Linear Programming Applications

Structural & Water Resources Problems
Introduction

- LP has been applied to formulate and solve several types of problems in engineering field
- LP finds many applications in the field of water resources and structural design which include
  - Planning of urban water distribution
  - Reservoir operation
  - Crop water allocation
  - Minimizing the cost and amount of materials in structural design
Objectives

- To discuss the applications of LP in the plastic design of frame structures
- To discuss the applications of LP in deciding the optimal pattern of irrigation
Example – Structural Design

- A beam column arrangement of a rigid frame is shown.
- Moment in beam is represented by $M_b$.
- Moment in column is denoted by $M_c$.
- $l = 8$ units and $h = 6$ units.
- Forces $F_1 = 2$ units and $F_2 = 1$ unit.

Assuming that plastic moment capacity of beam and columns are linear functions of their weights; the objective function is to minimize weights of the materials.
Example - Structural Design ...contd.

Solution:

- In the limit design, it is assumed that at the points of peak moments, plastic hinges will be developed.
- Points of development of peak moments are numbered in the above figure from 1 through 7.
- Development of sufficient hinges makes the structure unstable known as a collapse mechanism.
- For the design to be safe the energy absorbing capacity of the frame \( U \) should be greater than the energy imparted by externally applied load \( E \) for the various collapse mechanisms of the structure.
Example - Structural Design ...contd.

- The objective function can be written as
  
  Minimize \( f = \text{weight of beam} + \text{weight of column} \)

  \[
  f = w(2lM_b + 2hM_c)
  \]  \( (1) \)

  where \( w \) is weight per unit length over unit moment in material

- Since \( w \) is constant, optimizing (1) is same as optimizing

  \[
  f = (2lM_b + 2hM_c) = 16M_b + 12M_c
  \]
Example - Structural Design ...contd.

- Four possible collapse mechanisms are shown in the figure below with the corresponding $U$ and $E$ values.

![Diagrams showing four possible collapse mechanisms](image)

(a) $E = F_1 \delta_1 = 12 \theta$

$U = 2M_b \theta + 2M_c \theta$

(b) $E = F_2 \delta = 8 \theta$

$U = 4M_b \theta$
Example - Structural Design …contd.

\[ B = F_1 \delta_1 = 2 \theta \]
\[ U = 2 M_b \theta + 2 M_c \theta \]

(c)

\[ B = F_1 \delta_1 = 12 \theta \]
\[ U = 4 M_b \theta + 2 M_c \theta \]

(d)
Example - Structural Design ...contd.

- The optimization problem can be stated as

\[
\text{Minimize } f = 16M_b + 12M_c
\]

subject to

\[
\begin{align*}
M_c & \geq 3 \\
M_b & \geq 2 \\
2M_b + M_c & \geq 10 \\
M_b + M_c & \geq 6 \\
M_b & \geq 0; \quad M_c \geq 0
\end{align*}
\]
Example - Structural Design …contd.

- Introducing slack variables $X_1, X_2, X_3, X_4$ all, the system of equations can be written in canonical form as

\[
\begin{align*}
16M_B + 12M_C - f &= 0 \\
-M_c + X_1 &= -3 \\
-M_b + X_2 &= -2 \\
-2M_b - M_c + X_3 &= -10 \\
-M_b - M_c + X_4 &= -6 \\
16M_B + 12M_C - f &= 0
\end{align*}
\]
Example - Structural Design …contd.

- This model can be solved using Dual Simplex algorithm
- The final tableau is shown below

*Iteration 2:*

The optimal value of decision variables are $M_B = 7/2$; $M_C = 3$

And the total weight of the material required $f = 92w$ units

| Basic Variables | Variables |  |  |  |  | $b_2$
|-----------------|-----------|---|---|---|---|---|
|                 | $M_B$     | $M_C$ | $X_1$ | $X_2$ | $X_3$ | $X_4$
| $f$             | 0         | 0    | -4    | 0     | -8    | 0    | 92  |
| $M_C$           | 0         | 1    | -1    | 0     | 0     | 0    | 3   |
| $X_2$           | 0         | 0    | $\frac{1}{2}$ | 1     | $-\frac{1}{2}$ | 0    | 3/2 |
| $M_B$           | 1         | 0    | $\frac{1}{2}$ | 0     | $-\frac{1}{2}$ | 0    | 7/2 |
| $X_4$           | 0         | 0    | $-\frac{1}{2}$ | 0     | $-\frac{1}{2}$ | 1    | 1   |
| Ratio           |           |      |       |       |       |      |     |
Example - Irrigation Allocation

- Consider two crops 1 and 2. One unit of crop 1 produces four units of profit and one unit of crop 2 brings five units of profit. The demand of production of crop 1 is A units and that of crop 2 is B units. Let $x$ be the amount of water required for A units of crop 1 and $y$ be the same for B units of crop 2.

- The amount of production and the amount of water required can be expressed as a linear relation as shown below:

\[
A = 0.5(x - 2) + 2
\]
\[
B = 0.6(y - 3) + 3
\]
Example - Irrigation Allocation ...contd.

- Consider two crops 1 and 2. One unit of crop 1 produces four units of profit and one unit of crop 2 brings five units of profit. The demand of production of crop 1 is A units and that of crop 2 is B units. Let $x$ be the amount of water required for A units of crop 1 and $y$ be the same for B units of crop 2.

- The amount of production and the amount of water required can be expressed as a linear relation as shown below

$$A = 0.5(x - 2) + 2$$

$$B = 0.6(y - 3) + 3$$
Example - Irrigation Allocation ...contd.

Solution:

- Objective: Maximize the profit from crop 1 and 2
  
  \[ \text{Maximize } f = 4A + 5B; \]

- Expressing as a function of the amount of water,

  \[ \text{Maximize } f = 4[0.5(x - 2) + 2] + 5[0.6(y - 3) + 3] \]
  \[ f = 2x + 3y + 10 \]
Example - Irrigation Allocation ...contd.

subject to

- $x + y \leq 10$ ; Maximum availability of water
- $x \geq 2$ ; Minimum amount of water required for crop 1
- $y \geq 3$ ; Minimum amount of water required for crop 2
- The above problem is same as maximizing

$$f' = 2x + 3y$$

subject to same constraints.
Example - Irrigation Allocation ...contd.

- Changing the problem into standard form by introducing slack variables $S_1, S_2, S_3$

  $$\text{Maximize } f'' = 2x + 3y$$

  subject to

  $$x + y + S_1 = 10$$
  $$-x + S_2 = -2$$
  $$-y + S_3 = -3$$

This model is solved using simplex method
Example - Irrigation Allocation ...contd.

The final tableau is as shown

<table>
<thead>
<tr>
<th>Basic Variables</th>
<th>Variables</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$</td>
<td>$y$</td>
<td>$s_1$</td>
</tr>
<tr>
<td>$f'$</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The solution is $x = 2; \ y = 8; \ f' = 28$

Therefore, $f = 28 + 10 = 38$

Water allocated to crop A is 2 units and to crop B is 8 units and total profit yielded is 38 units.
Example – Water Quality Management

- Waste load allocation for water quality management in a river system can be defined as
  - Determination of optimal treatment level of waste, which is discharged to a river
  - By maintaining the water quality standards set by Pollution Control Agency (PCA), through out the river
- Conventional waste load allocation involves minimization of treatment cost subject to the constraint that the water quality standards are not violated
Consider a simple problem of \( M \) dischargers, who discharge waste into the river, and \( I \) checkpoints, where the water quality is measured by PCA.

- Let \( x_j \) is the treatment level and \( a_j \) is the unit treatment cost for \( j^{th} \) discharger (\( j=1,2,...,M \)).
- \( c_i \) is the dissolved oxygen (DO) concentration at checkpoint \( i \) (\( i=1,2,...,I \)), which is to be controlled.
- Decision variables for the waste load allocation model are \( x_j \) (\( j=1,2,...,M \)).
Example - Waster Quality Management ...contd.

- Objective function can be expressed as

  \[ \text{Maximize} \quad f = \sum_{j=1}^{M} a_j x_j \]

- Relationship between the water quality indicator, \( c_i \) (DO) at a checkpoint and the treatment level upstream to that checkpoint is linear (based on Streeter-Phelps Equation)

- Let \( g(x) \) denotes the linear relationship between \( c_i \) and \( x_j \).

- Then, \( c_i = g(x_j) \quad \forall i, j \)
Example - Waster Quality Management ... contd.

- Let $c_p$ be the permissible DO level set by PCA, which is to be maintained throughout the river.
- Therefore, $c_i \geq c_p \quad \forall i$
- This model can be solved using simplex algorithm which will give the optimal fractional removal levels required to maintain the water quality of the river.
Thank You