Module 5: Design of Sampled Data Control Systems
Lecture Note 1

So far we have discussed about the modelling of a discrete time system by pulse transfer function, various stability tests and time domain performance criteria. The main objective of a control system is to design a controller either in forward or in feedback path so that the closed loop system is stable with some desired performance. Two most popular design techniques for continuous time LTI systems are using root locus and frequency domain methods.

1 Design based on root locus method

- The effect of system gain and/or sampling period on the absolute and relative stability of the closed loop system should be investigated in addition to the transient response characteristics. Root locus method is very useful in this regard.
- The root locus method for continuous time systems can be extended to discrete time systems without much modifications since the characteristic equation of a discrete control system is of the same form as that of a continuous time control system.

In many LTI discrete time control systems, the characteristics equation may have either of the following two forms.

$$\begin{aligned} 1 + G(z)H(z) &= 0 \\ 1 + GH(z) &= 0 \end{aligned}$$

To combine both, let us define the characteristics equation as:

$$1 + L(z) = 0 \tag{1}$$

where,$L(z) = G(z)H(z)$ or $L(z) = GH(z)$. $L(z)$ is popularly known as the loop pulse transfer function. From equation (1), we can write

$$L(z) = -1$$

Since $L(z)$ is a complex quantity it can be split into two equations by equating angles and magnitudes of two sides. This gives us the angle and magnitude criteria as

$$\text{Angle Criterion:} \qquad \angle L(z) = \pm 180^0(2k+1), \quad k = 0, 1, 2....$$

$$\text{Magnitude Criterion:} \qquad |L(z)| = 1$$

The values of z that satisfy both criteria are the roots of the characteristics equation or close loop poles. Before constructing the root locus, the characteristics equation $1 + L(z) = 0$ should be rearranged in the following form

$$1 + K\frac{(z + z_1)(z + z_2)....(z + z_m)}{(z + p_1)(z + p_2)....(z + p_n)} = 0$$

where z_i's and p_i's are zeros and poles of open loop transfer function, m is the number of zeros n is the number of poles.

1.1 Construction Rules for Root Locus

Root locus construction rules for digital systems are same as that of continuous time systems.

1. The root locus is symmetric about real axis. Number of root locus branches equals the number of open loop poles.

2. The root locus branches start from the open loop poles at $K = 0$ and ends at open loop zeros at $K = \infty$. In absence of open loop zeros, the locus tends to ∞ when $K \to \infty$. Number of branches that tend to ∞ is equal to difference between the number of poles and number of zeros.

3. A portion of the real axis will be a part of the root locus if the number of poles plus number of zeros to the right of that portion is odd.

4. If there are n open loop poles and m open loop zeros then $n - m$ root locus branches tend to ∞ along the straight line asymptotes drawn from a single point $s = \sigma$ which is called centroid of the loci.

   $$\sigma = \frac{\sum \text{real parts of the open loop poles} - \sum \text{real parts of the open loop zeros}}{n - m}$$

   Angle of asymptotes

   $$\phi_q = \frac{180^o(2q + 1)}{n - m}, \quad q = 0, 1, \ldots, n - m - 1$$

5. Breakaway (Break in) points or the points of multiple roots are the solution of the following equation:

   $$\frac{dK}{dz} = 0$$

   where K is expressed as a function of z from the characteristic equation. This is a necessary but not sufficient condition. One has to check if the solutions lie on the root locus.

6. The intersection (if any) of the root locus with the unit circle can be determined from the Routh array.

7. The angle of departure from a complex open loop pole is given by

$$\phi_p = 180^o + \phi$$

where ϕ is the net angle contribution of all other open loop poles and zeros to that pole.

$$\phi = \sum_i \psi_i - \sum_{j \neq p} \gamma_j$$

ψ_i's are the angles contributed by zeros and γ_j's are the angles contributed by the poles.

8. The angle of arrival at a complex zero is given by

$$\phi_z = 180^o - \phi$$

where ϕ is same as in the above rule.

9. The gain at any point z_0 on the root locus is given by

$$K = \frac{\prod_{j=1}^{n} |z_0 + p_j|}{\prod_{i=1}^{m} |z_0 + z_i|}$$

1.2 Root locus diagram of digital control systems

We will first investigate the effect of controller gain K and sampling time T on the relative stability of the closed loop system as shown in Figure 1.

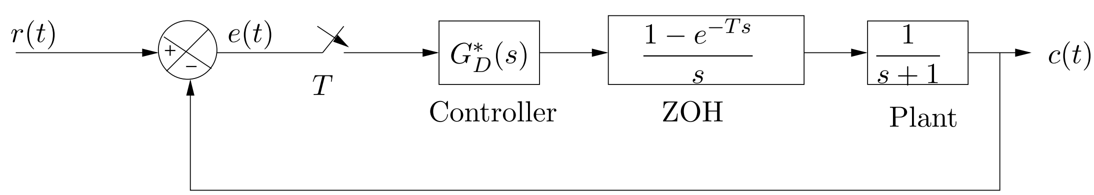

Figure 1: A discrete time control system

Let us first take T=0.5 sec.

$$\begin{aligned}
Z\left[G_{ho}(s)G_p(s)\right] &= Z\left[\frac{1-e^{-Ts}}{s}.\frac{1}{s+1}\right] \\
&= (1-z^{-1})Z\left[\frac{1}{s(s+1)}\right] \\
&= (1-z^{-1})Z\left[\frac{1}{s} - \frac{1}{s+1}\right] \\
&= \frac{z-1}{z}\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] \\
&= \frac{1-e^{-T}}{z-e^{-T}}
\end{aligned}$$

Let us assume that the controller is an integral controller, i.e., $G_D(z) = \dfrac{Kz}{z-1}$. Thus,

$$\begin{aligned} G(z) &= G_D(z) \cdot G_hG_p(z) \\ &= \frac{Kz}{z-1} \cdot \frac{1-e^{-T}}{z-e^{-T}} \end{aligned}$$

The characteristic equation can be written as

$$\begin{aligned} & 1 + G(z) = 0 \\ \Rightarrow \quad & 1 + \frac{Kz(1-e^{-T})}{(z-1)(z-e^{-T})} = 0 \\ \text{when } T = 0.5sec, \quad & L(z) = \frac{0.3935Kz}{(z-1)(z-0.6065)} \end{aligned}$$

$L(z)$ has poles at $z = 1$ and $z = 0.605$ and zero at $z = 0$.

Break away/ break in points are calculated by putting $\dfrac{dK}{dz} = 0$.

$$\begin{aligned} K &= -\frac{(z-1)(z-0.6065)}{0.3935z} \\ \frac{dK}{dz} &= -\frac{z^2-0.6065}{0.3935z^2} = 0 \\ \Rightarrow z^2 = 0.6065 &\Rightarrow z_1 = 0.7788 \text{ and } z_2 = -0.7788 \end{aligned}$$

Critical value of K can be found out from the magnitude criterion.

$$\left|\frac{0.3935z}{(z-1)(z-0.6065)}\right| = \frac{1}{K}$$

Critical gain corresponds to point $z = -1$. Thus

$$\begin{aligned} \left|\frac{-0.03935}{(-2)(-1.6065)}\right| &= \frac{1}{K} \\ \text{or,} \quad K &= 8.165 \end{aligned}$$

Figure 2 shows the root locus of the system for $K = 0$ to $K = 10$. Two root locus branches start from two open loop poles at $K = 0$. If we further increase K one branch will go towards the zero and the other one will tend to infinity. The blue circle represents the unit circle. Thus the stable range of K is $0 < K < 8.165$.

If $T = 1$ sec,

$$G(z) = \frac{0.6321Kz}{(z-1)(z-0.3679)}$$

Break away/ break in points:
$z^2 = 0.3679 \Rightarrow z_1 = 0.6065$ and $z_2 = -0.6065$ Critical gain $(K_c) = 4.328$ Figure 3 shows

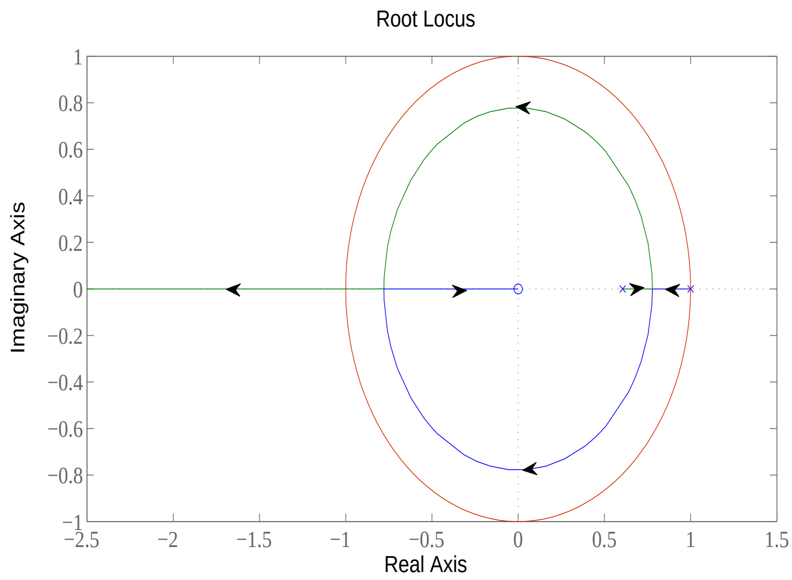

Figure 2: Root Locus when T=0.5 sec

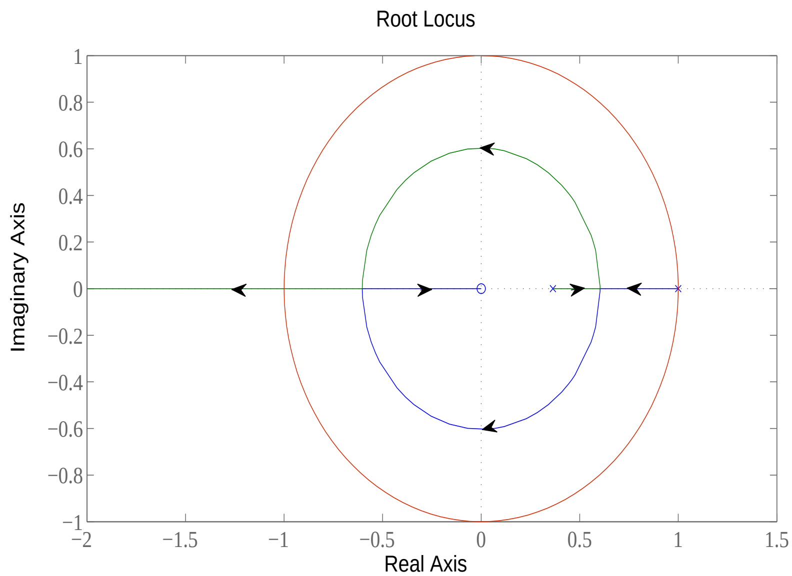

Figure 3: Root Locus when T=1 sec

the root locus for $K = 0$ to $K = 10$. It can be seen from the figure that the radius of the inside circle reduces and the maximum value of stable K also decreases to $K = 4.328$.

Similarly if $T = 2$ sec,

$$G(z) = \frac{0.8647Kz}{(z-1)(z-0.1353)}$$

One can find that the critical gain in this case further reduces to 2.626.

1.2.1 Effect of sampling period T

As can be seen from the previous example, large T has detrimental effect on relative stability. A thumb rule is to sample eight to ten times during a cycle of the damped sinusoidal oscillation of the output if it is underdamped. If overdamped 8/10 times during rise time.
As seen from the example making the sampling period smaller allows the critical gain to be larger, i.e., maximum allowable gain can be made larger by increasing sampling frequency /rate. It seems from the example that damping ratio decreases with the decrease in T. But one should take a note that damping ratio of the closed loop poles of a digital control system indicates the relative stability only if the sampling frequency is sufficiently high (8 to 10 times). If it is not the case, prediction of overshoot from the damping ratio will be erroneous and in practice the overshoot will be much higher than the predicted one.

Next, we may investigate the effect of T on the steady state error. Let us take a fixed gain $K = 2$.

When $T = 0.5$ sec. and $K = 2$,

$$G(z) = \frac{0.787z}{(z-1)(z-0.6065)}$$

Since this is a second order system, velocity error constant will be a non zero finite quantity.

$$Kv = \lim_{z\to 1} \frac{(1-z^{-1})G(z)}{T} = 4$$

$$\text{Thus,}\ \ e_{ss} = \frac{1}{4} = 0.25$$

When $T = 1$ sec. and $K = 2$

$$G(z) = \frac{1.2642z}{(z-1)(z-0.3679)}$$

$$Kv = \lim_{z\to 1} \frac{(1-z^{-1})G(z)}{T} = 2$$

$$e_{ss} = \frac{1}{2} = 0.5$$

When $T = 2$ sec. and $K = 2$

$$G(z) = \frac{1.7294z}{(z-1)(z-0.1353)}$$

$$Kv = \lim_{z\to 1} \frac{(1-z^{-1})G(z)}{T} = 1$$

$$e_{ss} = \frac{1}{1} = 1$$

Thus, increasing sampling period (decreasing sampling frequency) has an adverse effect on the steady state error as well.