Dynamic Programming
Applications

Water Allocation
Introduction and Objectives

- **Dynamic Programming**: Sequential or multistage decision making process
- **Water Allocation problem** is solved as a sequential process using dynamic programming

Objectives
- To discuss the Water Allocation Problem
- To explain and develop recursive equations for backward approach
- To explain and develop recursive equations for forward approach
Water Allocation Problem

- Consider a canal supplying water for three different crops
- Maximum capacity of the canal is $Q$ units of water.
- Amount of water allocated to each field as $x_i$
Water Allocation Problem … contd.

- Net benefits from producing the crops can be expressed as a function of the water allotted.

\[
NB_1(x_1) = 5x_1 - 0.5x_1^2 \\
NB_2(x_2) = 8x_2 - 1.5x_2^2 \\
NB_3(x_3) = 7x_3 - x_3^2
\]

- Optimization Problem: Determine the optimal allocations \( x_i \) to each crop that maximizes the total net benefits from all the three crops.
Solution using Dynamic Programming

- Structure the problem as a sequential allocation process or a multistage decision making procedure.
- Allocation to each crop is considered as a decision stage in a sequence of decisions.
- Amount of water allocated to crop $i$ is $x_i$
- Net benefit from this allocation is $NB_i(x_i)$
- Introduce one state variable $S_i$ :- Amount of water available to the remaining $(3-i)$ crops
- State transformation equation can be written as

  $$S_{i+1} = S_i - x_i$$

defines the state in the next stage
Sequential Allocation Process

The allocation problem is shown as a sequential process.

Available Quantity, $S_1 = Q$

- $S_2 = S_1 - x_1$
  - Crop 1
    - Net Benefits, $NB_1(x_1)$

- $S_3 = S_2 - x_2$
  - Crop 2
    - Net Benefits, $NB_2(x_2)$

- $S_4 = S_3 - x_3$
  - Crop 3
    - Net Benefits, $NB_3(x_3)$

- Remaining Quantity,
**Backward Recursive Equations**

- **Objective function:** To maximize the net benefits

  \[
  \max \sum_{i=1}^{3} NB_i(x_i)
  \]

- **Subjected to the constraints**

  \[
  x_1 + x_2 + x_3 \leq Q \\
  0 \leq x_i \leq Q \quad \text{for } i = 1, 2, 3
  \]

- Let \( f_1(Q) \) be the maximum net benefits that can be obtained from allocating water to crops 1, 2 and 3

  \[
  f_1(Q) = \max \left[ \sum_{i=1}^{3} NB_i(x_i) \right] \\
  \text{subject to } \begin{cases} 
  x_1 + x_2 + x_3 \leq Q \\
  x_1, x_2, x_3 \geq 0
  \end{cases}
  \]
Backward Recursive Equations ... contd.

Transforming this into three problems each having only one decision variable

\[ f_1(Q) = \max_{x_1} \left[ NB_1(x_1) + \max_{x_2} \left\{ NB_2(x_2) + \max_{x_3} \left[ NB_3(x_3) \right]\right\}\right] \]

Now starting from the last stage, let \( f_3(S_3) \) be the maximum net benefits from crop 3.

State variable \( S_3 \) for this stage can vary from 0 to Q

Thus,

\[ f_3(S_3) = \max_{x_3} NB_3(x_3) \]

\( 0 \leq x_3 \leq S_3 \)
But $S_3 = S_2 - x_2$. Therefore $f_3(S_3) = f_3(S_2 - x_2)$

Hence,

$$f_1(Q) = \max_{0 \leq x_1 \leq Q} \left[ NB_1(x_1) + \max_{0 \leq x_2 \leq Q-x_1=S_2} \{NB_2(x_2) + f_3(S_2 - x_2)\} \right]$$

Now, let $f_2(S_2)$ be the maximum benefits derived from crops 2 and 3 for a given quantity $S_2$ which can vary between 0 and $Q$

Therefore,

$$f_2(S_2) = \max_{0 \leq x_2 \leq Q-x_1=S_2} \{NB_2(x_2) + f_3(S_2 - x_2)\}$$
Backward Recursive Equations ... contd.

Now since $S_2 = Q - x_1$, $f_1(Q)$ can be rewritten as

$$f_1(Q) = \max_{0 \leq x_1 \leq Q} \left[ NB_1(x_1) + f_2(Q - x_1) \right]$$

Once the value of $f_3(S_3)$ is calculated, the value of $f_2(S_2)$ can be determined, from which $f_1(Q)$ can be determined.
**Forward Recursive Equations**

- Let the function \( f_i(S_i) \) be the total net benefit from crops 1 to \( i \) for a given input of \( S_i \) which is allocated to those crops.
- Considering the first stage,
  \[
  f_1(S_1) = \max_{x_1 \leq S_1} NB_1(x_1)
  \]
- Solve this equation for a range of \( S_1 \) values from 0 to \( Q \)
- Considering the first two crops, for an available quantity of \( S_2 \), \( f_2(S_2) \) can be written as
  \[
  f_2(S_2) = \max_{x_2 \leq S_2} [NB_2(x_2) + f_1(S_2 - x_2)]
  \]
Forward Recursive Equations ... contd.

- $S_2$ ranges from 0 to $Q$
- Considering the whole system, $f_3(S_3)$ can be expressed as,

$$f_3(S_3) = \max_{x_3 \leq S_3 = Q} \left[ NB_3(x_3) + f_2(S_3 - x_3) \right]$$

- If the whole $Q$ units of water should be allocated then the value of $S_3$ can be taken as equal to $Q$
- Otherwise, $S_3$ will take a range of values from 0 to $Q$
Conclusion

The basic equations for the water allocation problem using both the approaches are discussed.

A numerical problem and its solution will be described in the next lecture.
Thank You