Module 4: Time Response of discrete time systems
Lecture Note 1

1 Time Response of discrete time systems

Absolute stability is a basic requirement of all control systems. Apart from that, good relative stability and steady state accuracy are also required in any control system, whether continuous time or discrete time. **Transient response** corresponds to the system closed loop poles and **steady state response** corresponds to the excitation poles or poles of the input function.

1.1 Transient response specifications

In many practical control systems, the desired performance characteristics are specified in terms of time domain quantities. Unit step input is most commonly used in analysis of a system since it is easy to generate and represent a sufficiently drastic change thus providing useful information on both transient and steady state responses.

The transient response of a system depends on the initial conditions. It is a common practice to consider the system initially at rest.
Consider the digital control system shown in Figure1.

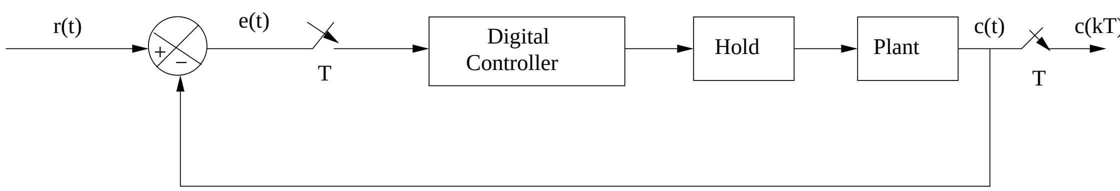

Figure 1: Block Diagram of a closed loop digital system

Similar to the continuous time case, transient response of a digital control system can also be characterized by the following.

1. Rise time (t_r): Time required for the unit step response to rise from 0% to 100% of its final value in case of underdamped system or 10% to 90% of its final value in case of overdamped system.

2. Delay time (t_d): Time required for the the unit step response to reach 50% of its final value.

3. Peak time (t_p): Time at which maximum peak occurs.

4. Peak overshoot (M_p): The difference between the maximum peak and the steady state value of the unit step response.

5. Settling time (t_s): Time required for the unit step response to reach and stay within 2% or 5% of its steady state value.

However since the output response is discrete the calculated performance measures may be slightly different from the actual values. Figure 2 illustrates this. The output has a maximum value $c_{\max}$ whereas the maximum value of the discrete output is $c^*_{\max}$ which is always less than or equal to $c_{\max}$. If the sampling period is small enough compared to the oscillations of the response then this difference will be small otherwise $c^*_{\max}$ may be completely erroneous.

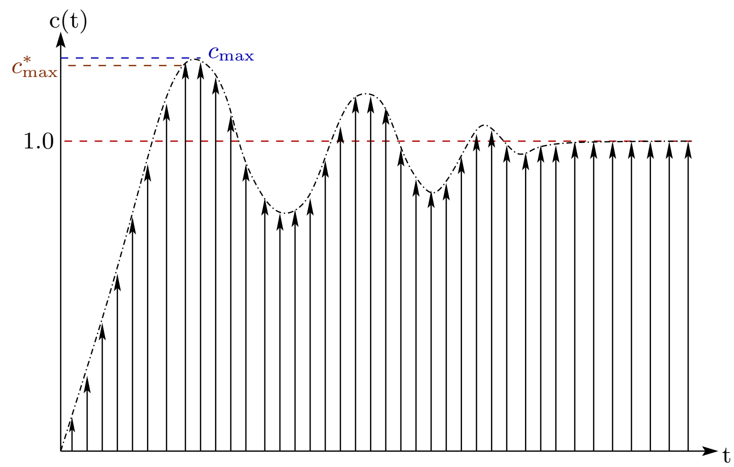

Figure 2: Unit step response of a discrete time system

1.2 Steady state error

The steady state performance of a stable control system is measured by the steady error due to step, ramp or parabolic inputs depending on the system type. Consider the discrete time system as shown in Figure 3.

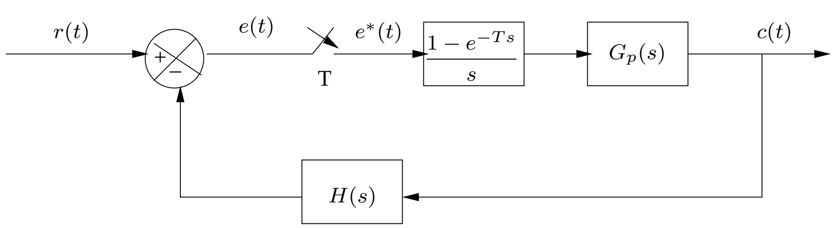

Figure 3: Block Diagram 2

From Figure 2, we can write

$$E(s) = R(s) - H(s)C(s)$$

We will consider the steady state error at the sampling instants.
From final value theorem

$$\begin{aligned}
\lim_{k\to\infty} e(kT) &= \lim_{z\to 1}\left[(1-z^{-1})E(z)\right] \\
G(z) &= (1-z^{-1})Z\left[\frac{G_p(s)}{s}\right] \\
GH(z) &= (1-z^{-1})Z\left[\frac{G_p(s)H(s)}{s}\right] \\
\frac{C(z)}{R(z)} &= \frac{G(z)}{1+GH(z)} \\
\text{Again, } E(z) &= R(z) - GH(z)E(z) \\
\text{or, } E(z) &= \frac{1}{1+GH(z)}R(z) \\
\Rightarrow e_{ss} &= \lim_{z\to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}R(z)\right]
\end{aligned}$$

The steady state error of a system with feedback thus depends on the input signal $R(z)$ and the loop transfer function $GH(z)$.

1.2.1 Type-0 system and position error constant

Systems having a finite nonzero steady state error with a zero order polynomial input (step input) are called **Type-0** systems. The position error constant for a system is defined for a step input.

$$\begin{aligned}
r(t) &= u_s(t) \quad \text{unit step input} \\
R(z) &= \frac{1}{1-z^{-1}} \\
e_{ss} &= \lim_{z\to 1}\frac{1}{1+GH(z)} = \frac{1}{1+K_p}
\end{aligned}$$

where $K_p = \lim_{z\to 1} GH(z)$ is known as the **position error constant**.

1.2.2 Type-1 system and velocity error constant

Systems having a finite nonzero steady state error with a first order polynomial input (ramp input) are called **Type-1** systems. The velocity error constant for a system is defined for a ramp input.

$$\begin{aligned}
r(t) &= u_r(t) \quad \text{unit ramp} \\
R(z) &= \frac{Tz}{(z-1)^2} \quad = \frac{TZ^{-1}}{(1-Z^{-1})^2} \\
e_{ss} &= \lim_{z\to 1}\frac{T}{(z-1)GH(z)} \quad = \frac{1}{K_v}
\end{aligned}$$

where $K_v = \frac{1}{T}\lim_{z\to 1}\left[(z-1)GH(z)\right]$ is known as the **velocity error constant**.

1.2.3 Type-2 system and acceleration error constant

Systems having a finite nonzero steady state error with a second order polynomial input (parabolic input) are called **Type-2** systems. The acceleration error constant for a system is defined for a parabolic input.

$$\begin{aligned} R(z) &= \frac{T^2 z(z+1)}{2(z-1)^3} \quad = \frac{T^2(1+z^{-1})z^{-1}}{2(1-z^{-1})^3} \\ e_{ss} &= \frac{T^2}{2}\lim_{z\to 1}\frac{(z+1)}{(z-1)^2\left[1+GH(z)\right]} \quad = \frac{1}{\lim_{z\to 1}\frac{(z-1)^2}{T^2}GH(z)} = \frac{1}{K_a} \end{aligned}$$

where $K_a = \lim\limits_{z\to 1}\dfrac{(z-1)^2}{T^2}GH(z)$ is known as the **acceleration error constant**.

Table 1 shows the steady state errors for different types of systems for different inputs.

Table 1: Steady state errors

System	Step input	Ramp input	Parabolic input
Type-0	$\frac{1}{1+K_p}$	∞	∞
Type-1	0	$\frac{1}{K_v}$	∞
Type-2	0	0	$\frac{1}{K_a}$

Example 1: Calculate the steady state errors for unit step, unit ramp and unit parabolic inputs for the system shown in Figure 4.

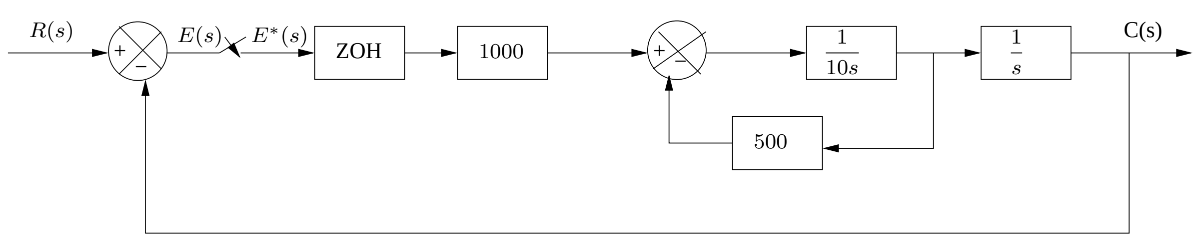

Figure 4: Block Diagram for Example 1

Solution: The open loop transfer function is:

$$\begin{aligned} G(s) &= \frac{C(s)}{E^*(s)} = G_{ho}(s)G_p(s) \\ &= \frac{1-e^{-Ts}}{s}\frac{1000/10}{s(s+500/10)} \end{aligned}$$

Taking Z-transform

$$\begin{aligned}
G(z) &= 2(1-z^{-1})\ \mathrm{Z}\left[\frac{1}{s^2} - \frac{10}{500s} + \frac{10}{500(s+5000)}\right] \\
&= 2(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{10z}{500(z-1)} + \frac{10z}{500(z-e^{-50T})}\right] \\
&= \frac{1}{250}\left[\frac{(500T-10+10e^{-50T})z-(500T+10)e^{-50T}+10}{(z-1)(z-e^{-50T})}\right]
\end{aligned}$$

Steady state error for step input $= \dfrac{1}{1+K_p}$ where $K_p = \lim_{z\to 1} G(z) = \infty$. $\Rightarrow e_{ss}^{step} = 0$.

Steady state error for ramp input $= \dfrac{1}{K_v}$ where $K_v = \frac{1}{T}\lim_{z\to 1}\left[(z-1)G(z)\right] = 2$. $\Rightarrow e_{ss}^{ramp} = 0.5$.

Steady state error for parabolic input $= \dfrac{1}{K_a}$ where $K_a = \frac{1}{T^2}\lim_{z\to 1}\left[(z-1)^2 G(z)\right] = 0$. $\Rightarrow$ $e_{ss}^{para} = \infty$.