Advanced Topics in Optimization

Piecewise Linear Approximation of a Nonlinear Function
Introduction and Objectives

Introduction

- There exists no general algorithm for nonlinear programming due to its irregular behavior.
- Nonlinear problems can be solved by first representing the nonlinear function (both objective function and constraints) by a set of linear functions and then apply simplex method to solve this using some restrictions.

Objectives

- To discuss the various methods to approximate a nonlinear function using linear functions.
- To demonstrate this using a numerical example.
Piecewise Linearization

- A nonlinear single variable function $f(x)$ can be approximated by a piecewise linear function.
- Geometrically, $f(x)$ can be shown as a curve being represented as a set of connected line segments.
Consider an optimization function having only one nonlinear term \( f(x) \)

Let the x-axis of the nonlinear function \( f(x) \) be divided by ‘\( p \)’ breaking points \( t_1, t_2, t_3, \ldots, t_p \)

Corresponding function values be \( f(t_1), f(t_2), \ldots, f(t_p) \)

If ‘\( x \)’ can take values in the interval \( 0 \leq x \leq X \), then the breaking points can be shown as

\[
0 \equiv t_1 < t_2 < \ldots < t_p \equiv X
\]
Piecewise Linearization: Method 1 …contd.

- Express ‘x’ as a weighted average of these breaking points
  \[ x = w_1 t_1 + w_2 t_2 + \ldots + w_p t_p \]
  
  i.e., \[ x = \sum_{i=1}^{p} w_i t_i \]

- Function \( f(x) \) can be expressed as
  \[ f(x) = w_1 f(t_1) + w_2 f(t_2) + \ldots + w_p f(t_p) = \sum_{i=1}^{p} w_i f(t_i) \]
  
  where \( \sum_{i=1}^{p} w_i = 1 \)
Finally the model can be expressed as

$$Max \ or \ Min \ f(x) = \sum_{i=1}^{p} w_i f(t_i)$$

subject to the additional constraints

$$\sum_{i=1}^{p} w_i t_i = x$$

$$\sum_{i=1}^{p} w_i = 1$$
Piecewise Linearization: Method 1 …contd.

- This linearly approximated model can be solved using simplex method with some restrictions
- Restricted condition:
  - There should not be more than two ‘\( w_i \)’ in the basis and
  - Two ‘\( w_i \)’ can take positive values only if they are adjacent. i.e., if ‘x’ takes the value between \( t_i \) and \( t_{i+1} \), then only \( w_i \) and \( w_{i+1} \) (contributing weights to the value of ‘x’) will be positive, rest all weights be zero
- In general, for an objective function consisting of ‘n’ variables (‘n’ terms) represented as

\[
\text{Max or Min } f(x) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)
\]
subjected to ‘m’ constraints
\[ g_{1j}(x_1) + g_{2j}(x_2) + \ldots + g_{nj}(x_n) \leq b_j \quad \text{for } j = 1, 2, \ldots, m \]

The linear approximation of this problem is

\[
\begin{align*}
\text{Max or Min} & \quad \sum_{k=1}^{n} \sum_{i=1}^{p} w_{ki} f_k(t_{ki}) \\
\text{subjected to} & \quad \sum_{k=1}^{n} \sum_{i=1}^{p} w_{ki} g_{kj}(t_{ki}) \leq b_j \quad \text{for } j = 1, 2, \ldots, m \\
& \quad \sum_{i=1}^{p} w_{ki} = 1 \quad \text{for } k = 1, 2, \ldots, n
\end{align*}
\]
Piecewise Linearization: Method 2

- 'x' is expressed as a sum, instead of expressing as the weighted sum of the breaking points as in the previous method
  
  \[ x = t_1 + u_1 + u_2 + \ldots + u_{p-1} = t_1 + \sum_{i=1}^{p-1} u_i \]

  where \( u_i \) is the increment of the variable 'x' in the interval \( (t_i, t_{i+1}) \) i.e., the bound of \( u_i \) is \( 0 \leq u_i \leq t_{i+1} - t_i \)

- The function \( f(x) \) can be expressed as
  
  \[ f(x) = f(t_1) + \sum_{i=1}^{p-1} \alpha_i u_i \]

  where \( \alpha_i \) represents the slope of the linear approximation between the points \( t_{i+1} \) and \( t_i \)

  \[ \alpha_i = \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i} \]
Finally the model can be expressed as

\[ \text{Max or Min } f(x) = f(t_1) + \sum_{i=1}^{p-1} \alpha_i u_i \]

subjected to additional constraints

\[ t_1 + \sum_{i=1}^{p-1} u_i = x \]

\[ 0 \leq u_i \leq t_{i+1} - t_i, \quad i = 1, 2, \ldots, p - 1 \]
The example below illustrates the application of method 1

Consider the objective function

$$\text{Maximize } f = x_1^3 + x_2$$

subject to

$$2x_1^2 + 2x_2 \leq 15$$
$$0 \leq x_1 \leq 4$$
$$x_2 \geq 0$$

The problem is already in separable form (i.e., each term consists of only one variable).
Piecewise Linearization: Numerical Example …contd.

- Split up the objective function and constraint into two parts
  \[ f = f_1(x_1) + f_2(x_2) \]
  \[ g_1 = g_{11}(x_1) + g_{12}(x_2) \]

  where

  \[ f_1(x_1) = x_1^3; \quad f_2(x_2) = x_2 \]
  \[ g_{11}(x_1) = 2x_1^2; \quad g_{12}(x_2) = 2x_2 \]

- \( f_2(x_2) \) and \( g_{12}(x_2) \) are treated as linear variables as they are in linear form
Consider five breaking points for $x_1$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t_{li}$</th>
<th>$f_i(t_{li})$</th>
<th>$g_{li}(t_{li})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>27</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>64</td>
<td>32</td>
</tr>
</tbody>
</table>

$f_1(x_1)$ can be written as,

$$f_1(x_1) = \sum_{i=1}^{5} w_{li} f_i(t_{li})$$

$$= w_{11} \times 0 + w_{12} \times 1 + w_{13} \times 8 + w_{14} \times 27 + w_{15} \times 64$$
Piecewise Linearization: Numerical Example
…contd.

- $g_{11}(x_1)$ can be written as,

$$g_{11}(x_1) = \sum_{i=1}^{5} w_i g_i(t_i)$$

$$= w_{11} \times 0 + w_{12} \times 2 + w_{13} \times 8 + w_{14} \times 18 + w_{15} \times 32$$

- Thus, the linear approximation of the above problem becomes

$$\text{Maximize } f = w_{12} + 8w_{13} + 27w_{14} + 64w_{15} + x_2$$

subject to

$$2w_{12} + 8w_{13} + 18w_{14} + 32w_{15} + 2x_2 + s_1 = 15$$

$$w_{11} + w_{12} + w_{13} + w_{14} + w_{15} = 1$$

$$w_{ii} \geq 0 \text{ for } i = 1, 2, ..., 5$$
This can be solved using simplex method in a restricted basis condition

The simplex tableau is shown below

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basis</th>
<th>$f$</th>
<th>$w_{11}$</th>
<th>$w_{12}$</th>
<th>$w_{13}$</th>
<th>$w_{14}$</th>
<th>$w_{15}$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$b_r$</th>
<th>$b_r/c_{rs}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$f$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-8</td>
<td>-27</td>
<td>-64</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>18</td>
<td>32</td>
<td>2</td>
<td>1</td>
<td>15</td>
<td>1.87</td>
</tr>
<tr>
<td>$w_{11}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Piecewise Linearization: Numerical Example …contd.

- From the table, it is clear that \( w_{15} \) should be the entering variable.
- \( s_1 \) should be the exiting variable.
- But according to restricted basis condition, \( w_{15} \) and \( w_{11} \) cannot occur together in basis as they are not adjacent.
- Therefore, consider the next best entering variable \( w_{14} \).
- This also is not possible, since \( s_1 \) should be exited and \( w_{14} \) and \( w_{11} \) cannot occur together.
- The next best variable \( w_{13} \), it is clear that \( w_{11} \) should be the exiting variable.
Piecewise Linearization: Numerical Example …contd.

- The simplex tableau is shown below

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basis</th>
<th>$f$</th>
<th>Variables</th>
<th>$b_r$</th>
<th>$\frac{b_r}{c_{rz}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$w_{11}$</td>
<td>$w_{12}$</td>
<td>$w_{13}$</td>
</tr>
<tr>
<td>1</td>
<td>$f$</td>
<td>1</td>
<td>8</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$s_1$</td>
<td>0</td>
<td>-8</td>
<td>-6</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$w_{13}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- The entering variable is $w_{15}$. Then the variable to be exited is $S_1$ and this is not acceptable since $w_{15}$ is not an adjacent point to $w_{13}$

- Next variable $w_{14}$ can be admitted by dropping $S_1$. 
Piecewise Linearization: Numerical Example …contd.

The simplex tableau is shown below

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basis</th>
<th>$f$</th>
<th>Variables</th>
<th>$b_r$</th>
<th>$\frac{b_r}{c_{ij}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$w_{11}$</td>
<td>$w_{12}$</td>
<td>$w_{13}$</td>
</tr>
<tr>
<td>1</td>
<td>$f$</td>
<td>1</td>
<td>-7.2</td>
<td>-4.4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$w_{14}$</td>
<td>0</td>
<td>-0.8</td>
<td>-0.6</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$w_{13}$</td>
<td>0</td>
<td>1.8</td>
<td>1.6</td>
<td>1</td>
</tr>
</tbody>
</table>

- Now, $w_{15}$ cannot be admitted since $w_{14}$ cannot be dropped
- Similarly $w_{11}$ and $w_{12}$ cannot be entered as $w_{13}$ cannot be dropped
Since there is no more variable to be entered, the process ends
- Therefore, the best solution is
  \[ w_{13} = 0.3; \quad w_{14} = 0.7 \]
- Now,
  \[ x_1 = \sum_{i=1}^{5} w_{i1} t_{1i} = w_{13} \times 2 + w_{14} \times 3 = 2.7 \]
  
  and \( x_2 = 0 \)
- The optimum value is \( f = 21.3 \)
- This may be an approximate solution to the original nonlinear problem
- However, the solution can be improved by taking finer breaking points
Thank You