Graph Theory: Lecture No. 1

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References

1. Reinhard Diestel : Graph Theory (Springer)
2. Douglas B. West: Introduction to Graph Theory (Prentice-Hall India)
3. A. Bondy and U.S.R. Murty: Graph Theory (Springer)
4. B. Bollobas : Modern Graph Theory (Springer)
What is a Graph?

It is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$ and a relation that associates with each edge two vertices (not necessarily distinct) called its end points.
- A loop
- Multiple edge.
- Simple Graph
- Finite Graph
Some simple graphs

- Complete Graph.
- Cycle
- Path
Subgraph and Induced Subgraph

- $H$ is a subgraph of $G$: Then $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. The assignment of end points to edges in $H$ is the same as that in $G$.

- $H$ is an induced subgraph of $G$ on $S$, where $S \subseteq V(G)$: Then $V(H) = S$ and $E(H)$ is the set of edges of $G$ such that both the end points belong to $S$. 
A graph $G$ is connected if each pair of vertices belongs to a path.
An Isomorphism from a simple graph $G$ to a simple graph $H$ is a bijection $f : V(G) \rightarrow V(H)$ such that $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$.
Forest and Tree

- A graph without any cycle is acyclic.
- A forest is an acyclic graph
- A tree is a connected acyclic graph.
Bipartite Graph

A graph $G$ is bipartite if $V(G)$ is the union of two disjoint (possibly empty) independent sets called partite sets of $G$. (A subset $S \subseteq V(G)$ is an independent set if the induced subgraph on $S$ contains no edges.)
A tree is a bipartite graph.

Can we say that the complete graph $K_n$ is bipartite?

The complete bipartite graph.
A set $S \subseteq V(G)$ is a vertex cover of $G$ (of the edges of $G$) if every edge of $G$ is incident with a vertex in $S$. A vertex cover of the minimum cardinality is called a minimum vertex cover. We will denote this set by MVC(G).
What is the cardinality of MVC in the complete graph $K_n$?
What about the complete bipartite graph $K_{m,n}$?
The cycle $C_n$, when $n$ is even and odd?
The cardinality of a biggest independent set in $G$ is called the independence number (or stability number) of $G$ and is denoted by $\alpha(G)$. 
Is there any relation between \(|MVC(G)|\) and \(\alpha(G)\)?

- If we remove a VC from \(G\), the rest is an independent set.
- So, if we remove MVC from \(G\), the rest, i.e. \(V - MVC\) is an independent set.
- So, \(\alpha(G) \geq n - |MVC(G)|\). Thus \(|MVC(G)| \geq n - \alpha(G)\).
- Similarly if we remove any independent set from \(G\), the rest is VC, and so \(|MVC| \leq n - \alpha(G)\).
- Thus we get \(|MVC| = n - \alpha(G)\).
- If we denote —MVC\((G)\)— by \(\beta(G)\), then we have \(\beta(G) + \alpha(G) = n\).
Matching

- A set $M$ of independent edges in a graph is called a matching.
- $M$ is a matching of $U \subseteq V(G)$, if every vertex in $U$ is incident with an edge in $M$.
- Then a vertex in $U$ is a matched vertex. The vertices which are not incident with any edge of $M$ is unmatched.
A matching $M$ is a perfect matching of $G$, if every vertex in $G$ is matched by $M$.
If $G$ has $n$ vertices, what is the cardinality of a perfect matching $M$ of $G$?
The cardinality of the biggest matching in $G$ can be denoted by $\alpha'(G)$. 
What is the value of $\alpha'(G)$ for:

- Cycle $C_n$
- Path $P_n$
- Complete Graph $K_n$
- Complete Bipartite graph $K_{m,n}$