$i_{\alpha q} = \frac{-v_\beta}{v_\alpha^2 + v_\beta^2} q = \frac{-(\sqrt{3} V \cos \omega t)}{3V^2} \sqrt{3} V I \sin 2(\omega t + 30^\circ) = I \cos \omega t \sin 2(\omega t + 30^\circ) \tag{4.95}$$

$$i_{\beta q} = \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} q = \frac{\sqrt{3} V \sin \omega t}{3V^2} \sqrt{3} V I \sin 2(\omega t + 30^\circ) = I \sin \omega t \sin 2(\omega t + 30^\circ) \tag{4.96}$$

Thus knowing $i_{\alpha p}, i_{\alpha q}, i_{\beta p}$ and $i_{\beta q}$ we can determine active and reactive components of currents on the source side as given below.

$$\begin{bmatrix} i_{ap} \\ i_{bp} \end{bmatrix} = [C]^{-1} \begin{bmatrix} i_{\alpha p} \\ i_{\beta p} \end{bmatrix}$$
Where,

\[
[C]^{-1} = \begin{bmatrix}
\frac{\sqrt{3}}{2} & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}^{-1} = \begin{bmatrix}
\frac{\sqrt{3}}{2} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

Using above equation, we can find out the active and reactive components of the current, as given below.

\[
\begin{bmatrix}
i_{ap} \\
i_{bp}
\end{bmatrix} = \begin{bmatrix}
\frac{\sqrt{3}}{2} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}
\end{bmatrix} \begin{bmatrix}
I \sin \omega t \{1 - \cos 2(\omega t + 30^\circ)\} \\
-I \cos \omega t \{1 - \cos 2(\omega t + 30^\circ)\}
\end{bmatrix}
\]

From the above,

\[
i_{ap} = \sqrt{\frac{2}{3}} I \sin \omega t \{1 - \cos 2(\omega t + 30^\circ)\} = \sqrt{\frac{2}{3}} \frac{I}{\sqrt{6}} \{2 \sin \omega t - 2 \sin \omega t \cos 2(\omega t + 30^\circ)\} = \frac{I}{\sqrt{6}} \{2 \sin \omega t - \sin(3\omega t + 60^\circ) - \sin(-\omega t - 60^\circ)\}
\]

\[
i_{bp} = -\frac{1}{\sqrt{6}} i_{ap} + \frac{1}{\sqrt{2}} i_{bp} = -\frac{1}{\sqrt{6}} I \sin \omega t \{1 - \cos 2(\omega t + 30^\circ)\} + \frac{1}{\sqrt{2}} (-I \cos \omega t) \{1 - \cos 2(\omega t + 30^\circ)\} = -I \{1 - \cos 2(\omega t + 30^\circ)\} \left[ \frac{\sin \omega t + \cos \omega t}{\sqrt{6}} \right] = -\frac{2I}{\sqrt{6}} \{1 - \cos 2(\omega t + 30^\circ)\} \left[ \frac{1}{2} \sin \omega t + \frac{\sqrt{3}}{2} \cos \omega t \right] = -\frac{2I}{\sqrt{6}} \{1 - \cos 2(\omega t + 30^\circ)\} \sin(\omega t + 60^\circ) = -\frac{2I}{\sqrt{6}} \{\sin(\omega t + 60^\circ) - \sin(\omega t + 60^\circ) \cos 2(\omega t + 30^\circ)\} = -\frac{2I}{\sqrt{6}} \left\{ \sin(\omega t) \frac{1}{2} + \cos(\omega t) \frac{\sqrt{3}}{2} + \frac{1}{2} \sin(\omega t) - \frac{1}{2} \sin(3\omega t + 120^\circ) \right\} = -\frac{I}{\sqrt{6}} \{\sin(\omega t) + 2 \sin(\omega t + 60^\circ) - \sin(3\omega t + 120^\circ)\}
\]
\[ i_{aq} = \sqrt{\frac{2}{3}} i_{aq} \]
\[ = \sqrt{\frac{2}{3}} \frac{I}{2} \{2 \sin(2\omega t + 30^\circ) \cos \omega t\} \]
\[ = \frac{I}{\sqrt{6}} \{\sin(\omega t + 60^\circ) + \sin(3\omega t + 60^\circ)\} \]

\[ i_{bq} = -\frac{1}{\sqrt{6}} i_{aq} + \frac{1}{\sqrt{2}} i_{bq} \]
\[ = -\frac{1}{\sqrt{6}} I \cos \omega t \{\sin 2(\omega t + 30^\circ)\} + \frac{1}{\sqrt{2}} (I \sin \omega t) \{\sin 2(\omega t + 30^\circ)\} \]
\[ = -\frac{I}{\sqrt{6}} 2 \sin(\omega t + 30^\circ) \left\{ -\frac{1}{2} \cos \omega t + \frac{\sqrt{3}}{2} \sin \omega t \right\} \]
\[ = \frac{I}{\sqrt{6}} 2 \sin(\omega t + 30^\circ) \sin(\omega t - 30^\circ) \]
\[ = \frac{I}{\sqrt{6}} \{\cos(\omega t + 90^\circ) - \cos(3\omega t + 30^\circ)\} \]
\[ = \frac{I}{\sqrt{6}} \{ -\sin(\omega t) - \cos(3\omega t + 30^\circ)\} \]

Thus we have,
\[ i_{ap} = \frac{I}{\sqrt{6}} \{2 \sin \omega t + \sin(\omega t + 60^\circ) - \sin(3\omega t + 60^\circ)\} \] (4.97)
\[ i_{bp} = -\frac{I}{\sqrt{6}} \{\sin(\omega t) + 2 \sin(\omega t + 60^\circ) - \sin(3\omega t + 120^\circ)\} \] (4.98)
\[ i_{aq} = \frac{I}{\sqrt{6}} \{\sin(\omega t + 60^\circ) + \sin(3\omega t + 60^\circ)\} \] (4.99)
\[ i_{bq} = \frac{I}{\sqrt{6}} \{-\sin(\omega t) - \cos(3\omega t + 30^\circ)\} \] (4.100)

From the above equations, the names instantaneous active current and instantaneous reactive current given in the pq theory do not have commonality with the notion of active and reactive currents used in electrical engineering. Also, the reactive current \(i_q\) occurs in supply lines of load in spite of the absence of the reactive element of the in the load. Furthermore, the nature of load is linear and harmonics are absent, still resolutions of active and reactive components of the current based on pq theory gives harmonics. For example, in the above discussion,
\[ i_{ap} = \frac{I}{\sqrt{6}} \{2 \sin \omega t + \sin(\omega t + 60^\circ) - \sin(3\omega t + 60^\circ)\} \] (4.101)
is the active current component in the line ‘a’ and it contains the third order harmonic. This contradicts the basic notion of the active current that was introduced to electrical engineering by
Fryze [5]. Thus, it seems a major misconception of electrical phenomenon in three phase circuits with balanced sinusoidal voltages for linear load that do not have harmonics. Moreover, the active current \( i_p \) that results from \( pq \) theory is not the current that should remain in the supply lines after the load is compensated to unity power factor as defined by Fryze. Therefore it can not be a compensation goal.

Also, it is evident that the instantaneous reactive power \( q(t) \) as defined by \( pq \) theory does not really identify the power properties of load instantaneously. For example for the above discussion, the active and reactive power are given as following.

\[
p(t) = \sqrt{3}VI \{1 - \cos 2(\omega t + 30^\circ)\} \\
q(t) = \sqrt{3}VI \sin 2(\omega t + 30^\circ)
\]

The following points are noted.

1. The active components of currents \((i_{ap}, i_{bp}, i_{cp})\) and reactive components of currents, \((i_{aq}, i_{bq}, i_{cq})\) contain third harmonic, which is not possible for a linear load as discussed above.

2. The sum of reactive components of currents, \(i_{aq}\) and \(i_{bq}\) is not equal to zero, i.e., \(i_{aq} + i_{bq} \neq 0\), even though no overall reactive power is required from the load.

3. The instantaneous reactive power \(q(t)\) defined by \(pq\) theory does not identify the power properties of the load instantaneously. Both powers \(p(t)\) and \(q(t)\) are time varying quantities, so that a pair of their values at any single point of time does not identify the power properties of load. The possibility of instantaneous identification of active and reactive power \(p(t)\) and \(q(t)\) does not mean that power properties of load are identified instantaneously. For example,

\[
\text{At } (\omega \tau + 30^\circ) = 90^\circ, \begin{cases} p(t) = 2\sqrt{3}VI \\ q(t) = 0 \end{cases}
\]
The above implies that as if it is resistive load.

Similarly at \((\omega \tau + 30^\circ) = 0^\circ, \begin{cases} p(t) = 0 \\ q(t) = 0 \end{cases}\)
Which implies as there is no load.

And when \((\omega \tau + 30^\circ) = 105^\circ, \begin{cases} p(t) = \sqrt{3}VI \left(1 + \sqrt{3} \right) \\ q(t) = -\sqrt{3}VI \left(\frac{1}{2} \right) \end{cases}\)
implies as it is capacitive load.

Similarly when \((\omega \tau + 30^\circ) = 75^\circ, \begin{cases} p(t) = \sqrt{3}VI \left(1 + \sqrt{3} \right) \\ q(t) = \sqrt{3}VI \left(\frac{1}{2} \right) \end{cases}\)
implies as if the load is inductive.
We therefore conclude that power properties cannot be identified without monitoring of the of \(p(t)\) and \(q(t)\) powers over the entire cycle period. For example in above case, the instantaneous reactive power \(q(t)\) has occurred is not because of load reactive power \(Q\) but because of voltage unbalance. This unbalance nature of load can not be identified by instantaneous reactive power \(q(t)\) values. Therefore, \(pq\) theory gives no advantage with respect to the time interval needed to identify the nature of load and its property over the the over power theories based on time domain or frequency domain approach that required the system to be monitored over one period.

Thus, we have seen that each phase has some reactive power. But there is no reactive element. This reactive power appear because of unbalance in the system and not because of reactive component. So this is an additional information what is required. From this illustration, it is evident that the instantaneous reactive power current has commonality with the load reactive power \(Q\). It also appears that the instantaneous active current in \(pq\) theory \((i_{ap}, i_{bp}, i_{cp})\) have no commonality with the load active power \(P\).

### Powers computation

The secondary side powers are given as following.

\[
\overline{S}_a = P_a + jQ_a = V_a T_a^* = 398.37 \angle 30^\circ = 39675 \text{ VA}
\]

Thus, \(P_a = 39675\) W, \(Q_a = 0\) VAr

\[
\overline{S}_b = P_b + jQ_b = V_b T_b^* = 398.37 \angle -90^\circ = 0 \text{ VA}
\]

\(P_b = 0\) W, \(Q_b = 0\) VAr

\[
\overline{S}_c = P_c + jQ_c = V_c T_c^* = 398.37 \angle 150^\circ = 0 \text{ VA}
\]

\(P_c = 0\) W, \(Q_c = 0\) VAr

The total active and reactive powers on the secondary side are given as following.

\[
P = P_a + P_b + P_c = 39675 \text{ W}
\]

\[
Q = Q_a + Q_b + Q_c = 0 \text{ VAr}
\]

\[
S_{vect} = S_{arith} = P = 39675 \text{ VA}
\]

\[
pf_{vect} = pf_{arith} = P/S = 1.0
\]

The primary side powers are given as following.

\[
\overline{S}_A = P_A + jQ_A = V_a T_a^* = 230 \angle 0^\circ = 19837.50 - j11453.16 \text{ VA}
\]

Thus, \(P_A = 19837.50\) W, \(Q_A = -11453.160\) VAr

\[
\overline{S}_B = P_B + jQ_B = V_b T_b^* = 230 \angle -120^\circ = 19837.50 - j11453.160 \text{ VAr}
\]

\[
\overline{S}_C = P_C + jQ_C = V_c T_c^* = 230 \angle 120^\circ = 0
\]

\(P_C = 0\) W, \(Q_C = 0\) VAr
The total active and reactive powers on the primary side are given as following.

\[ P = P_a + P_b + P_c = 39675 \text{ W} \]
\[ Q = Q_a + Q_b + Q_c = 0 \text{ VAr} \]
\[ S_{\text{vect}} = \left| s_A + s_B + s_C \right| = P = 39675 \text{ VA} \]
\[ S_{\text{arith}} = \left| s_A \right| + \left| s_B \right| + \left| s_C \right| = 22906 + 22906 + 0 = 45813 \text{ VA} \]
\[ p_{f_{\text{vect}}} = P/S_{\text{vect}} = 1.0 \]
\[ p_{f_{\text{arith}}} = P/S = 39675/45813 = 0.866 \]

**Example 4.2:** Assume an Inductive load connected as shown in the 4.7 below. It is supplied from a symmetrical source of a sinusoidal balanced voltage with \( v_a = \sqrt{2}V \sin \omega t \), with \( V = 230 \) Volts. Express the voltage and currents for primary and secondary side of the transformer. Express the active and reactive component of the currents, powers and discuss them.

**Solution:** With the above given values, the primary side phase voltages with respect to virtual ground could be expressed as the following.

\[ v_A = \sqrt{2} V \sin \omega t = 230\sqrt{2} \sin \omega t \]
\[ v_B = \sqrt{2} V \sin(\omega t - 120^\circ) = 230\sqrt{2}\sin(\omega t - 120^\circ) \]
\[ v_C = \sqrt{2} V \sin(\omega t + 120^\circ) = 230\sqrt{2}\sin(\omega t + 120^\circ) \]

Therefore, the primary side line-to-line voltage are expressed as following.

\[ v_{AB} = \sqrt{3} V \sin(\omega t + 30^\circ) \]
\[ v_{BC} = \sqrt{3} V \sin(\omega t - 90^\circ) \]
\[ v_{CA} = \sqrt{3} V \sin(\omega t + 150^\circ) \]

These voltages are transformed to the secondaries and are expressed below.

\[ v_a = \sqrt{3} V \sin \omega t = 398.37\sqrt{2}\sin(\omega t + 30^\circ) \]
\[ v_b = \sqrt{3} V \sin(\omega t - 120^\circ) = 398.37\sqrt{2}\sin(\omega t - 90^\circ) \]
\[ v_c = \sqrt{3} V \sin(\omega t + 120^\circ) = 398.37\sqrt{2}\sin(\omega t + 150^\circ) \]
In phasor form, the above voltages are expressed as below.

\[
\begin{align*}
V_a &= \sqrt{3} V \angle 30^\circ \\
V_b &= \sqrt{3} V \angle -90^\circ \\
V_c &= \sqrt{3} V \angle 150^\circ
\end{align*}
\]

Therefore, the currents on the secondary side are given below.

\[
\begin{align*}
i_a &= \frac{\sqrt{2} \sqrt{3} V}{X} \sin(\omega t - 60^\circ) = 99.59\sqrt{2} \sin(\omega t - 60^\circ) \\
i_b &= 0 \\
i_c &= 0
\end{align*}
\]

In phasor form, the above can be expressed as,

\[
\bar{I}_a = 99.59\angle -60^\circ \text{ A}.
\]

The above phase-\(a\) current \((i_a)\) is transformed to the primary of the delta connected winding. Since the currents should have \(90^\circ\) phase shift with respect to the voltages across the windings as given by (4.102), therefore the currents on the secondary side of transformer are given as following.

\[
\begin{align*}
i_A &= = i_{AB} = i_a = \sqrt{2} I \sin(\omega t - 60^\circ) \\
i_B &= -i_A = -\sqrt{2} I \sin(\omega t - 60^\circ) \\
i_C &= 0
\end{align*}
\]

In phasor form, the above can be expressed as,

\[
\begin{align*}
\bar{T}_A &= \bar{I}_a \angle -60^\circ = 99.59\angle -60^\circ \text{ A} \\
\bar{T}_B &= -\bar{I}_a \angle -60^\circ = 99.59\angle -60^\circ \text{ A} \\
\bar{T}_C &= 0 \text{ A}.
\end{align*}
\]

After, knowing the voltages and currents of the primary side of the transformer, their \(\alpha\) and \(\beta\) components are expressed as below.

\[
\begin{bmatrix}
v_\alpha \\
v_\beta
\end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} \begin{bmatrix} v_A \\
v_B
\end{bmatrix}
\]

Substituting \(v_A\) and \(v_B\) from (4.102) in the above equation, we get the following.

\[
\begin{bmatrix}
v_\alpha \\
v_\beta
\end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} V \sin \omega t \\
\sqrt{2} V \sin(\omega t - 120^\circ) \end{bmatrix} = \begin{bmatrix} \sqrt{3} V \sin \omega t \\
-\sqrt{3} V \cos(\omega t) \end{bmatrix}
\]

And,

\[
\begin{bmatrix}
i_\alpha \\
i_\beta
\end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} \begin{bmatrix} i_A \\
i_B
\end{bmatrix} = \begin{bmatrix} \sqrt{2} I \sin(\omega t - 60^\circ) \\
-\sqrt{2} I \sin(\omega t - 60^\circ) \end{bmatrix}
\]

\[
\begin{bmatrix}
i_\alpha \\
i_\beta
\end{bmatrix} = \begin{bmatrix} \frac{3}{2} \sqrt{I} \sin(\omega t - 60^\circ) \\
-I \sin(\omega t - 60^\circ) \end{bmatrix}
\]

(4.103)
Based on the above, the active and reactive powers are computed as below.

\[ p(t) = p_{\alpha\beta} = v_\alpha i_\alpha + v_\beta i_\beta 
= \sqrt{3}V \sin(\omega t) \sqrt{3}I \sin(\omega t - 60^\circ) + (-\sqrt{3}V \cos \omega t)(-I \sin(\omega t - 60^\circ)) 
= 2\sqrt{3}VI \sin(\omega t - 60^\circ) \left[ \frac{\sqrt{3}}{2} \sin \omega t + \frac{1}{2} \cos \omega t \right] 
= \sqrt{3}VI [2 \sin(\omega t - 60^\circ) \cos(\omega t - 60^\circ)] 
= \sqrt{3}VI \sin 2(\omega t - 60^\circ) \quad (4.104) \]

\[ q(t) = v_\alpha i_\beta - v_\beta i_\alpha 
= \sqrt{3}V \sin \omega t \{ -I \sin(\omega t - 60^\circ) \} - (-\sqrt{3}V \cos \omega t) \sqrt{3}I \sin(\omega t - 60^\circ) 
= -\sqrt{3}VI 2 \sin(\omega t - 60^\circ) \left[ \frac{1}{2} \sin \omega t - \frac{\sqrt{3}}{2} \cos \omega t \right] 
= -\sqrt{3}VI 2 \sin(\omega t - 60^\circ) (\sin(\omega t - 60^\circ)) 
= -\sqrt{3}VI 2 \sin^2(\omega t - 60^\circ) 
= -\sqrt{3}VI \{ 1 - \cos 2(\omega t - 60^\circ) \} \quad (4.105) \]

Based on above values of \( p \) and \( q \) powers, the \( \alpha \) and \( \beta \) components of active and reactive components are given below.

\[
\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \begin{bmatrix} i_{\alpha p} + i_{\alpha q} \\ i_{\beta p} + i_{\beta q} \end{bmatrix}
\]

Where,

\[ i_{\alpha p} = \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} p 
= \frac{\sqrt{3}V \sin \omega t}{(\sqrt{3}V \sin \omega t)^2 + (-\sqrt{3}V \cos \omega t)^2} p 
= \frac{1}{\sqrt{3}V} \sin \omega t \sqrt{3}I \sin 2(\omega t - 60^\circ) 
= I \sin \omega t \sin 2(\omega t - 60^\circ) \quad (4.106) \]

Similarly,
\[ i_{\beta p} = \frac{v_\beta}{v_\alpha^2 + v_\beta^2} \]
\[ = \frac{-\sqrt{3} V \cos \omega t}{(\sqrt{3} V I \sin \omega t)^2 + (-\sqrt{3} V \cos \omega t)^2} \]
\[ = \frac{-\sqrt{3} V \cos \omega t}{3V^2} \sqrt{3} V I \sin 2(\omega t - 60^\circ) \]
\[ = -I \cos \omega t \sin 2(\omega t - 60^\circ) \]  

(4.107)

\[ i_{\alpha q} = \frac{-v_\alpha}{v_\alpha^2 + v_\beta^2} q \]
\[ = \frac{-(-\sqrt{3} V \cos \omega t)}{3V^2} \left[ -\sqrt{3} V I \{1 - \cos 2(\omega t - 60^\circ)\} \right] \]
\[ = -I \cos \omega t (1 - \cos 2(\omega t - 60^\circ)) \]  

(4.108)

\[ i_{\beta q} = \frac{v_\alpha}{v_\alpha^2 + v_\beta^2} q \]
\[ = \frac{\sqrt{3} V \sin \omega t}{3V^2} \left[ -\sqrt{3} V I \{1 - \cos 2(\omega t - 60^\circ)\} \right] \]
\[ = -I \sin \omega t \{1 - \cos 2(\omega t - 60^\circ)\} \]  

(4.109)

Thus, knowing \( i_{\alpha p}, i_{\alpha q}, i_{\beta p} \) and \( i_{\beta q} \), we can determine active and reactive components of currents on the source side as given below.

\[
\begin{bmatrix}
  i_{\alpha p} \\
  i_{\beta p}
\end{bmatrix} = [C]^{-1} \begin{bmatrix}
  i_{\alpha p} \\
  i_{\beta p}
\end{bmatrix}
\]

Where,

\[
[C]^{-1} = \begin{bmatrix}
  \frac{\sqrt{3}}{2} & 0 \\
  \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2}
\end{bmatrix}^{-1} = \begin{bmatrix}
  \frac{\sqrt{2}}{3} & 0 \\
  \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

(4.110)

Using above equation, we can find out the active and reactive components of the current, as given below.

\[
\begin{bmatrix}
  i_{\alpha p} \\
  i_{\beta p}
\end{bmatrix} = \begin{bmatrix}
  \sqrt{\frac{3}{2}} & 0 \\
  \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}
\end{bmatrix} \begin{bmatrix}
  I \sin \omega t \sin 2(\omega t - 60^\circ) \\
  -I \cos \omega t \sin 2(\omega t - 60^\circ)
\end{bmatrix}
\]

From the above,
\[ i_{ap} = \sqrt{\frac{2}{3}} I \sin \omega t \sin 2(\omega t - 60^\circ) \]
\[ = \sqrt{\frac{2}{3}} I \left\{ 2 \sin \omega t \sin 2(\omega t - 60^\circ) \right\} \]
\[ = \frac{I}{\sqrt{6}} \left\{ \cos(\omega t - 2\omega t + 120^\circ) \right\} \]
\[ = \frac{I}{\sqrt{6}} \left\{ \cos(\omega t - 120^\circ) \right\} \]
\[ = \frac{I}{\sqrt{6}} \left\{ \cos(\omega t - 120^\circ) - \cos(3\omega t - 120^\circ) \right\} \] (4.111)

\[ i_{bp} = -\frac{1}{\sqrt{6}} i_{ap} + \frac{1}{\sqrt{2}} i_{\beta p} \]
\[ = -\frac{1}{\sqrt{6}} I \sin \omega t \left\{ \sin 2(\omega t - 60^\circ) \right\} + \frac{1}{\sqrt{2}} (-I \cos \omega t) \left\{ \sin 2(\omega t - 60^\circ) \right\} \]
\[ = -\frac{I}{\sqrt{6}} \left\{ 2 \sin 2(\omega t - 60^\circ) \right\} \left[ \frac{1}{2} \sin \omega t + \frac{\sqrt{3}}{2} \cos \omega t \right] \]
\[ = -\frac{I}{\sqrt{6}} \left\{ 2 \sin 2(\omega t - 60^\circ) \sin(\omega t + 60^\circ) \right\} \]
\[ = -\frac{I}{\sqrt{6}} \left\{ \cos(\omega t - 180^\circ) \right\} \]
\[ = \frac{I}{\sqrt{6}} \left\{ \cos \omega t + \cos(3\omega t - 60^\circ) \right\} \]

\[ i_{aq} = \sqrt{\frac{2}{3}} i_{aq} \]
\[ = -\sqrt{\frac{2}{3}} I \left\{ 1 - \cos 2(\omega t - 60^\circ) \right\} \cos \omega t \]
\[ = -\frac{I}{\sqrt{6}} \left\{ 2 \cos \omega t - 2 \cos \omega t \cos 2(\omega t - 60^\circ) \right\} \]
\[ = -\frac{I}{\sqrt{6}} \left\{ 2 \cos \omega t - \cos(3\omega t - 120^\circ) - \cos(\omega t - 120^\circ) \right\} \]
\[ = \frac{I}{\sqrt{6}} \left\{ -2 \cos \omega t + \cos(\omega t - 120^\circ) + \cos(3\omega t - 120^\circ) \right\} \] (4.112)
\[i_{bq} = -\frac{1}{\sqrt{6}} i_{aq} + \frac{1}{\sqrt{2}} i_{bq}\]

\[= -\frac{1}{\sqrt{6}} \left[-I \cos \omega t \{1 - \cos 2(\omega t - 60^\circ)\}\right] + \frac{1}{\sqrt{2}} \left[-I \sin \omega t \{1 - \cos 2(\omega t - 60^\circ)\}\right]\]

\[= -\frac{I}{\sqrt{6}} \left\{1 - \cos 2(\omega t - 60^\circ)\right\} \left\{-\frac{1}{2} \cos \omega t - \frac{\sqrt{3}}{2} \sin \omega t\right\}\]

\[= \frac{I}{\sqrt{6}} \left\{1 - \cos 2(\omega t - 60^\circ)\right\} \left\{\frac{1}{2} \cos \omega t + \frac{\sqrt{3}}{2} \sin \omega t\right\}\]

\[= \frac{I}{\sqrt{6}} \left\{1 - \cos 2(\omega t - 60^\circ)\cos(\omega t - 60^\circ)\right\}\]

\[= \frac{I}{\sqrt{6}} \left\{2 \cos(\omega t - 60^\circ) - 2 \cos(\omega t - 60^\circ) \cos 2(\omega t - 60^\circ)\right\}\]

\[= \frac{I}{\sqrt{6}} \left\{2 \cos(\omega t - 60^\circ) + 3 \omega t - \cos(\omega t - 60^\circ)\right\}\]

(4.113)

Thus we have,

\[i_{ap} = \frac{I}{\sqrt{6}} \left\{\cos(\omega t - 120^\circ) - \cos(3\omega t - 120^\circ)\right\}\]

\[i_{bp} = \frac{I}{\sqrt{6}} \left\{\cos(\omega t) + \cos(3\omega t - 60^\circ)\right\}\]

\[i_{aq} = \frac{I}{\sqrt{6}} \left\{-2 \cos \omega t + \cos(\omega t - 120^\circ) + \cos(3\omega t - 120^\circ)\right\}\]

\[i_{bq} = \frac{I}{\sqrt{6}} \left\{2 \cos(\omega t - 60^\circ) + 3 \omega t - \cos(\omega t - 60^\circ)\right\}\]

From above equations, it is clear that there exist active components of current which also have third harmonic component. Also, the reactive components too have third harmonics. This again does not match with definitions of active and reactive components of currents proposed by Fryze [5].

**Powers computation**

The secondary side powers are given as following.

\[\overline{S}_a = P_a + jQ_a = \overline{V}_a \overline{I}_a^* = 398.37 \angle \overline{30^\circ} 99.59 \angle 60^\circ = 39673.8361 \angle 90^\circ \text{ VA}\]

Thus, \(P_a = 0 \text{ W}, Q_a = 39673.8361 \text{ VAr}\)

\[\overline{S}_b = P_b + jQ_b = \overline{V}_b \overline{I}_b^* = 398.37 \angle -90^\circ \times 0 = 0 \text{ VA}\]

\(P_b = 0 \text{ W}, Q_b = 0 \text{ VAr}\)

\[\overline{S}_c = P_c + jQ_c = \overline{V}_c \overline{I}_c^* = 398.37 \angle 150^\circ \times 0 = 0 \text{ VA}\]

\(P_c = 0 \text{ W}, Q_c = 0 \text{ VAr}\)
The total active and reactive powers on the secondary side are given as following.

\[
P = P_a + P_b + P_c = 0 \text{ W} \\
Q = Q_a + Q_b + Q_c = 39673.83 \text{ VAr} \\
S_{vect} = S_{arith} = Q = 39673.83 \text{ VA} \\
pf_{vect} = p_{f_{arith}} = P/S = 0
\]

The primary side powers are given as following.

\[
S_A = P_A + jQ_A = V_a T_a^* = 230 \angle 0^\circ 99.59 \angle 60^\circ = 11452.85 + j19836.91 \text{ VA} \\
\text{Thus, } P_A = 11452.85 \text{ W}, \ Q_A = 19836.91 \text{ VAr}
\]

\[
S_B = P_B + jQ_B = V_b T_b^* = 230 \angle -120^\circ (-99.59 \angle -120^\circ)^* = \\
P_B = -11452.85 \text{ W}, \ Q_B = 19836.91 \text{ VAr}
\]

\[
S_C = P_C + jQ_C = V_C T_C^* = 230 \angle 120^\circ \times 0 \\
P_C = 0 \text{ W}, \ Q_C = 0 \text{ VAr}
\]

The total active and reactive powers on the primary side are given as following.

\[
P = P_a + P_b + P_c = 0 \text{ W} \\
Q = Q_a + Q_b + Q_c = 39673.82 \text{ VAr} \\
S_{vect} = |S_A + S_B + S_C| = 39673.82 \text{ VA} \\
S_{arith} = |S_A| + |S_B| + |S_C| = 22906 + 22906 + 0 = 45811.4 \text{ VA} \\
pf_{vect} = P/S_{vect} = 0 \\
pf_{arith} = P/S = 0
\]

**Example 4.3**: Consider the star-delta connected ideal transformer with 1:1 turn ratio as shown in Fig. 2. The secondary side of transformer, a load of 2 ohms is connected between the phase-\( a \) and \( b \). Compute the following.

(a) Time domain expressions of currents in each phase on both primary and secondary side.

(b) Does the load require reactive power from the source? If any, find its value. Also compute the reactive power on each phase of either side of transformer.

(c) Also determine active powers on each phase and overall active power on either side of the transformer.

(d) If you have similar arrangement with balanced load and same output power, comment upon the rating of line conductors and transformers.

**Solution** In this example, we have three phase star-delta connected transformer of turns ratio 1:1 with star side connected to three phase balanced voltage source and neutral connected to ground.
Thus, in this three phase star delta connected transformer, delta side phase voltages equal the star side line voltages. The primary side instantaneous phase voltages are given by,

\[ v_A = 230\sqrt{2} \sin(\omega t) = 325.27 \sin(\omega t) \]
\[ v_B = 230\sqrt{2} \sin(\omega t - 120^\circ) = 325.27 \sin(\omega t - 120^\circ) \]
\[ v_C = 230\sqrt{2} \sin(\omega t + 120^\circ) = 325.27 \sin(\omega t + 120^\circ) \]

Therefore, instantaneous line to line voltages at delta side (secondary) are given by,

\[ v_{ab} = 230\sqrt{2} \sin(\omega t) = 325.27 \sin(\omega t) \]
\[ v_{bc} = 230\sqrt{2} \sin(\omega t - 120^\circ) = 325.27 \sin(\omega t - 120^\circ) \]
\[ v_{ca} = 230\sqrt{2} \sin(\omega t + 120^\circ) = 325.27 \sin(\omega t + 120^\circ) \]

Therefore, instantaneous phase voltages with respect to ground are given as follows.

\[ v_a = \frac{230\sqrt{2}}{\sqrt{3}} \sin(\omega t - 30^\circ) = 132.79\sqrt{2} \sin(\omega t - 30^\circ) \]
\[ v_b = \frac{230\sqrt{2}}{\sqrt{3}} \sin(\omega t - 150^\circ) = 132.79\sqrt{2} \sin(\omega t - 150^\circ) \]
\[ v_c = \frac{230\sqrt{2}}{\sqrt{3}} \sin(\omega t + 90^\circ) = 132.79\sqrt{2} \sin(\omega t + 90^\circ) \]

(a) On delta side, we have a resistive load of \( R = 3 \Omega \) connected between terminals \( a \) and \( b \). Thus, expression for instantaneous currents flowing out of terminals \( a \), \( b \) and \( c \) of the transformer are given by,

\[ i_a = \frac{v_{ab}}{R} = \frac{230\sqrt{2}}{3} \sin \omega t = 76.67\sqrt{2} \sin \omega t \]
\[ i_b = -i_a = -\frac{v_{ab}}{R} = -\frac{230\sqrt{2}}{3} \sin \omega t = -76.67\sqrt{2} \sin \omega t \]
\[ i_c = 0 \]

Therefore, the winding currents on the secondary side are given as below.

\[ i_{ab} = 76.67\sqrt{2} \sin \omega t = 108.41 \sin \omega t \]
\[ i_{bc} = 0 \]
\[ i_{ca} = 0 \]

These currents are transformed to the primary windings. Thus, the time domain expressions of these currents are given as below.
\[i_A = 76.67\sqrt{2} \sin \omega t\]
\[i_B = 0\]
\[i_C = 0\]

(b) Load does not require any reactive power from the source because it is a purely resistive load. This fact can also be verified by looking at the expressions for instantaneous phase voltages and currents on star side.

(c) Similarly, no current (and hence power) is drawn by the load from phases-\(B\) and \(C\). Thus, various powers on the primary side are as follows.

For phase-A,
\[\overline{S}_A = P_A + jQ_A = \overline{V}_A \overline{T}_A^* = 230\angle0^\circ \times 76.67\angle0^\circ = 17634.1\ \text{VA}\]
\[P_A = 17634.1\ \text{W},\ Q_A = 0\ \text{VAr}\]

For phase-B,
\[\overline{S}_B = P_B + jQ_B = \overline{V}_B \overline{T}_B^* = 230\angle-120^\circ \times 0 = 0\ \text{VA}\]
\[P_B = 0\ \text{W},\ Q_B = 0\ \text{VAr}\]

For phase-C,
\[\overline{S}_C = P_C + jQ_C = \overline{V}_C \overline{T}_C^* = 230\angle120^\circ \times 0 = 0\ \text{VA}\]
\[P_C = 0\ \text{W},\ Q_C = 0\ \text{VAr}\]

Thus, various powers on the secondary side are as follows.

For phase-a,
\[\overline{S}_a = P_a + jQ_a = \overline{V}_a \overline{T}_a^* = 132.79\angle-30^\circ \times 76.67\angle0^\circ = 8817.05 - j5090.53\ \text{VA}\]
\[P_a = 8817.05\ \text{W},\ Q_a = -5090.53\ \text{VAr}\]

For phase-b,
\[\overline{S}_b = P_b + jQ_b = \overline{V}_b \overline{T}_b^* = 132.79\angle-150^\circ \times (-76.67\angle0^\circ) = 8817.05 + j5090.53\ \text{VA}\]
\[P_b = 8817.05\ \text{W},\ Q_b = 5090.53\ \text{VAr}\]

For phase-c,
\[\overline{S}_c = P_c + jQ_c = \overline{V}_c \overline{T}_c^* = 132.79\angle90^\circ \times 0 = 0\ \text{VA}\]
\[P_c = 0\ \text{W},\ Q_c = 0\ \text{VAr}\]

For the above analysis, it is observed that the total active power, \(P = P_A + P_B + P_C = P_a + P_b + P_c = 17634.1\ \text{W}\) and total reactive power \(Q = Q_A + Q_B + Q_C = Q_a + Q_b + Q_c = 0\ \text{VAr}\). However, due to unbalanced load, on delta side of the transformer phase-b and phase-c experience reactive power as calculated above. This creates some power factor in each phase.

(d) Because, the load which is currently getting power from one phase on delta side will be shared
by all the three phases phase voltages being the same. Thus, required current rating of the line conductors and transformers will be reduced.

**Example 4.4**
Consider the a three-phase balanced system shown in Fig. 4.9. The supply voltages are: $V_a = 220\angle0^\circ$, $V_b = 220\angle-120^\circ$, $V_c = 220\angle120^\circ$ with balance load resistance of 1.32Ω. The output power $P_o$ is 100 kW and the losses in the feeder are 5% of te output power. The reactance of feeder is not considered in the study.

(a) Determine voltages at the load points, phase currents and feeder resistance.

(b) Now let us say that the load is unbalanced by connecting a resistive load between phases-$a$ and $b$ with same power output. With this configuration, compute active, reactive, various apparent powers and power factor based on them. Compute losses in the system and comment upon the result.

(c) Now, with same losses and power output, find an equivalent three-phase balanced circuit and repeat (b). Comment on the result.

![Diagram](image)

Fig. 4.9 Study on three-phase unbalanced system

**Solution:**
(a) For the given system the phase voltages are $V_a = 220\angle0^\circ$, $V_b = 220\angle-120^\circ$, $V_c = 220\angle120^\circ$. The load active power is 100 kW with 5% losses in the feeder, i.e., $\Delta P_a = 5$ kW. Using above parameters the RMS value of the phase voltage is computed as following.

$$P_o = 100 \times 1000 = 3 \times \frac{V^2}{1.32}$$

From the above equation, value of $V$ is given as following.

$$V = \sqrt{\frac{100000 \times 1.32}{3}} = 209.76 \text{ V}$$

Thus, the rms value of phase currents is given by,

$$I = \frac{V}{R} = \frac{209.76}{1.32} = 159.9 \text{ A}$$
The voltage drop across the feeder is found using following equation.

\[
\Delta V = 230 - 209.76 = 20.23 \text{ V}
\]

For 5% losses in the feeder, the value of feeder resistance can be computed as below.

\[
3 \times (158.9)^2 \times r = 0.05 \times 100 \times 1000
\]

which implies, \( r = 0.066 \Omega \)

Since the load is balanced, the various apparent powers (arithmetic, vector and effective) are same. These are computed below.

\[
S_A = S_a + S_b + S_c = V_a I_a + V_b I_b + V_c I_c = 3 V_a I_a = 3 \times 209.5 \times 158.9 = 100 \text{ kVA}
\]

Now we compute the vector apparent power \( S_v \) as given below.

\[
S_v = \sqrt{P^2 + Q^2} = \sqrt{(V_a I_a \cos \phi_a + V_b I_b \cos \phi_b + V_c I_c \cos \phi_c)^2 + (V_a I_a \sin \phi_a + V_b I_b \sin \phi_b + V_c I_c \sin \phi_c)^2}
\]

\[
= \sqrt{(3 V_a I_a \cos \phi_a)^2 + (3 V_a I_a \sin \phi_a)^2}
\]

\[
= 3 V_a I_a = 100 \text{ kVA}
\]

Similarly, the effective apparent power can be computed as below.

\[
S_e = 3 V_e I_e = 3V_a I_a = 100 \text{ kVA}
\]

The power factors based on above apparent powers are given below.

\[
pf_A = pf_v = pf_e = 1.0
\]

Now an unbalanced circuit with same power output is considered. This is achieved by placing a resistance of 1.17 \( \Omega \) between any two phases (say between phases \( a \) and \( b \)). This is shown in Fig. 4.10.

For this circuit, the line currents are computed as below. Let \( I \) be the RMS value of the line current, then following equation must be satisfied.

\[
I^2 \times R = 100 \times 1000
\]

From the above equation, \( I \) is given as below.

\[
I = \sqrt{\frac{100000}{1.17}} = 292.35 \text{ A}
\]
Thus, phase currents are as following.

\[ I_a = 292.35 \angle 30^\circ \, A \]
\[ I_b = -I_a = -292.35 \angle 30^\circ \, A \]
\[ I_c = 0 \, A \]

Knowing the currents in the circuit we can compute the voltages \( V_a, V_b \) and \( V_c \) as following.

\[ V_a = 220 \angle 0^\circ - 292.25 \angle 30^\circ \times 0.066 \]
\[ = 220 - (16.67 + j9.9296) \]
\[ = 223.33 - j9.6296 \]
\[ = 203.55\angle -2.70^\circ \, V \]

Similarly, voltages \( V_b \) and \( V_c \) can be computed which are given below.

\[ V_b = 220\angle -120^\circ - (-292.25\angle 30^\circ) \times 0.066 \]
\[ = 203.55\angle -117.29^\circ \, V \]
\[ V_c = 220\angle 120^\circ - 0 \times 0.066 = 220\angle 120^\circ \]

Thus, \( V_a = V_b = 203.55 \, V \) and \( V_c = 0 \, V \). Knowing three-phase voltages and currents, the active and reactive powers are computed as following.

\[ S_a = |V_a I_a^*| = 203.55\angle -2.70^\circ \times 292.35\angle -30^\circ \]
\[ = 50088.79 - j32156.42 \]
\[ = P_a + jQ_a \]

Thus, \( S_a = \sqrt{P_a^2 + Q_a^2} = 59522.46 \, VA \)

\[ S_b = |V_b I_b^*| = 203.55\angle -117.3^\circ \times 292.35\angle 150^\circ \]
\[ = 50088.79 + j32156.42 \]
\[ = P_b + jQ_b \]

Thus, \( S_b = \sqrt{P_b^2 + Q_b^2} = 59522.46 \, VA \)
\[
\bar{S}_c = V_c I_c^* = 220 \angle 120^\circ \times 0 \\
= 0 + j 0 \\
= P_c + j Q_c
\]

Thus, \( S_c = \sqrt{P_c^2 + Q_c^2} = 0 \) VA

Based on the computations of active and reactive powers in the above, we shall compute arithmetic, vector and effective apparent powers and corresponding power factors.

\[
S_A = S_a + S_b + S_c = 59522.46 + 59522.46 + 0 = 119094.92 \text{ VA}
\]

\[
S_v = |\bar{S}_v| = |\bar{S}_a + \bar{S}_b + \bar{S}_c| = 100177.58 \text{ VA}
\]

The effective apparent power is computed as following.

\[
S_e = 3 \times V_e \times I_e \\
= 3 \times \sqrt{\frac{V_a^2 + V_b^2 + V_c^2}{3}} \times \sqrt{\frac{I_a^2 + I_b^2 + I_c^2}{3}} \\
= \sqrt{V_a^2 + V_b^2 + V_c^2} \times \sqrt{I_a^2 + I_b^2 + I_c^2} \\
= \sqrt{203.6^2 + 203.6^2 + 220^2} \times \sqrt{292.35^2 + 292.35^2 + 0^2} \\
= 149816.05 \text{ VA}
\]

Based on above apparent powers, the power factor are as following.

\[
pf_A = \frac{100000}{119044.92} = 0.84 \\
pf_v = \frac{100000}{100000} = 1.0 \\
pf_e = \frac{100000}{149816.05} = 0.667
\]

The power loss in the feeders is computed as below.

\[
P_{loss} = I_a^2 r + I_b^2 r + I_c^2 r \\
= 292.35^2 \times 0.066 + 292.35^2 \times 0.066 + 0 \\
= 11260 = 11.26 \text{ kW}
\]

Thus it is observed that when the energy is delivered to the unbalanced load, the power loss in the supply increases from 5 to 11.26 kW. It means that currents on the lines have increased and this implies that an unbalanced purely resistive load cannot be considered as unity power factor load. The calculation of the power factor using the vector apparent power leads to a value which is equal to unity. This disqualifies, the vector apparent power \( S_v \) as an acceptable definition of the apparent power in the presence of the load unbalance.

Since the value of the apparent power of balanced load is independent of this power definition, we could ask the question: what is apparent power of a balanced load with the active power \( P = 100 \text{ kW} \), that causes same power loss of 11.26 kW, as the unbalanced load discussed earlier. For
power loss of 11.26 kW the following equation holds true.

\[ 3 \times I^2 \times r = 11260 \]

Therefore, \[ I = \sqrt{\frac{11260}{3 \times 0.066}} = 283.01 \text{ A} \]

Now we use the condition of output power as below.

\[ P_0 = 100000 = 3 \times I^2 \times R \]

Implying that, \[ R = \sqrt{\frac{100000}{3 \times 238.015}} = 0.59 \Omega \]

The total input power must be equal to output power + losses. Therefore,

\[ P_i = 11260 = 3 \times 220 \times I \times \cos \phi \]

Implying \[ \cos \phi = \frac{112600}{220 \times 238.05} \]

\[ \approx 0.7078 \implies \phi = 44.94^\circ \]

From the above equation,

\[ \tan \phi = \tan 44.94^\circ = 0.9978 = \frac{X}{R+r} \]

Therefore, \[ X = 0.9978 \times (0.59 + 0.0659) = 0.65 \Omega. \]

Therefore, load impedance (Z) is given as below.

\[ Z = 0.59 + j0.65 \Omega \]

(4.115)

Knowing above parameters the voltages at the load points can be computed which are given below.

\[ V_a = 220 \angle 0^\circ - 238.015 \angle 44.94^\circ \times 0.0659 \]

\[ = 220 - 11.1 - j11.07 = 208.89 - j11.07 \]

\[ = 209.18 \angle -3.03^\circ \text{ V} \]

Similarly, \[ V_b = 209.18 \angle -123.03^\circ \text{ V} \]

\[ V_c = 209.18 \angle 116.97^\circ \text{ V} \]

The phase currents are as following.

\[ I_a = 238.015 \angle -44.94^\circ \text{ A} \]

\[ I_b = 238.015 \angle -164.94^\circ \text{ A} \]

\[ I_c = 238.015 \angle -75.06^\circ \text{ A} \]

For this equivalent balanced circuit (with same output power and power loss), the three apparent powers i.e., arithmetic, vector and effective are same and these are as following.

\[ S_A = S_v = S_e = 3 \times V_a \times I_a = 3 \times 209.18 \times 238.015 = 149.363 \text{ kVA} \]

and \[ P = P_a + P_b + P_c = 100 \text{ kW} \]

Thus the power factor based on the above apparent powers will also be same. Therefore,

\[ pf_a = pf_v = pf_e = 100/149.363 = 0.67 \]
4.3 Theory of Instantaneous Symmetrical Components

The theory of instantaneous symmetrical components can be used for the purpose of load balancing, harmonic suppression, and power factor correction [2], [4]. The control algorithms based on instantaneous symmetrical component theory can practically compensate any kind of unbalance and harmonics in the load, provided we have a high band width current source to track the filter reference currents. These algorithms have been derived in this section. For any set of three-phase instantaneous currents or voltages, the instantaneous symmetrical components are defined by,

\[
\begin{bmatrix}
\tilde{i}_{a0} \\
\tilde{i}_{a+} \\
\tilde{i}_{a-}
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix}i_a \\ i_b \\ i_c \end{bmatrix}
\] (4.116)

Similarly for three-phase instantaneous voltages, we have,

\[
\begin{bmatrix}
\tilde{v}_{a0} \\
\tilde{v}_{a+} \\
\tilde{v}_{a-}
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix}v_a \\ v_b \\ v_c \end{bmatrix}
\] (4.117)

In the above equations, \(a\) is a complex operator and it is given by \(a = e^{j \frac{2\pi}{3}}\) and \(a^2 = e^{j \frac{4\pi}{3}}\). It is to be noted that the instantaneous components of currents, \(\tilde{i}_{a+}\) and \(\tilde{i}_{a-}\) are complex time varying quantities also they are complex conjugate of each other. This same is true for \(\tilde{v}_{a+}\) and \(\tilde{v}_{a-}\) quantities. The terms \(\tilde{i}_{a0}\) and \(\tilde{v}_{a0}\) are real quantities, however (-) has been used as upper script for the sake of uniformity of notation. These instantaneous symmetrical components are used to formulate equations for load compensation. First a three-phase, four-wire system supplying star connected load is considered.

4.3.1 Compensating Star Connected Load

A three-phase four wire compensated system is shown in Fig. 4.11. In the figure, three-phase load currents \((i_{la}, i_{lb} \text{ and } i_{lc})\), can be unbalanced and nonlinear load. The objective in either three or four-wire system compensation is to provide balanced supply current such that its zero sequence component is zero. We therefore have,
\[ i_{sa} + i_{sb} + i_{sc} = 0 \] (4.118)

Using equations (4.116)-(4.117), instantaneous positive sequence voltage \((\overline{v}_{a+})\) and current \((\overline{i}_{a+})\) are computed from instantaneous values of \(v_{sa}, v_{sb}, v_{sc}\) and \(i_{sa}, i_{sb}, i_{sc}\) respectively. To have a predefined power factor from the source, the relationship between the angle of \(\overline{v}_{a+}\) and \(\overline{i}_{a+}\) is given as following.

\[ \angle \overline{v}_{a+} = \angle \overline{i}_{a+} + \phi_+ \] (4.119)

Where \(\phi_+\) is desired phase angle between \(\overline{v}_{a+}\) and \(\overline{i}_{a+}\). The above equation is rewritten as follows.

\[ \angle \left( \frac{1}{3} [v_{sa} + a v_{sb} + a^2 v_{sc}] \right) = \angle \left( \frac{1}{3} [i_{sa} + a i_{sb} + a^2 i_{sc}] \right) + \phi_+ \]

L.H.S = R.H.S

L.H.S of the above equation is expressed as below

\[
\text{L.H.S} = \angle \left[ \frac{1}{3} \left\{ v_{sa} + \left( -\frac{1}{2} + j \frac{\sqrt{3}}{2} \right) v_{sb} + \left( -\frac{1}{2} - j \frac{\sqrt{3}}{2} \right) v_{sc} \right\} \right] \\
= \angle \left[ \frac{1}{3} \left\{ \left( v_{sa} - v_{sb} \right) \left( \frac{2}{2} \right) + j \frac{\sqrt{3}}{2} \left( v_{sb} - v_{sc} \right) \right\} \right] \\
= \tan^{-1} \left( \frac{\sqrt{3}/2 (v_{sb} - v_{sc})}{(v_{sa} - v_{sb}/2 - v_{sc}/2)} \right) \\
= \tan^{-1} \frac{K_1}{K_2} \quad (4.120)
\]

Where, \(K_1 = (\sqrt{3}/2) (v_{sb} - v_{sc})\) and \(K_2 = (v_{sa} - v_{sb}/2 - v_{sc}/2)\). Similarly R.H.S of the equation is expanded as below.

\[
\text{R.H.S} = \angle \left[ \frac{1}{3} \left\{ i_{sa} + \left( -\frac{1}{2} + j \frac{\sqrt{3}}{2} \right) i_{sb} + \left( -\frac{1}{2} - j \frac{\sqrt{3}}{2} \right) i_{sc} \right\} \right] + \phi_+ \\
= \angle \left[ \frac{1}{3} \left\{ \left( i_{sa} - i_{sb} \right) \left( \frac{2}{2} \right) + j \frac{\sqrt{3}}{2} \left(i_{sb} - i_{sc} \right) \right\} \right] + \phi_+ \\
= \tan^{-1} \left( \frac{\sqrt{3}/2 (i_{sb} - i_{sc})}{(i_{sa} - i_{sb}/2 - i_{sc}/2)} \right) + \phi_+ \\
= \tan^{-1} \frac{K_3}{K_4} + \phi_+ \quad (4.121)
\]
Where, \( K_3 = \left( \frac{\sqrt{3}}{2} \right) (i_{sb} - i_{sc}) \) and \( K_4 = (i_{sa} - i_{sb}/2 - i_{sc}/2) \). Equating (4.120) and (4.121), we get the following.

\[
\tan^{-1} \frac{K_1}{K_2} = \tan^{-1} \frac{K_3}{K_4} + \phi_+
\]

Taking tangent on both sides, the following is obtained.

\[
\tan \left( \tan^{-1} \frac{K_1}{K_2} \right) = \tan \left( \tan^{-1} \frac{K_3}{K_4} + \phi_+ \right)
\]

Therefore,

\[
\frac{K_1}{K_2} = \frac{(K_3/K_4) + \tan \phi_+}{1 - (K_3/K_4) \times \tan \phi_+}
\]

The above equation implies that,

\[
K_1 K_4 - K_1 K_3 \tan \phi_+ - K_2 K_3 - K_2 K_4 \tan \phi_+ = 0
\]

Substituting the values of \( K_1, K_2, K_3, K_4 \) in the above equation, the following expression is obtained.

\[
\frac{\sqrt{3}}{2} \left( v_{sb} - v_{sc} \right) \left( i_{sa} - \frac{i_{sb} - i_{sc}}{2} \right) - \frac{3}{4} \left( v_{sb} - v_{sc} \right) \left( i_{sb} - i_{sc} \right) \tan \phi_+ \\
-\frac{\sqrt{3}}{2} \left( v_{sa} - \frac{v_{sb} - v_{sc}}{2} \right) \left( i_{sb} - i_{sc} \right) - \left( v_{sa} - \frac{v_{sb} - v_{sc}}{2} \right) \left( i_{sa} - \frac{i_{sb} - i_{sc}}{2} \right) \tan \phi_+ = 0
\]

Above equation can be arranged with terms associated with \( i_{sa}, i_{sb} \) and \( i_{sc} \). This is given below.

\[
\left\{ \frac{\sqrt{3}}{2} \left( v_{sb} - v_{sc} \right) + \frac{\tan \phi_+}{2} \left( v_{sb} + v_{sc} - 2 v_{sa} \right) \right\} i_{sa} \\
+ \left\{ \frac{\sqrt{3}}{2} \left( v_{sc} - v_{sa} \right) + \frac{\tan \phi_+}{2} \left( v_{sc} + v_{sa} - 2 v_{sb} \right) \right\} i_{sb} \\
+ \left\{ \frac{\sqrt{3}}{2} \left( v_{sa} - v_{sb} \right) + \frac{\tan \phi_+}{2} \left( v_{sa} + v_{sb} - 2 v_{sc} \right) \right\} i_{sc} = 0
\]

Dividing above equation by \( \frac{\sqrt{3}}{2} \), it can be written as follows.

\[
\left\{ \left( v_{sb} - v_{sc} \right) + \frac{\tan \phi_+}{\sqrt{3}} \left( v_{sb} + v_{sc} - 2 v_{sa} \right) \right\} i_{sa} \\
+ \left\{ \left( v_{sc} - v_{sa} \right) + \frac{\tan \phi_+}{\sqrt{3}} \left( v_{sc} + v_{sa} - 2 v_{sb} \right) \right\} i_{sb} \\
+ \left\{ \left( v_{sa} - v_{sb} \right) + \frac{\tan \phi_+}{\sqrt{3}} \left( v_{sa} + v_{sb} - 2 v_{sc} \right) \right\} i_{sc} = 0
\]
Assume \( \beta = \tan \phi \sqrt{3} \), the above equation is further simplified to,

\[
\left\{ (v_{sb} - v_{sc}) + \beta (v_{sb} + v_{sc} - 2 v_{sa}) \right\} i_{sa}
+ \left\{ (v_{sc} - v_{sa}) + \beta (v_{sc} + v_{sa} - 2 v_{sb}) \right\} i_{sb}
+ \left\{ (v_{sa} - v_{sb}) + \beta (v_{sa} + v_{sb} - 2 v_{sc}) \right\} i_{sc} = 0.
\]

Adding and subtracting \( v_{sa}, v_{sb} \) and \( v_{sc} \) in \( \beta \) terms and expressing \( v_{s0} = 3(v_{sa} + v_{sb} + v_{sc}) \) in above equation, we get the following.

\[
\left\{ (v_{sb} - v_{sc}) - 3 \beta (v_{sa} - v_{s0}) \right\} i_{sa}
+ \left\{ (v_{sc} - v_{sa}) - 3 \beta (v_{sb} - v_{s0}) \right\} i_{sb}
+ \left\{ (v_{sa} - v_{sb}) - 3 \beta (v_{sc} - v_{s0}) \right\} i_{sc} = 0.
\]

The third objective of compensation is that the power supplied from the source \( p_s \) must be equal to average load power \( P_{lavg} \). Therefore the following holds true.

\[
p_s = v_{sa} i_{sa} + v_{sb} i_{sb} + v_{sc} i_{sc} = P_{lavg}
\]

The above equation has important implications. For example when supply voltage are balanced, the above equation is satisfied for balanced source currents. However if supply voltage are unbalanced and distorted, above equation gives set of currents which are also not balanced and sinusoidal in order to supply constant power.

Equations (4.118), (4.124) and (4.123), can be written in matrix form as given below.

\[
\begin{bmatrix}
(v_{sb} - v_{sc}) + \beta (v_{sb} + v_{sc} - 2 v_{sa}) & (v_{sc} - v_{sa}) + \beta (v_{sc} + v_{sa} - 2 v_{sb}) & (v_{sa} - v_{sb}) + \beta (v_{sa} + v_{sb} - 2 v_{sc})
\end{bmatrix}
\begin{bmatrix}
i_{sa} \\
i_{sb} \\
i_{sc}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
P_{lavg}
\end{bmatrix}
\]

Which can be further written as,

\[
[A] [i_{sabc}] = [P_{lavg}]
\]

Therefore,

\[
[i_{sabc}] = [A^{-1}] [P_{lavg}] = \frac{1}{\Delta A} \begin{bmatrix}
a_{c11} & a_{c12} & a_{c13} \\
a_{c21} & a_{c22} & a_{c23} \\
a_{c31} & a_{c32} & a_{c33}
\end{bmatrix}^T \begin{bmatrix}
0 \\
0 \\
P_{lavg}
\end{bmatrix}
= \frac{1}{\Delta A} \begin{bmatrix}
a_{c11} & a_{c21} & a_{c31} \\
a_{c12} & a_{c22} & a_{c32} \\
a_{c13} & a_{c23} & a_{c33}
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
P_{lavg}
\end{bmatrix}
\]

Where \( a_{cij} \) is the cofactor of \( i^{th} \) row and \( j^{th} \) column of \( A \) matrix in (4.125) and \( \Delta A \) is the determinant of matrix \( A \). Due to the presence of zero elements in first two rows of column vector with
power elements, the cofactors in first two columns need not to be computed. These are indicated by dots in the following matrix.

\[
\begin{bmatrix}
i_{sabc} \\
i_{sa} \\
i_{sb} \\
i_{sc}
\end{bmatrix} = \frac{1}{\Delta A} \begin{bmatrix}
a_{c31} & \cdot & \cdot & a_{c31} \\
\cdot & a_{c32} & \cdot & \cdot \\
\cdot & \cdot & a_{c33} & \cdot
\end{bmatrix} \begin{bmatrix}
0 \\
0
\end{bmatrix} = \frac{1}{\Delta A} \begin{bmatrix}
a_{c31} \\
a_{c32} \\
a_{c33}
\end{bmatrix} P_{lavg}
\]

The determinant of matrix \( A \) is computed as below.

\[
\Delta_A = [(v_{sc} - v_{sa}) + \beta (v_{sc} + v_{sa} - 2v_{sb})]v_{sc} - [(v_{sa} - v_{sb}) + \beta (v_{sa} + v_{sb} - 2v_{sc})]v_{sb}
- [(v_{sb} - v_{sc}) + \beta (v_{sb} + v_{sc} - 2v_{sa})]v_{sc} + [(v_{sa} - v_{sb}) + \beta (v_{sa} + v_{sb} - 2v_{sc})]v_{sa}
+ [(v_{sb} - v_{sc}) + \beta (v_{sb} + v_{sc} - 2v_{sa})]v_{sb} - [(v_{sc} - v_{sa}) + \beta (v_{sc} + v_{sa} - 2v_{sb})]v_{sa}
\]

\[
= \beta [v_{sc}^2 + v_{sa}v_{sc} - 2v_{sb}v_{sc} - v_{sa}v_{sc} - v_{sb}^2 + 2v_{sb}v_{sc} - v_{sc}v_{sa} - v_{sa}^2 + 2v_{sa}v_{sc} + v_{sa}^2 + v_{sc}^2]
+ v_{sa}^2 + v_{sb}v_{sa} - 2v_{sc}v_{sa} + v_{sc}^2 + v_{sb}v_{sc} - v_{sc}v_{sa} - v_{sa}^2 + 2v_{sa}v_{sc}
+ (v_{sc} - v_{sa})v_{sc} - (v_{sa} - v_{sb})v_{sb} - (v_{sb} - v_{sc})v_{sc}
+ (v_{sa} - v_{sb})v_{sa} + (v_{sb} - v_{sc})v_{sb} - (v_{sc} - v_{sa})v_{sa}
\]

The above equation can be further simplified to,

\[
\Delta_A = \beta \cdot 0 + v_{sc}^2 - v_{sa}v_{sc} - v_{sb}v_{sa} + v_{sb}^2 - v_{sa}v_{sc} + v_{sc}^2 + v_{sa}^2
- v_{sa}v_{sb} + v_{sb}^2 - v_{sc}v_{sa} - v_{sa}v_{sc} + v_{sa}^2
= 2v_{sa}^2 + 2v_{sb}^2 + 2v_{sc}^2 - 2v_{sa}v_{sc} - 2v_{sb}v_{sc} - 2v_{sa}v_{sc}
= v_{sa}^2 + v_{sb}^2 - 2v_{sa}v_{sb} + v_{sc}^2 - 2v_{sb}v_{sc} + v_{sa}^2 + v_{sc}^2 - 2v_{sa}v_{sc}
= (v_{sa} - v_{sb})^2 + (v_{sa} - v_{sc})^2 + (v_{sc} - v_{sa})^2
= (v_{sa} + v_{sb} + v_{sc})^2
\]

Further adding and subtracting, \( v_{sa}^2 + v_{sb}^2 + v_{sc}^2 \) in above equation, we get the following.

\[
\Delta_A = 3(v_{sa}^2 + v_{sb}^2 + v_{sc}^2) - (v_{sa}^2 + v_{sb}^2 + v_{sc}^2 + 2v_{sa}v_{sb} + 2v_{sb}v_{sc} + 2v_{sc}v_{sa}) = (4.126)
\]

Further,

\[
(v_{sa} + v_{sb} + v_{sc})^2 = v_{sa}^2 + v_{sb}^2 + v_{sc}^2 + 2v_{sa}v_{sb} + 2v_{sb}v_{sc} + 2v_{sc}v_{sa} = (4.127)
\]

Using equations (4.126) and (4.127), we obtain the following.

\[
\Delta_A = 3(v_{sa}^2 + v_{sb}^2 + v_{sc}^2) - (v_{sa} + v_{sb} + v_{sc})^2
= 3 \sum_{j=a,b,c} v_{s}^2 - 9v_{s0}^2
= 3 \sum_{j=a,b,c} v_{s}^2 - 3v_{s0}^2 = (4.128)
\]

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In above equation, the term \( v_{so} \) is the instantaneous zero sequence component of the source voltage and it is given as following.

\[
v_{so} = \frac{(v_{sa} + v_{sb} + v_{sc})}{3}
\]  

(4.129)

The determinant of matrix \( A \), using (4.128), can also be expressed as,

\[
\Delta_A = 3 \left[ v_{sa}^2 + v_{sb}^2 + v_{sc}^2 - 3 v_{so} \frac{(v_{sa} + v_{sb} + v_{sc})}{3} \right] = 3 \left[ v_{sa}^2 + v_{sb}^2 + v_{sc}^2 + 3v_{so}^2 - 2 \times 3v_{so}^2 \right] = 3\left[ v_{sa}^2 + v_{sb}^2 + v_{sc}^2 + v_{so}^2 + v_{so}^2 - 2v_{so}(v_{sa} + v_{sb} + v_{sc}) \right] = 3\left[ (v_{sa}^2 + v_{so}^2 - 2v_{sa}v_{so}) + (v_{sb}^2 + v_{so}^2 - 2v_{sb}v_{so}) + (v_{sc}^2 + v_{so}^2 - 2v_{sc}v_{so}) \right] = 3\left[ (v_{sa}^2 - v_{so}^2) + (v_{sb}^2 - v_{so}^2) + (v_{sc}^2 - v_{so}^2) \right]
\]

The cofactors \( a_{c31} \), \( a_{c32} \), and \( a_{c33} \) are computed as below.

\[
a_{c31} = - (v_{sc} - v_{sa}) - \beta(v_{sc} + v_{sa} - 2v_{sb}) + (v_{sa} - v_{sb}) + \beta(v_{sa} + v_{sb} - 2v_{sc}) = (2v_{sa} - v_{sb} - v_{sc}) + \beta(-v_{sc} - v_{sa} + 2v_{sb} + v_{sa} + v_{sb} - 2v_{sc}) = (2v_{sa} + v_{sa} - v_{sc} - v_{sc} - v_{sa}) + 3\beta(v_{sb} - v_{sc}) = 3(v_{sa} - v_{sa}) + 3\beta(v_{sb} - v_{sc})
\]

Similarly,

\[
a_{c32} = -(v_{sa} - v_{sb}) - \beta(v_{sa} + v_{sb} - 2v_{sc}) + (v_{sb} - v_{sc}) + \beta(v_{sb} + v_{sc} - 2v_{sa}) = (2v_{sb} - v_{sc} - v_{sa}) + \beta(-v_{sa} - v_{sb} + 2v_{sc} + v_{sb} + v_{sc} - 2v_{sa}) = 3(v_{sb} - v_{sc}) + 3\beta(v_{sc} - v_{sa})
\]

And,

\[
a_{c33} = (v_{sc} - v_{sa}) + \beta(v_{sc} + v_{sa} - 2v_{sb}) - (v_{sb} - v_{sc}) - \beta(v_{sb} + v_{sc} - 2v_{sa}) = 3(v_{sc} - v_{sa}) + 3\beta(v_{sc} - v_{sa})
\]

Knowing the value of cofactors, we now have,

\[
\begin{bmatrix}
  i_{sa} \\
  i_{sb} \\
  i_{sc}
\end{bmatrix} = \frac{1}{3 \sum_{j=a,b,c} v_{sj}^2 - 3v_{so}^2} \begin{bmatrix}
  3(v_{sa} - v_{so}) + 3\beta(v_{sb} - v_{sc}) \\
  3(v_{sb} - v_{so}) + 3\beta(v_{sc} - v_{sa}) \\
  3(v_{sc} - v_{so}) + 3\beta(v_{sa} - v_{sb})
\end{bmatrix} [P_{lavg}]
\]

\[
\begin{bmatrix}
  i_{sa} \\
  i_{sb} \\
  i_{sc}
\end{bmatrix} = \frac{1}{3 \sum_{j=a,b,c} v_{sj}^2 - 3v_{so}^2} \begin{bmatrix}
  (v_{sa} - v_{so}) + \beta(v_{sb} - v_{sc}) \\
  (v_{sb} - v_{so}) + \beta(v_{sc} - v_{sa}) \\
  (v_{sc} - v_{so}) + \beta(v_{sa} - v_{sb})
\end{bmatrix} [P_{lavg}]
\]

(4.130)
From the above equation, the desired source currents can be written as following.

\[
i_{sa} = \frac{(v_{sa} - v_{so}) + \beta(v_{sb} - v_{sc})}{\sum_{j=a,b,c} v_{sj}^2 - 3 v_{so}^2} P_{avg} \tag{4.131}
\]

\[
i_{sb} = \frac{(v_{sb} - v_{so}) + \beta(v_{sc} - v_{sa})}{\sum_{j=a,b,c} v_{sj}^2 - 3 v_{so}^2} P_{avg} \tag{4.132}
\]

\[
i_{sc} = \frac{(v_{sc} - v_{so}) + \beta(v_{sa} - v_{sb})}{\sum_{j=a,b,c} v_{sj}^2 - 3 v_{so}^2} P_{avg} \tag{4.133}
\]

Applying Kirchoff’s current law at the point of common coupling (PCC), we have,

\[
i_{fa}^* = i_{la} - i_{sa} \tag{4.134}
\]

\[
i_{fb}^* = i_{lb} - i_{sb} \tag{4.135}
\]

\[
i_{fc}^* = i_{lc} - i_{sc} \tag{4.136}
\]

Replacing \(i_{sa}, i_{sb}\) and \(i_{sc}\) from equations (4.131)-(4.133), we obtain the reference filter currents as given in the following.

\[
i_{fa}^* = i_{la} - i_{sa} = \frac{(v_{sa} - v_{so}) + \beta(v_{sb} - v_{sc})}{\sum_{j=a,b,c} v_{sj}^2 - 3 v_{so}^2} P_{avg} \tag{4.137}
\]

\[
i_{fb}^* = i_{lb} - i_{sb} = \frac{(v_{sb} - v_{so}) + \beta(v_{sc} - v_{sa})}{\sum_{j=a,b,c} v_{sj}^2 - 3 v_{so}^2} P_{avg} \tag{4.138}
\]

\[
i_{fc}^* = i_{lc} - i_{sc} = \frac{(v_{sc} - v_{so}) + \beta(v_{sa} - v_{sb})}{\sum_{j=a,b,c} v_{sj}^2 - 3 v_{so}^2} P_{avg} \tag{4.139}
\]

### 4.3.2 Compensating Delta Connected Load

The balancing of an unbalanced \(\Delta\)-connected load is a generic problem and the theory of instantaneous symmetrical components can be used to balance the load. The schematic diagram of this compensated scheme is shown in Fig. 4.12. This compensator is connected between the phasors of

![Fig. 4.12 A compensation for delta connected load](image-url)

this scheme. The aim is to generate the three reference current waveforms denoted by \(i_{fa}^*, i_{fb}^*, i_{fc}^*\).
Applying Kirchoff's current law at nodes, we can express

\[ i_{sa} + i_{sb} + i_{sc} = 0 \]  \hspace{1cm} (4.140)

Applying Kirchoff's current law at nodes, we can express \( i_{sa}, i_{sb} \) and \( i_{sc} \) respectively as following.

\[ i_{sa} = (i_{lab} - i_{faba}^{*}) - (i_{fca} - i_{fca}^{*}) \]
\[ i_{sb} = (i_{lbc} - i_{fba}^{*}) - (i_{lab} - i_{fca}^{*}) \]  \hspace{1cm} (4.141)
\[ i_{sc} = (i_{lca} - i_{fca}^{*}) - (i_{lbc} - i_{fba}^{*}) \]

As can be seen from above equations, equation (4.140) is satisfied. Since in \( \Delta \) connected load, zero sequence current cannot flow, therefore

\[ (i_{lab} - i_{faba}^{*}) + (i_{lbc} - i_{fba}^{*}) + (i_{lca} - i_{fca}^{*}) = 0 \]  \hspace{1cm} (4.142)

The source supplies the average load power, \( P_{lavg} \) and following equation is satisfied.

\[ v_{sa}i_{sa} + v_{sb}i_{sb} + v_{sc}i_{sc} = P_{lavg} \]  \hspace{1cm} (4.143)

From equation (4.123), the power factor between the source voltages and currents should be met. Thus we have,

\[ \{(v_{sb} - v_{sc}) - 3\beta(v_{sa} - v_{s0})\}i_{sa} \]
\[ +\{(v_{sc} - v_{sa}) - 3\beta(v_{sb} - v_{s0})\}i_{sb} \]
\[ +\{(v_{sa} - v_{sb}) - 3\beta(v_{sc} + v_{s0})\}i_{sc} = 0. \]  \hspace{1cm} (4.144)

Replacing \( i_{sa}, i_{sb} \) and \( i_{sc} \) from (4.141), above equation can be simplified to the following.

\[ (v_{sb} - v_{sc}) - 3\beta(v_{sa} - v_{s0})[(i_{lab} - i_{faba}^{*}) - (i_{lca} - i_{fca}^{*})] \]
\[ +(v_{sc} - v_{sa}) - 3\beta(v_{sb} - v_{s0})[(i_{lbc} - i_{fba}^{*}) - (i_{lab} - i_{fca}^{*})] \]
\[ +(v_{sa} - v_{sb}) - 3\beta(v_{sc} - v_{s0})[(i_{lca} - i_{fca}^{*}) - (i_{lbc} - i_{fba}^{*})] = 0 \]  \hspace{1cm} (4.145)

Simplifying above expression we get

\[ \begin{align*}
&\left\{(v_{sb} - v_{sc}) + \beta(v_{sa} - v_{sc} - 2v_{sa})\right\}[i_{lab} - i_{faba}^{*}] \\
&+\left\{(v_{sc} - v_{sa}) + \beta(v_{sa} + v_{sc} - 2v_{sa})\right\}[i_{lbc} - i_{fba}^{*}] \\
&+\left\{(v_{sa} - v_{sb}) + \beta(v_{sa} - v_{sc} - 2v_{sa})\right\}[i_{lca} - i_{fca}^{*}] \\
\end{align*} \]  \hspace{1cm} (4.146)

The first term I is as follows:

\[ I = \{(v_{sb} - v_{sc} - v_{sa}) + \beta(v_{sa} + v_{sc} - 2v_{sa} - v_{sc} - v_{sa} + 2v_{sb})\} (i_{lab} - i_{faba}^{*}) \]
\[ = \{(v_{sa} + v_{sb} - 2v_{sc}) - 3\beta(v_{sa} - v_{sb})\} (i_{lab} - i_{faba}^{*}) \]
\[ = -3 \{(v_{sc} - v_{sa}) + \beta(v_{sa} - v_{sb})\} (i_{lab} - i_{faba}^{*}) \]  \hspace{1cm} (4.147)
Similarly, the second and third terms are given as below.

\[
\text{II} = \{(v_{sc} - v_{sa} - v_{sb} + v_{sb}) + \beta (v_{sc} + v_{sa} - 2v_{sb} - v_{sa} + 2v_{sc})\} (i_{ibc} - i_{fbc}^*)
\]
\[
= \{(v_{sb} + v_{sc} - 2v_{sa}) - 3\beta (v_{sb} - v_{sc})\} (i_{ibc} - i_{fbc}^*)
\]
\[
= -3 \{(v_{sa} - v_{sb}) + \beta (v_{sb} - v_{sc})\} (i_{ibc} - i_{fbc}^*) \quad (4.148)
\]

\[
\text{III} = \{(v_{sa} - v_{sb} - v_{sb} + v_{sc}) + \beta (v_{sa} + v_{sb} - 2v_{sc} - v_{sb} - v_{sc} + 2v_{sa})\} (i_{ica} - i_{fca}^*)
\]
\[
= \{(v_{sc} + v_{sa} - 2v_{sb}) - 3\beta (v_{sc} - v_{sa})\} (i_{ica} - i_{fca}^*)
\]
\[
= -3 \{(v_{sb} - v_{sa}) + \beta (v_{sc} - v_{sa})\} (i_{ica} - i_{fca}^*) \quad (4.149)
\]

Summing above three terms and simplifying we get,

\[
\{(v_{sc} - v_{sb}) + \beta (v_{sa} - v_{sb})\} (i_{lab} - i_{fab}^*)
\]
\[+ \{(v_{sa} - v_{sb}) + \beta (v_{sb} - v_{sc})\} (i_{ibc} - i_{fbc}^*)
\]
\[+ \{(v_{sb} - v_{sa}) + \beta (v_{sc} - v_{sa})\} (i_{ica} - i_{fca}^*) = 0 \quad (4.150)
\]

The third condition for load compensation ensures that the average load power should be supplied from the sources. Therefore,

\[
v_{sa}i_{sa} + v_{sb}i_{sb} + v_{sc}i_{sc} = P_{lavg} \quad (4.151)
\]

The terms \(i_{sa}, i_{sb}\) and \(i_{sc}\) are substituted from (4.141) in the above equation and the modified equation is given below.

\[
v_{sa} \{(i_{lab} - i_{fab}^*) - (i_{ica} - i_{fca}^*)\} + v_{sb} \{(i_{ibc} - i_{fbc}^*) - (i_{lab} - i_{fab}^*)\}
\]
\[+ v_{sc} \{(i_{ica} - i_{fca}^*) - (i_{ibc} - i_{fbc}^*)\} = 0 \quad (4.152)
\]

The above is simplified to,

\[
(v_{sa} - v_{sb})(i_{lab} - i_{fab}^*) + (v_{sb} - v_{sc})(i_{ibc} - i_{fbc}^*) + (v_{sc} - v_{sa})(i_{ica} - i_{fca}^*) = 0 \quad (4.154)
\]

Equations (4.142), (4.150), (4.154) can be written in the matrix form as given below.

\[
\begin{bmatrix}
1 & 1 & 1 \\
(v_{sc} - v_{sa}) + \beta (v_{sa} - v_{sb}) & (v_{sa} - v_{sb}) + \beta (v_{sb} - v_{sc}) & (v_{sb} - v_{sa}) + \beta (v_{sc} - v_{sa})
\end{bmatrix}
\begin{bmatrix}
i_{lab} - i_{fab}^* \\
i_{ibc} - i_{fbc}^* \\
i_{ica} - i_{fca}^*
\end{bmatrix}
\]
\[= P_{lavg} \quad (4.155)
\]

The above equation can be written in the following form.

\[
[A_{\Delta}]
\begin{bmatrix}
i_{lab} - i_{fab}^* \\
i_{ibc} - i_{fbc}^* \\
i_{ica} - i_{fca}^*
\end{bmatrix}
= P_{lavg} \quad (4.156)
\]

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Therefore,

\[
\begin{bmatrix}
\ell_{ab} - \ell_{f_{ab}}^* \\
\ell_{bc} - \ell_{f_{bc}}^* \\
\ell_{ca} - \ell_{f_{ca}}^*
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
0 \\
F_{\text{avg}}
\end{bmatrix}
\] (4.157)

The above equation is solved by finding the determinate of \( A_{\Delta} \) and the cofactors transpose as given below.

\[
\begin{bmatrix}
\ell_{ab} - \ell_{f_{ab}}^* \\
\ell_{bc} - \ell_{f_{bc}}^* \\
\ell_{ca} - \ell_{f_{ca}}^*
\end{bmatrix} = \frac{1}{|A_{\Delta}|} \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}^{T} \begin{bmatrix}
0 \\
0 \\
F_{\text{avg}}
\end{bmatrix}
\] (4.158)

The determinant \(|A_{\Delta}|\) and cofactors in above equation are calculated below.

\[
|A_{\Delta}| = [(v_{sa} - v_{sa})(v_{sc} - v_{sa}) - (v_{sa} - v_{so})(v_{sb} - v_{sc}) - (v_{sc} - v_{sa})(v_{sb} - v_{sc})
- [(v_{sc} - v_{so}) + \beta (v_{sb} - v_{sc})] (v_{sb} - v_{sa}) + [(v_{sb} - v_{so}) + \beta (v_{sc} - v_{sa})] (v_{sa} - v_{sb})
+ [(v_{sc} - v_{sa}) + \beta (v_{sa} - v_{sb})] (v_{sb} - v_{sc}) - [(v_{sa} - v_{so}) + \beta (v_{sb} - v_{sc})] (v_{sa} - v_{sb})
\]

Separating all the terms containing \( \beta \) and rearranging the above equation, we get.

\[
|A_{\Delta}| = (v_{sa} - v_{so})(v_{sc} - v_{sa} - v_{sa} + v_{sc}) + (v_{sb} - v_{so})(v_{sa} - v_{sc} + v_{sc} - v_{sb})
+ (v_{sc} - v_{so})(v_{sc} - v_{sa} + v_{sb} - v_{sc}) + \beta \times 0
\] (4.159)

In the above, using \( v_{sa} + v_{sc} + v_{sb} = 3v_{so} \),

\[
v_{sa} - v_{so} - v_{sa} + v_{sb} = v_{sa} + v_{sb} + v_{sc} - 3v_{sa} = -3(v_{sa} - v_{so})
\]

\[
-v_{sb} + v_{sa} + v_{sc} - v_{sb} = v_{sa} + v_{sb} + v_{sc} - 3v_{sb} = -3(v_{sb} - v_{so})
\]

\[
-v_{sc} + v_{sa} + v_{sb} - v_{sc} = v_{sa} + v_{sb} + v_{sc} - 3v_{sc} = -3(v_{sc} - v_{so})
\]

Replacing above term in (4.159), we get the following.

\[
A_{\Delta} = -3[(v_{sa} - v_{so})(v_{sa} - v_{so}) + (v_{sb} - v_{so})(v_{sb} - v_{so}) + (v_{sc} - v_{so})(v_{sc} - v_{so})]
= -3[(v_{sa} - v_{so})^2 + (v_{sb} - v_{so})^2 + (v_{sc} - v_{so})^2]
= -3 \sum_{j=a,b,c} (v_{sj} - v_{so})^2
\] (4.160)
The above equation can also be written as,

$$|A\Delta| = -3 \left[ v_{sa}^2 + v_{sb}^2 + v_{sc}^2 + 3 v_{s0}^2 - 2 v_{s0} (v_{sa} + v_{sb} + v_{sc}) \right]$$

$$= -3 \left[ v_{sa}^2 + v_{sb}^2 + v_{sc}^2 + 3 v_{s0}^2 - 6 v_{s0}^2 \right]$$

$$= -3 \left[ v_{sa}^2 + v_{sb}^2 + v_{sc}^2 - 3 v_{s0}^2 \right]$$

$$= -3 \left( \sum_{j=a,b,c} v_{sj}^2 - 3 v_{s0}^2 \right)$$

(4.161)

**Calculation of cofactors of** $A\Delta$

We need to calculate $a_{c31}$, $a_{c32}$ and $a_{c33}$. These are computed as following.

$$a_{c31} = \left[ (v_{sb} - v_{so}) + \beta(v_{sc} - v_{sa}) - (v_{sa} - v_{so}) + \beta(v_{sb} - v_{sc}) \right]$$

$$= \left[ (v_{sb} - v_{sa}) + \beta(v_{sa} + v_{sb} - 2 v_{sc}) \right]$$

$$= \left[ (v_{sb} - v_{sa}) - \beta(v_{sa} + v_{sb} + v_{sc} - 3 v_{sc}) \right]$$

$$= [v_{sbc} - 3 \beta(v_{sa} - v_{so})]$$

Similarly,

$$a_{c32} = \left[ (v_{sb} - v_{so}) + \beta(v_{sc} - v_{sa}) - (v_{sc} - v_{so}) + \beta(v_{sa} - v_{sb}) \right]$$

$$= \left[ (v_{sb} - v_{sc}) + \beta(v_{sc} + v_{sb} - 2 v_{sa}) \right]$$

$$= \left[ (v_{sb} - v_{sc}) + \beta(v_{sa} + v_{sb} + v_{sc} - 3 v_{sa}) \right]$$

$$= [v_{sbc} - 3 \beta(v_{sa} - v_{so})]$$

and

$$a_{c33} = \left[ (v_{sa} - v_{so}) + \beta(v_{sb} - v_{sc}) - (v_{sc} - v_{so}) + \beta(v_{sa} - v_{sb}) \right]$$

$$= \left[ (v_{sa} - v_{sc}) - \beta(v_{sa} + v_{sc} - 2 v_{sb}) \right]$$

$$= \left[ (v_{sa} - v_{sc}) - \beta(v_{sa} + v_{sb} + v_{sc} - 3 v_{sb}) \right]$$

$$= [v_{sca} - 3 \beta(v_{sb} - v_{so})]$$

Therefore, the solution of the equation is given by,

$$\begin{bmatrix}
  i_{lab} - i_{f_{lab}}^* \\
  i_{lbc} - i_{f_{lbc}}^* \\
  i_{lca} - i_{f_{lca}}^*
\end{bmatrix} = \frac{1}{A\Delta} \begin{bmatrix}
  a_{c11} & a_{c12} & a_{c13} \\
  a_{c21} & a_{c22} & a_{c23} \\
  a_{c31} & a_{c32} & a_{c33}
\end{bmatrix}^T \begin{bmatrix}
  0 \\
  0 \\
  P_{\text{avg}}
\end{bmatrix}$$

(4.162)
From the above equation and substituting the values of cofactors obtained above, we get the following.

\[ i_{lab} - i^*_{fab} = \frac{a_{c31}}{|A_3|} P_{avg} \]

\[ = - \left[ v_{sab} - 3\beta(v_{sc} - v_{s0}) \right] \frac{P_{avg}}{-3 \sum_{j=a,b,c} v_{sij}^2 - 3 v_{s0}^2} \]

\[ = \frac{[v_{sab} / 3 - \beta(v_{sc} - v_{s0})]}{\sum_{j=a,b,c} v_{sij}^2 - 3 v_{s0}^2} P_{avg} \]

From the above equation, the reference compensator current \((i^*_{fab})\) can be given as follows.

\[ i^*_{fab} = i_{lab} - \frac{v_{sab}}{3} - \beta(v_{sc} - v_{s0}) \sum_{j=a,b,c} v_{sij}^2 - 3 v_{s0}^2 P_{avg} \] (4.163)

Similarly,

\[ i^*_{fbc} = i_{lbc} - \frac{v_{sbc}}{3} - \beta(v_{sa} - v_{s0}) \sum_{j=a,b,c} v_{sij}^2 - 3 v_{s0}^2 P_{avg} \] (4.164)

and

\[ i^*_{fca} = i_{lca} - \frac{v_{sca}}{3} - \beta(v_{sb} - v_{s0}) \sum_{j=a,b,c} v_{sij}^2 - 3 v_{s0}^2 P_{avg} \] (4.165)

When the source power factor is unity, \(\beta = 0\), for balanced source voltages (fundamental) \(v_{s0} = 0\). Substituting these values in above equations, we get,

\[ i^*_{fab} = i_{lab} - \frac{v_{sab}}{3} \sum_{j=a,b,c} v_{sij}^2 P_{avg} \]

\[ i^*_{fbc} = i_{lbc} - \frac{v_{sbc}}{3} \sum_{j=a,b,c} v_{sij}^2 P_{avg} \] (4.166)

\[ i^*_{fca} = i_{lca} - \frac{v_{sca}}{3} \sum_{j=a,b,c} v_{sij}^2 P_{avg} \]

Further it can be seen that,

\[ v_{sab}^2 + v_{sbc}^2 + v_{sca}^2 = (v_{sa} - v_{sb})^2 + (v_{sb} - v_{sc})^2 + (v_{sc} - v_{sa})^2 \]

\[ = v_{sa}^2 + v_{sb}^2 - 2v_{sa}v_{sb} + v_{sc}^2 + v_{sa}^2 - 2v_{sb}v_{sc} + v_{sc}^2 + v_{sa}^2 - 2v_{sc}v_{sa} \]

\[ = 2(v_{sa}^2 + v_{sb}^2 + v_{sc}^2) - (2v_{sa}v_{sb} + 2v_{sb}v_{sc} + 2v_{sc}v_{sa}) \]

\[ = 3(v_{sa}^2 + v_{sb}^2 + v_{sc}^2) - (v_{sa}^2 + v_{sb}^2 + v_{sc}^2 + 2v_{sa}v_{sb} + 2v_{sb}v_{sc} + 2v_{sc}v_{sa}) \]

\[ = 3(v_{sa}^2 + v_{sb}^2 + v_{sc}^2) - (v_{sa}^2 + v_{sb}^2 + v_{sc}^2)^2 \]

\[ = 3[(v_{sa}^2 + v_{sb}^2 + v_{sc}^2) - 3 v_{s0}^2] \] (4.167)
Replacing $3 \left[ \sum_{j=a,b,c} v_{sj}^2 - 3 v_{s0}^2 \right]$ in equations (4.163), (4.164) and (4.165), we get the following.

\[
\begin{align*}
  i_{fab}^* &= i_{lab} - \frac{v_{sab} - 3\beta (v_{sc} - v_{s0})}{3(v_{sab}^2 + v_{sbc}^2 + v_{sca}^2)} P_{lavg} \\
  i_{fbc}^* &= i_{lbc} - \frac{v_{sbc} - 3\beta (v_{sa} - v_{s0})}{3(v_{sab}^2 + v_{sbc}^2 + v_{sca}^2)} P_{lavg} \\
  i_{fca}^* &= i_{lca} - \frac{v_{sca} - 3\beta (v_{sb} - v_{s0})}{3(v_{sab}^2 + v_{sbc}^2 + v_{sca}^2)} P_{lavg}
\end{align*}
\]

(4.168)

For unity power factor and balanced source voltages (fundamental), the reference compensator currents are given as following.

\[
\begin{align*}
  i_{fab}^* &= i_{lab} - \frac{v_{sab}}{27V^2} P_{lavg} \\
  i_{fbc}^* &= i_{lbc} - \frac{v_{sbc}}{27V^2} P_{lavg} \\
  i_{fca}^* &= i_{lca} - \frac{v_{sca}}{27V^2} P_{lavg}
\end{align*}
\]

(4.169)

Where $V$ is the rms value of phase voltages.

**References**


Chapter 5

SERIES COMPENSATION: VOLTAGE COMPENSATION USING DVR
(Lectures 36-44)

5.1 Introduction

Power system should ensure good quality of electric power supply, which means voltage and current waveforms should be balanced and sinusoidal. Furthermore, the voltage levels on the system should be within reasonable limits, generally within $100 \pm 5\%$ of their rated value. If the voltage is more or less than this pre-specified value, performance of equipments is sacrificed. In case of low voltages, picture on television starts rolling, the torque of induction motor reduces to the square of voltage and therefore there is need for voltage compensation.

5.2 Conventional Methods to Regulate Voltage

In order to keep load bus voltage constant, many conventional compensating devices such as listed below can be used. In general, these can be referred as VAR compensator.

1. Shunt Capacitors
2. Series Capacitors
3. Synchronous Capacitor
4. Tap Changing Transformer
5. Booster Transformer
6. Static Synchronous Series Capacitor
7. Dynamic Voltage Restorer
The first six methods are employed at transmission level while the last method is by employing Dynamic Voltage Restorer (DVR), is mostly employed in power distribution network to protect any voltage variation at the load bus connected to the sensitive and critical electrical units. The DVR is a series connected custom power device used to mitigate the voltage unbalance, sags, swells, harmonics and any abrupt changes due to abnormal conditions in the system. In the following section, dynamic voltage restorer will be described in detail.

5.3 Dynamic Voltage Restorer (DVR)

A dynamic voltage restorer (DVR) is a solid state inverter based on injection of voltage in series with a power distribution system [1], [2]. The DC side of DVR is connected to an energy source or an energy storage device, while its ac side is connected to the distribution feeder by a three-phase interfacing transformer. A single line diagram of a DVR connected power distribution system is shown in the figure below. In this figure, \(v_s(t)\) represents supply voltage, \(v_t(t)\) represents terminal voltage and \(v_l(t)\) represents the load voltage. Since DVR is a series connected device, the source current, \(i_s(t)\) is same as load current, \(i_l(t)\). Also note in the figure, \(v_f(t)\) is DVR injected voltage in series with line such that the load voltage is maintained at sinusoidal nominal value.

![Fig. 5.1 A single-line diagram of DVR compensated system](image)

The three-phase DVR compensated system is shown in Fig. 5.2 below. It is assumed that the transmission line has same impedance in all three phases. A DVR unit which is represented in Fig. 5.1, have following components [3]- [5].

1. Voltage Source Inverter
2. Filter capacitors and inductors
3. Injection transformer
4. DC storage system

These components are shown in Fig. 5.3. Some other important issues i.e., how much voltage should be injected in series using appropriate algorithm, choice of suitable power converter topology to synthesize voltage and design of filter capacitor and inductor components have to be addressed while designing the DVR unit.
5.4 Operating Principle of DVR

Consider a DVR compensated single phase system as shown in Fig. 5.4. Let us assume that source voltage is 1.0 pu and we want to regulate the load voltage to 1.0 pu. Let us denote the phase angle between $V_s$ and $V_l$ as $\delta$. In this analysis, harmonics are not considered. Further we assume that during DVR operation, real power is not required except some losses in the inverter and the non-ideal filter components. These losses for the time being are considered to be zero. This condition implies that the phase difference between $V_f$ and $I_s$ should be 90°. Let us first consider a general case to understand the concept.

For the circuit shown in Fig. 5.4, applying Kirchoff’s voltage law,

$$V_s + V_f = I_s (R_s + jX_s) + V_l$$

$$= I_s Z_s + V_l.$$  (5.1)
Note that in above circuit $I_s = I_l = I$. The load voltage $V_l$ can be written in terms of load current and load impedance as given below.

$$V_s + V_f = T (Z_s + Z_l)$$

(5.2)

Therefore equation (5.1) can be written as following.

$$V_s + V_f = T (R_s + jX_s) + V_l$$

$$= T Z_s + V_l$$

(5.3)

Above equation can be re-written as following.

$$V_s = V_l + TR_s - (V_f - jT X_s)$$

(5.4)

With the help of above equation, the relationship between load voltage and the source and DVR voltages can be expressed as below.

$$V_l = \left( \frac{V_s + V_f}{Z_s + Z_l} \right) Z_l$$

(5.5)

**Example 5.1** Let us apply condition to maintain load voltage same as source voltage i.e., $V_l = V_s$ or $V_l = V_s$. Discuss the feasibility of injected voltage in series with the line as shown in Fig. 5.4, to obtain load voltage same as source voltage. Consider the following cases.

a. Line resistance is negligible with $Z_s = j0.25$ pu and $Z_l = 0.5 + j0.25$ pu.

b. When the load is purely resistive with $Z_s = 0.45 + j0.25$ pu and $Z_l = 0.5$ pu.

**Solution:**

(a) When line resistance is negligible
The above condition implies that, \( X_s = 0 \). Without DVR, the load terminal voltage \( V_l \) can be given as following.

\[
\bar{I}_l = \frac{V_s}{Z_l + jZ_s} = \frac{1.0\angle0^o}{0.5 + j0.5} = 1.0 - j1.0 = 1.4142\angle -45^o \text{ pu}
\]

Therefore the load voltage is given as following. \( \bar{V}_l = Z_l \bar{I}_l = 0.5590\angle26.56^o \times 1.4142\angle45^o = 0.7906\angle -18.43^o \text{ pu} \). This is illustrated in Fig. 5.5(a). Thus the load voltage has reduced by 21%.

Now it is desired to maintain load voltage same as supply voltage in magnitude and phase angle. Thus, substituting \( V_s = V_l \) in equation (5.3), we get,

\[
\begin{align*}
\bar{V}_s + \bar{V}_f &= \bar{I} (R_s + jX_s) + \bar{V}_l \\
\Rightarrow \bar{V}_f &= \bar{I} (R_s + jX_s) \\
\bar{V}_f &= \frac{jX_s}{Z_l} \bar{V}_l, \text{ since } R_s = 0 \text{ and } \bar{I} = \bar{V}_l/Z_l
\end{align*}
\]

Neglecting resistance part of the feeder impedance, \( Z_s = j0.25 \), the DVR voltage can be computed as above.

\[
\bar{V}_f = \frac{j0.25}{0.5 + j0.25} \times 1.0\angle0^o \text{ for } \bar{V}_l = 1.0\angle0^o = 0.4472\angle63.4349^o \text{ pu}.
\]

From the above the, line current is computed as following.

\[
\bar{I}_s = \frac{\bar{V}_l}{Z_l} = \frac{1.0\angle0^o}{0.5590\angle26.56^o} = 1.7889\angle -26.56^o \text{ pu}.
\]

It is to be noted that, although \( V_s = V_l = 1.0\angle0^o \text{ pu} \), it does not imply that no power flows from source to load. In fact the total effective source voltage is \( \bar{V}_s = \bar{V}_s + \bar{V}_f = 1.2649\angle18.8^o \text{ pu} \). Therefore it implies that the effective source voltage is leading the load voltage by an angle of 18.43\(^o\). This ensures the power flow from the source to load. This is illustrated by drawing phasor diagram in the Fig. 5.5(b) below.

**b) When load is purely resistive**

For this case \( X_l = 0 \), therefore \( Z_l = R_l = 0.5 \text{ pu} \). Substituting \( \bar{V}_s = \bar{V}_l \) in (5.1), we get the following.

\[
\bar{V}_f = (R_s + jX_s) \bar{I}
\]

Substituting \( \bar{I} = \bar{V}_l/R_l \), we get,

\[
\bar{V}_f = \frac{R_s + jX_s}{R_l} \bar{V}_l = R_s \left( \frac{\bar{V}_l}{R_l} \right) + jX_s \left( \frac{\bar{V}_l}{R_l} \right) = R_s \bar{I}_s + jX_s \bar{I}_s.
\]

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From this above equation, it is indicated that DVR voltage has two components, the one is in phase with $I_s$ and the other is in phase quadrature with $I_s$. This implies that for purely resistive load, it is not possible to maintain $V_l = V_s$ without active power supplied from the DVR to the load. This is due to the presence of in phase component of the DVR voltage in the above equation. This is illustrated in the phasor diagram given in Fig. 5.6 below.

Although it is not possible to maintain $V_l = V_s$ without injection of active power from the DVR, however it is possible to maintain the magnitudes of load voltage and source voltages to the same value i.e., $V_s = V_l$. However, this may be true for a limited range of the load resistance.

5.4.1 General Case

In general, it is desired to maintain the magnitude of the load voltage equal to the source voltage i.e., 1.0 pu. The voltage equation in general relating the source, load and DVR has been expressed in (5.4) and is given below.

$$\bar{V}_s = \bar{V}_l + \bar{T}R_s - (\bar{V}_f - j\bar{X}X_s)$$

The above equation is illustrated using phasor diagram description in Fig. 5.7 given below. Three cases of voltage compensation are discussed below.

**Case 1:** When $R_s I < CD$

For this case, it is always possible to maintain load voltage same as source voltage i.e., $V_l = V_s$. The DVR is expected to supply enough range reactive power to meet this condition. When $R_s I_s$
is quite smaller than CD, the above condition can be met by supplying less reactive power from the DVR. For this condition there are two solutions. Graphically, these solutions are represented by points A and B in the Fig. 5.7.

**Case 2:** When $R_s I > CD$

For this condition, it is not possible to meet $V_l = V_s$. This is shown by lines passing through points $D$ and $D_1$. This may take place due to the higher feeder resistance or high current, thus making product of $I R_s$ relatively large.

**Case 3:** When $R_s I = CD$

This is limiting case of compensation to obtain $V_s = V_l$. This condition is now satisfied at only one point when $CD=R_s I$. This is indicated by point $D$ in the Fig. 5.7.

Now let us set the following objective for the load compensation.

$$V_l = V_s = V = 1.0 \text{ pu} \quad (5.6)$$

From Fig. 5.7, $OA = V \cos \phi_l = \cos \phi_l$.

Therefore, $CD = OD - OC = V(1 - \cos \phi_l) = (1 - \cos \phi_l) \text{ pu}$. In order to meet the condition
given by (5.6), the following must be satisfied.

\[ R_s I \leq V (1 - \cos \phi_l) \]  \hspace{1cm} (5.7)

The above implies that

\[ R_s \leq \frac{V (1 - \cos \phi_l)}{I} \]  \hspace{1cm} (5.8)

or \[ I \leq \frac{V (1 - \cos \phi_l)}{R_s} \]  \hspace{1cm} (5.9)

Thus it is observed that for a given power factor, the DVR characteristics can be obtained by varying \( R_s \) and keeping \( I \) constant or vice versa. This is described below. Let us consider three conditions \( R_s = 0.04 \) pu, \( R_s = 0.1 \) pu and \( R_s = 0.4 \) pu. For these values of feeder resistance, the line currents are expressed as following using (5.9).

\[
\begin{align*}
I &= 25 (1 - \cos \phi_l) \text{ pu} \quad \text{for} \quad R_s = 0.04 \text{ pu} \\
I &= 10 (1 - \cos \phi_l) \text{ pu} \quad \text{for} \quad R_s = 0.1 \text{ pu} \\
I &= 2.5 (1 - \cos \phi_l) \text{ pu} \quad \text{for} \quad R_s = 0.4 \text{ pu}.
\end{align*}
\]

The above currents are plotted as function of load power factor and are shown in Fig. 5.8. Since \( R_s I = V_l(1 - \cos \phi_l) \), when \( R_s \) increases, \( I \) has to decrease to make \( V_l(1 - \cos \phi_l) \) to be a constant for a given power factor. Thus if the load requires more current than the permissible value, the DVR will not be able to regulate the load voltage at the nominal value, i.e., 1.0 pu. However we can regulate bus voltage less than 1.0 pu. For regulating the load voltage less than 1.0 pu the current drawing capacity of the load increases.

![Fig. 5.8 DVR characteristics for different load power factor and feeder resistance](image-url)
5.5 Mathematical Description to Compute DVR Voltage

The previous section explains DVR characteristics and describes the feasibility of realizing DVR voltage graphically under different operating conditions. In this section, a feasible solution for the DVR voltage is presented with a mathematical description. This plays significant role while implementing DVR on real time basis. Reproducing equation (5.3) for sake of completeness,

\[ V_s + V_f = V_l + (R_s + jX_s) I. \]  

(5.10)

Denoting, \((R_s + jX_s) I = a_2 + jb_2\) and \(V_f = V_f \angle \angle V_f = V_f (a_1 + jb_1)\), the above equation can be written as following.

\[ V_s = V_l + (a_2 + jb_2) - V_f = V_l + (a_2 + jb_2) - V_f (a_1 + jb_1) \]  

(5.11)

Since, source voltage and load voltage have to be maintained at nominal value i.e., 1.0 pu, therefore \(V_s = V_s \angle \delta = 1.0 \angle \delta\). Substituting this value of \(V_s\) in above equation, we get,

\[ V_s = 1.0 \angle \delta = \cos \delta + j \sin \delta = \{ (1 + a_2) - V_f a_1 \} + j (b_2 - V_f b_1) \]  

(5.12)

Squaring and adding the real and imaginary parts from both the sides of the above equation, we get,

\[ (1 + a_2)^2 + V_f^2 a_1^2 - 2(1 + a_2) a_1 V_f + b_2^2 + V_f^2 b_1^2 - 2 b_1 b_2 V_f - 1.0 = 0 \]  

(5.13)

Since \(a_1^2 + b_1^2 = 1\), therefore summation of underlines terms, \(V_f^2 (a_1^2 + b_1^2) = V_f^2\). Using this and rearranging above equation in the power of \(V_f\), we get the following.

\[ V_f^2 - 2 \{ (1 + a_2) a_1 + b_1 b_2 \} V_f + (1 + a_2)^2 + b_2^2 - 1.0 = 0 \]  

(5.14)

The above equation gives two solutions for \(V_f\). These are equivalent to two points A and B shown in the Fig. 5.7. However, the feasible value of the voltage is chosen on the basis of the rating of the DVR.

Example 5.2 Consider a system with supply voltage 230 V = 1.0 pu, 50 Hz as shown in the Fig. 5.9. Consider feeder impedance as \(Z_s = 0.05 + j0.3\) pu and load impedance \(Z_l = 0.5 + j0.3\) pu.

1. Compute the load voltage without DVR.
2. Compute the current and DVR voltage such that \(V_l = V_s\).
3. Compute the effective source voltage including DVR. Explain the power flow in the circuit.
4. Compute the terminal voltage with DVR compensation.
Solution: 1. When DVR is not connected.

The system parameters are given as following. The supply voltage $V_s = 1.0 \angle 0^\circ \, \text{pu}$, $Z_s = R_s + jX_s = 0.05 + j0.3$ and $Z_l = R_l + jX_l = 0.5 + j0.3 \, \text{pu}$. The current in the circuit is given by,

$$I_s = \frac{V_s}{Z_s + Z_l} = \frac{1.0}{0.55 + j0.6} = 0.83 - j0.91 = 1.2286 \angle -47.49^\circ \, \text{pu}$$

The load voltage is therefore given by,

$$V_l = Z_s I_s = 1.2286 \angle -47.49^\circ \times 0.5 + j0.3 = 0.7164 \angle 16.53^\circ \, \text{pu}$$

Thus we observe that the load voltage is 71% of the rated value. Due to reduction in the load voltage, the load may not perform to the expected level.

2. When DVR is connected

It is desired to maintain $V_l = V_s$ by connecting the DVR. Taking $V_l$ as reference phasor i.e., $V_l = 1.0 \angle 0^\circ$, The line current is computed as below.

$$I = \frac{1.0\angle 0}{0.5 + j0.3} = 1.47 - j0.88 = 1.715 \angle -30.96^\circ \, \text{pu}$$

Writing KVL for the circuit shown in (5.9),

$$V_s + V_f = V_l + (R_s + jX_s) I.$$

The DVR voltage $V_f$ can be expressed as following.

$$V_f = V_f \angle (\angle I_s + 90^\circ)$$
The angle of $\nabla_f$ is taken as $\angle I_s + 90^\circ$ so that DVR do not exchange any active power with the system.

$$\nabla_f = V_f \angle (\angle I_s + 90^\circ)$$

$$= V_f \angle (-30.96^\circ + 90^\circ)$$

$$= V_f \angle 59.04^\circ = V_f (0.51 + j0.86) = V_f (a_1 + jb_1) \text{ pu}$$

The above equation implies that $a_1 = 0.51$, $b_1 = 0.86$.

Let us now compute $(R_s + jX_s) \bar{I}$.

$$(R_s + jX_s) \bar{I} = (0.05 + j0.3) (1.46 - j0.86)$$

$$= 0.3041 \angle 80.54^\circ \times 1.715 \angle -30.96^\circ$$

$$= 0.5199 \angle 49.58^\circ$$

$$= 0.3370 + j0.3958 \text{ pu}$$

The above implies that $a_2 = 0.337$ and $b_2 = 0.3958$. As discussed in previous section, the equation

$$\nabla_s + \nabla_f = \nabla_l + (R_s + jX_s) \bar{I}$$

can be written in following form.

$$V_f^2 - 2 \{(1 + a_2) a_1 + b_1 b_2\} V_f + (1 + a_2)^2 + b_2^2 - 1.0 = 0.$$ 

Substituting $a_1$, $b_1$, $a_2$, $b_2$ in the above equation, we get the following quadratic equation for the DVR.

$$V_f^2 - 2.0463 V_f + 0.9442 = 0$$

Solving the above equation, we get $V_f = 0.7028, 1.3434$ pu as two values of the DVR voltage. These two values correspond to the points A and B respectively in Fig. 5.7. However, the feasible solution is $V_f = 0.7028$ pu, as it ensures less rating of the DVR.

Therefore,

$$\nabla_f = 0.7028 \angle 59.04^\circ$$

$$= 0.3614 + j0.6028 \text{ pu.}$$

The source voltage can be computed using the following equation.

$$\nabla_s = \nabla_l + (R_s + jX_s) \bar{I} - \nabla_f.$$ 

$$= 1.0 \angle 0^\circ + (0.05 + j0.3) 1.715 \angle -30.96^\circ - 0.7028 \angle 59.04^\circ$$

$$= 0.9767 - j0.2056 = 1.0 \angle -11.89^\circ \text{ pu}$$

3. Effective source voltage

It is seen that the magnitude of $\nabla_s$ is 1.0 pu which is satisfying the condition $V_s = V_l$. However the angle of $\nabla_s$ is $\angle -11.89^\circ$ which implies that power is flowing from load to the source. This is not true because the effective source voltage is now $\nabla'_s = \nabla_s + \nabla_f$. This is computed below.

$$\nabla_s + \nabla_f = \nabla'_s = 0.9767 - j0.2056 + 0.3614 + j0.6028$$

$$= 1.3382 + j0.397$$

$$= 1.3958 \angle 16.52^\circ \text{ pu}$$

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From above it is evident that the effective source voltage has magnitude of 1.3958 pu and an angle of $\angle 16.52^\circ$ which ensures that power flows from source to the load. For this the equivalent circuit is shown in the Fig. 5.10 below.

$$\bar{V}_s = \bar{V}_s + \bar{V}_f = 1.39\angle 16.52^\circ \quad R_s + jX_s \quad \bar{V}_l = 1.0\angle 0^\circ$$

![Fig. 5.10 A DVR compensated system](image)

4. **Terminal voltage with DVR compensation**

The terminal voltage can also be computed as following.

$$\bar{V}_t = \bar{V}_s - Z_s \bar{I} = \bar{V}_l - \bar{V}_f$$

$$= 1.0\angle 0^\circ - 0.6983\angle 59.06^\circ$$

$$= 0.8796\angle -43.28^\circ \text{ pu}$$

This indicates that for rated current flowing in the load, the terminal voltage is less than the 1.0 pu and needs compensation. After compensation the load voltage is 1.0 pu as shown in the Fig. 5.10. The details of voltages are depicted in the following figure.

5.6 **Transient Operation of the DVR**

In the previous section the operation of the DVR in the steady state was discussed with assumption that full system information is available. While implementing the DVR compensation scheme, the above discussed method should be implemented on the real time basis. For the single phase DVR operation, following steps are required.

1. Define a reference quantity such as the terminal voltage $\bar{V}_l(t)$ and other quantities are synchronized to it.

2. To compute phase angle of the DVR voltage, a fundamental of line current is extracted with respect to reference quantity.

3. Then DVR voltage is computed using Equation (5.14), which is reproduced below.

$$V_f^2 - 2 \left\{ (1 + a_2) a_1 + b_1 b_2 \right\} V_f + (1 + a_2)^2 + b_2^2 - 1.0 = 0$$
4. DVR voltage $V_f$ is then synthesized using magnitude $V_f$ from the above equation and phase angle that leads the fundamental of the line current by $90^\circ$.

The above method can be refereed as Type 1 control [1]. The method assumes that all circuit parameters are known along with the information of the source impedance. This however may not be feasible in all circumstances. To solve this problem Type 2 control is suggested. In Type 2 control only local quantities are required to compute the DVR voltage. The method is described below.

The terminal voltage, which is local quantity to the DVR as shown in Fig. 5.1 can be expressed as following.

$$V_t = V_l - V_f = V_l \angle 0^\circ - V_f (a_1 + j b_1)$$

$$= (V_l - a_1 V_f) - j b_1 V_f$$  \hspace{1cm} (5.15)

Since, $V_t = V_l \angle \delta_t = V_l \cos \delta_t + j V_l \sin \delta_t$, the above equation is written as following.

$$V_l \cos \delta_t + j V_l \sin \delta_t = (V_l - a_1 V_f) - j b_1 V_f$$  \hspace{1cm} (5.16)

Squaring adding both sides we get,

$$V_t^2 = (V_l - a_1 V_f)^2 + b_1^2 V_f^2$$

$$= V_l^2 + a_1^2 V_f^2 + b_1^2 V_f^2 - 2 a_1 V_l V_f$$

$$= V_l^2 + V_f^2 - 2 a_1 V_l V_f$$  \hspace{1cm} (since $a_1^2 + b_1^2 = 1$).

The above equation can be arranged in the powers of the DVR voltage as given below.

$$V_f^2 - 2 a_1 V_l V_f + V_l^2 - V_t^2 = 0$$  \hspace{1cm} (5.18)

To implement DVR for unbalanced three-phase system without harmonics, the positive sequence currents ($I_a^+, I_b^+$ and $I_c^+$) of line currents are extracted using Fourier transform. Based on these values of currents the angles of DVR voltages are found by shifting current angles by $90^\circ$ i.e.,

$$\angle \nabla_{fa} = \angle I_a^+ + 90^\circ$$

$$\angle \nabla_{fb} = \angle I_b^+ + 90^\circ$$

$$\angle \nabla_{fc} = \angle I_c^+ + 90^\circ$$  \hspace{1cm} (5.19)

The magnitude of DVR voltage can be found using equations (5.14) and (5.18) for Type 1 and Type 2 control respectively.

Based on above the DVR voltages $v_{fa}, v_{fb}, v_{fc}$ can be expressed in time domain as given below.

$$v_{fa} = \sqrt{2} V_{fa} \sin(\omega t + \angle \nabla_{fa})$$

$$v_{fb} = \sqrt{2} V_{fb} \sin(\omega t - 120^\circ + \angle \nabla_{fb})$$

$$v_{fc} = \sqrt{2} V_{fc} \sin(\omega t + 120^\circ + \angle \nabla_{fc})$$  \hspace{1cm} (5.20)
5.6.1 Operation of the DVR With Unbalance and Harmonics

In the previous analysis, it was assumed that the supply voltages are unbalanced without harmonics. In this section the operation of the DVR with harmonics will be discussed. The terminal voltages \((v_{ta}, v_{tb} \text{ and } v_{tc})\) are resolved into their fundamental positive sequence voltages and the rest part, as given below.

\[
\begin{align*}
v_{ta} &= v_{ta1}^+ + v_{ta rest} \\
v_{tb} &= v_{tb1}^+ + v_{tb rest} \\
v_{tc} &= v_{tc1}^+ + v_{tc rest}
\end{align*}
\]  

(5.21)

The angles of fundamental DVR voltages \(\angle V_{fa1}, \angle V_{fb1} \text{ and } \angle V_{fc1}\) can be extracted as explained above. The magnitudes of the fundamental DVR voltages \(V_{fa1}, V_{fb1} \text{ and } V_{fc1}\) can be computed using equations (5.14) and (5.18) for Type 1 and Type 2 control respectively. For example, using Type 2 control the fundamental phase-\(a\) DVR voltage is computed as per following equation.

\[
V_{fa1}^2 - 2 a_{a1} V_l V_{fa1} + V_l^2 - V_{fa1}^+ = 0
\]  

(5.22)

In above equation \(a_{a1} + jb_{a1} = \angle V_{fa1}\) and \(V_{fa1}^+\) is fundamental positive sequence phase-\(a\) terminal voltage as given above in (5.21). Similar expression can be written for phase-\(b\) and phase-\(c\). This equation gives solution only for fundamental component of the DVR voltage. The rest of the DVR voltages which consist of harmonics and unbalance must be equal and opposite to that of the rest part of the terminal voltages i.e., \(v_{ta rest}, v_{tb rest} \text{ and } v_{tc rest}\). Therefore these can be given using following equations.

\[
\begin{align*}
v_{fa rest} &= -v_{ta rest} \\
v_{fb rest} &= -v_{tb rest} \\
v_{fc rest} &= -v_{tc rest}
\end{align*}
\]  

(5.23)

Thus, the total DVR voltage to be injected can be given as following.

\[
\begin{align*}
v_{fa} &= v_{fa1} + v_{fa rest} \\
v_{fa} &= v_{fb1} + v_{fb rest} \\
v_{fa} &= v_{fc1} + v_{fc rest}
\end{align*}
\]  

(5.24)

In above equation, \(v_{fa1}, v_{fb1}, v_{fc1}\) are constructed using equation (5.20). Once \(v_{fa}, v_{fb} \text{ and } v_{fc}\) are known, these voltages are synthesized using suitable power electronic circuit. It will be discussed in the following section.

5.7 Realization of DVR voltage using Voltage Source Inverter

In the previous section, a reference voltage of DVR was extracted using discussed control algorithms. This DVR voltage however should be realized in practice. This is achieved with the help of power electronic converter which is also known as voltage source inverter. Various components of the DVR were listed in the beginning of chapter. They are shown in detail in the Fig. 5.11.
The transformer injects the required voltage in series with the line to maintain the load bus voltage at the nominal value. The transformer not only reduces the voltage requirement but also provides isolation between the inverters. The filter components of the DVR such as external inductance ($L_t$) which also includes the leakage of the transformer on the primary side and ac filter capacitor on the secondary side play significant role in the performance of the DVR [Sasitharan Thesis].

The same DC link can be extended to other phases as shown in Fig. 5.11. The single phase equivalent of the DVR is shown in the Fig. 5.12.

In Figs. 5.11 and 5.12, $v_{inv}$ denotes the switched voltage generated at the inverter output terminals, the inductance, $L_t$ represents the total inductance and resistance including leakage inductance and resistance of transformer. The resistance, $R_t$ models the switching losses of the inverter and the copper loss of the connected transformer. The voltage source inverter (VSI) is operated in a switching band voltage control mode to track the reference voltages generated using control logic as discussed below.

Let $V_f^*$ be the reference voltage of a phase that DVR needs to inject in series with the line with help
of the VSI explained above. We form a voltage hysteresis band of $\pm h$ over this reference value. Thus, the upper and lower limits within which the DVR has to track the voltage can be given as following.

$$v_{f_{up}} = v_f^* + h$$
$$v_{f_{dn}} = v_f^* - h$$

(5.25)

The following switching logic is used to synthesize the reference DVR voltage.

If $v_f \geq v_{f_{up}}$
- $S_1 - S_2$ OFF and $S_3 - S_4$ ON (‘-1’ state)
else if $v_f \leq v_{f_{dn}}$
- $S_1 - S_2$ ON and $S_3 - S_4$ OFF (‘+1’ state)
else if $v_{f_{dn}} \geq v_f \leq v_{f_{up}}$
- retain the current switching status of switches
end.

It is to be noted that switches status $S_1 - S_2$ ON and $S_3 - S_4$ OFF is denoted by ‘+1’ state and it gives $v_{inv} = +V_{dc}$. The switches status $S_1 - S_2$ OFF and $S_3 - S_4$ ON corresponds to ‘-1’ state providing $v_{inv} = -V_{dc}$ as shown in Fig. 5.11. The above switching logic is very basic and has scope to be refined. For example ‘0’ state of the switches of the VSI as shown in Fig. 5.11, can also be used to have smooth switching and to minimize switching losses. In the zero state, $v_{inv} = 0$ and refers switches status as $S_3 \ D_1$ or $S_4 \ D_2$ for positive inverter current ($i_{inv} > 0$). Similarly, for negative inverter current ($i_{inv} < 0$), ‘0’ state is obtained through $S_1 \ D_3$ or $S_2 \ D_4$. With the addition of ‘0’ state, the switching logic becomes as follows.

If $v_f^* > 0$
- if $v_f \geq v_{f_{up}}$
  - ‘0’ state
- else if $v_f \leq v_{f_{dn}}$
  - ‘+1’ state
end
else if $v_f^* < 0$
- if $v_f \geq v_{f_{up}}$
  - ‘-1’ state
- else if $v_f \leq v_{f_{dn}}$
  - ‘0’ state
end

In order to improve the switching performance one more term is added in the above equation.
based on the feedback of filter capacitor current.

\[ v_{f_{up}} = v_f^* + h + \alpha i_{fac} \]
\[ v_{f_{dn}} = v_f^* - h + \alpha i_{fac} \]  \hspace{1cm} (5.26)

Where \( \alpha \) is a proportional gain given to smoothen and stabilize the switching performance of the VSI [2]. The dimension of \( \alpha \) is \( \Omega \) and is thus is equivalent to virtual resistance, whose effect to damp out and smoothen the DVR voltage trajectory resulted from the switching of the inverter [4]. The value of hysteresis band \( (h) \) should be chosen in such a way that it limits switching frequency within the prescribed maximum value. This kind of voltage control using VSI is called as switching band control. The actual DVR voltage is compared with these upper and lower bands of the voltage \( (V_{f_{up}}, V_{f_{dn}}) \) and accordingly switching commands to the power switch are generated. The switching control logic is described in the Table 5.1. To minimize switching frequency of the VSI, three level logic has been used. For this an additional check of polarity of the reference voltage has been taken into consideration. Based on this switching status, the inverter supplies \(+V_{dc}, 0\) and \(-V_{dc}\) levels of voltage corresponding to the 1, 0 and -1 given in the table, in order to synthesis the reference DVR voltage.

**Table 5.1 Three level switching logic for the VSI**

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Switching value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_f^* \geq 0 ) \hspace{0.2cm} ( V_f &gt; V_{up} )</td>
<td>0</td>
</tr>
<tr>
<td>( V_f^* \geq 0 ) \hspace{0.2cm} ( V_f &lt; V_{dn} )</td>
<td>1</td>
</tr>
<tr>
<td>( V_f^* &lt; 0 ) \hspace{0.2cm} ( V_f &gt; V_{up} )</td>
<td>-1</td>
</tr>
<tr>
<td>( V_f^* &lt; 0 ) \hspace{0.2cm} ( V_f &lt; V_{dn} )</td>
<td>0</td>
</tr>
</tbody>
</table>

In addition to switching band control, an additional loop is required to correct the voltage in the dc storage capacitor against losses in the inverter and transformer. During transients, the dc capacitor voltage may rise or fall from the reference value due to real power flow for a short duration. To correct this voltage deviation, a small amount of real power must be drawn from the source to replenish the losses. To accomplish this, a simple proportional-plus-integral controller (PI) is used. The signal \( u_c \) is generated from this PI controller as given below.

\[ u_c = K_p e_{V_{dc}} + K_i \int e_{V_{dc}} dt \]  \hspace{1cm} (5.27)

Where, \( e_{V_{dc}} = V_{dcref} - V_{dc} \). This control loop need not to be too fast. It may be updated once in a cycle preferably synchronized to positive zero crossing of phase-\( a \) voltage. Based on this information the variable \( u_c \) will be included in generation of the fundamental of DVR voltage as given below.

\[ V_{f1} = V_{f1} \angle (\angle I_s + 90^o - u_c) = V_{f1} (\tilde{a}_1 + j\tilde{b}_1) \]  \hspace{1cm} (5.28)

Then the equation (5.18), is modified to the following.

\[ V_{f1}^2 - 2\tilde{a}_1 V_i V_{f1} + V_i^2 - V_{i1}^2 = 0 \]  \hspace{1cm} (5.29)

The above equation is used to find the DVR voltage. It can be found that the phase difference between line current and DVR voltage differs slightly from 90\(^o\) in order to account the losses in the inverter.
5.8 Maximum Compensation Capacity of the DVR Without Real Power Support from the DC Link

There is direct relationship between the terminal voltage, power factor of the load and the maximum possible achievable load voltage, with assumption that no real power is required from the dc bus. Referring to quadratic equation in (5.29), for given value of \( V_{t1} \) and a target load bus voltage \( V_l \), the equation gives two real values of \( V_{f1} \) for feasible solution. In case solution is not feasible, the equation gives two complex conjugate roots. This concludes that the maximum voltage that DVR can compensate corresponds to the single solution of the above equation, which is given below. This solution corresponds to point ‘D’ in Fig. 5.7.

\[
V_{f1} = \frac{2 \bar{a}_1 V_l \pm \sqrt{(2 \bar{a}_1 V_l)^2 - 4 (V_l^2 - V_{t1}^2)}}{2} \tag{5.30}
\]

Since, voltage should not be complex number, the value of the terms within square root must not be negative. Therefore

\[
(2 \bar{a}_1 V_l)^2 \geq 4 (V_l^2 - V_{t1}^2) \tag{5.31}
\]

The above equation implies that

\[
V_l = \frac{V_{t1}}{\sqrt{1 - \bar{a}_1^2}}. \tag{5.32}
\]

And therefore, the DVR voltage is given by the following equation.

\[
V_{f1} = \bar{a}_1 V_l \tag{5.33}
\]

With no losses in the VSI, \( u_c = 0 \),

\[
V_l = \frac{V_{t1}}{\sqrt{1 - \bar{a}_1^2}} = \frac{V_{t1}}{\sqrt{1 - \bar{a}_1^2}} \tag{5.34}
\]

Since, \( \bar{a}_1 + j\bar{b}_1 = 1 \angle (90^\circ + \phi_l) = \cos(90^\circ + \phi_l) + j \sin(90^\circ + \phi_l) = -\sin \phi_l + j \cos \phi_l \). This implies \( \bar{a}_1 = -\sin \phi_l \), therefore \( \sqrt{1 - \bar{a}_1^2} = \sqrt{1 - (-\sin \phi_l)^2} = \cos \phi_l \). Using this relation, the above equation can be written as following.

\[
V_l = \frac{V_{t1}}{\cos \phi_l} \tag{5.35}
\]

**Example 5.3** A DVR is shown in Fig. 5.13. The feeder impedance of the line \( 0.1 + j0.5 \) pu. Assume \( i_h \) to be load current represented by square waveform approximated by the following expression.

\[
i_h = 1. \sin(\omega t - 30^\circ) + 0.3 \sin(3 \omega t - 90^\circ) \] pu

1. Find the load voltage \( v(t) \) without DVR compensation i.e., \( v_f = 0 \).
2. Is it possible to maintain load voltage, \( V_l \) to be 1.0 pu sinusoidal waveform? If yes what is the DVR voltage, \( v_f(t) \)?
3. If no, how much maximum voltage can be maintained at load terminal with the DVR without taking any real power from the dc bus?
Fig. 5.13 A DVR compensated system

Solution:

1. When $V_f = 0$

$$v_t = v_s - \sum_{h=1,3} Z_{sh} i_h$$

The impedance at the fundamental frequency, $Z_{s1} = 0.1 + j0.5 = 0.51\angle78.7^\circ$ pu.
The impedance at third harmonic, $Z_{s3} = 0.1 + j1.5 = 1.50\angle86.18^\circ$ pu.
Therefore the voltage drop due to fundamental component of the current,

$$V_{z_{s1}} = (0.1 + j0.5) \times 0.707 \angle -30^\circ$$
$$= 0.51 \angle 78.69^\circ \times 0.707 \angle -30^\circ$$
$$= 0.36 \angle 48.69 \text{ pu}.$$  

The voltage drop due to third harmonic component of the current,

$$V_{z_{s3}} = (0.1 + j1.5) \times 0.21 \angle -90^\circ$$
$$= 0.31 \angle -3.82^\circ \text{ pu}.$$  

The load voltage thus can be given by

$$v_t = v_s - (i_{s1} Z_{s1} + i_{s3} Z_{s3})$$
$$= 1.0 \sin \omega t - 0.51 \sin(\omega t + 48.69^\circ) - 0.45 \sin(3\omega t - 3.82^\circ)$$
$$= 0.7947 \sin(\omega t - 28.81^\circ) - 0.45 \sin(3\omega t - 3.82^\circ) \text{ pu}$$
$$= v_{t1}(t) + v_{th}(t).$$

Implying that,

$$V_{t1} = \frac{0.7947}{\sqrt{2}} \angle -28.81^\circ = 0.5619 \angle -28.81^\circ \text{ pu}.$$  

2. With DVR
From the above equation, $V_{t1} = 0.5619$. With load voltage $v_l = 1.0 \sin(\omega t - \phi_l)$, the DVR voltage $V_{f1}$ can be solved using quadratic equation as mentioned in Type 2 control. Further,

$$V_{f1} = V_{f1} \angle (\angle T_{s1} + 90^o) = V_{f1} \angle (-30^o + 90^o) = V_{f1} \angle 60^o$$

$$= V_{f1} (\cos 60^o + j \sin 60^o) = V_{f1} (0.5 + j0.8666) = V_{f1} (a_1 + jb_1) \text{pu.}$$

The above implies $a_1 = 0.5$, $b_1 = 0.866$. Knowing this, we can solve $V_{f1}$ using following quadratic equation.

$$V_{f1}^2 - 2a_1 V_l V_{f1} + V_l^2 - V_{t1}^2 = 0$$

From the above,

$$V_{f1} = a_1 V_l \pm \sqrt{a_1^2 V_l^2 - (V_l^2 - V_{t1}^2)}$$

$$= 0.5 \times \frac{1}{\sqrt{2}} \pm \sqrt{(0.5)^2 \times \left(\frac{1}{\sqrt{2}}\right)^2 - \left\{\left(\frac{1}{\sqrt{2}}\right)^2 - (0.5619)^2\right\}}$$

$$= 0.35 \pm \sqrt{0.125 - 0.1842}$$

The above solution is complex quantity, which implies that it is not possible to maintain load voltage at $1.0 \sin(\omega t - \phi_l)$.

3. Maximum possible load voltage

The maximum load voltage that can be obtained with the DVR, without any real power from the dc bus can be given as following.

$$V_l = \frac{V_{t1}}{\sqrt{1 - a_1^2}} = \frac{0.5619}{\sqrt{1 - 0.5^2}} = 0.6488 \text{pu.}$$

In the time domain the load voltage $v_l = v_{t1} = \sqrt{2} \times 0.6488 \sin(\omega t - \phi_l) = 0.9175 \sin(\omega t - \phi_l)$.

For this load voltage the DVR voltage is given as following.

$$V_{f1} = a_1 V_l = 0.5 \times 0.6488 = 0.3244 \text{pu.}$$

This implies

$$V_{f1} = 0.3244 \angle 60^o \text{pu.}$$

The time domain repression for the fundamental DVR voltage is given as,

$$v_{f1}(t) = \sqrt{2} \times 0.3244 \sin(\omega t + 60^o) = 0.4587 \sin(\omega t + 60^o) \text{pu.}$$

The harmonic voltage that DVR compensates is as following.

$$v_{fh}(t) = -v_{th} = 0.45 \sin(3 \omega t - 3.82^o) \text{pu.}$$

The total DVR voltage is given as below.

$$v_f(t) = v_{f1}(t) + v_{th}(t)$$

$$= 0.4587 \sin(\omega t + 60^o) + 0.45 \sin(3 \omega t - 3.82^o) \text{pu}$$
References


