Linear Programming

Simplex method - 1
Introduction and Objectives

*Simplex method* is the most popular method used for the solution of *Linear Programming Problems* (LPP).

**Objectives**
- To discuss the motivation of simplex method
- To discuss Simplex algorithm
- To demonstrate the construction of simplex tableau
Motivation of Simplex method

- Solution of a LPP, if exists, lies at one of the vertices of the feasible region.
- All the basic solutions can be investigated one-by-one to pick up the optimal solution.
- For 10 equations with 15 variables there exists $15C_{10} = 3003$ basic feasible solutions!
- Too large number to investigate one-by-one.
- This can be overcome by simplex method
Simplex Method: Concept in 3D case

- In 3D, a feasible region (i.e., volume) is bounded by several surfaces.
- Each vertex (a basic feasible solution) of this volume is connected to the three other adjacent vertices by a straight line to each, being intersection of two surfaces.
- *Simplex algorithm* helps to move from one vertex to another adjacent vertex which is closest to the optimal solution among all other adjacent vertices.
- Thus, it follows the shortest route to reach the optimal solution from the starting point.
Simplex Method: Concept in 3D case

Thus there is no need to investigate all the basic feasible solutions.

Sequence of basic feasible solutions on the shortest route is generated by \textit{simplex algorithm}.
General procedure of simplex method

Simplex method involves following steps

1. General form of given LPP is transformed to its *canonical form* (refer Lecture notes 1).
2. Find a basic feasible solution of the LPP (there should exist at least one).
3. Move to an adjacent basic feasible solution which is closest to the optimal solution among all other adjacent vertices.
4. Repeat until optimum solution is achieved.

Step three involves ‘Simplex Algorithm’
Simplex Algorithm

Let us consider the following LPP

Maximize \( Z = 4x_1 - x_2 + 2x_3 \)
subject to \( 2x_1 + x_2 + 2x_3 \leq 6 \)
\( x_1 - 4x_2 + 2x_3 \leq 0 \)
\( 5x_1 - 2x_2 - 2x_3 \leq 4 \)
\( x_1, x_2, x_3 \geq 0 \)
**Simplex Algorithm … contd.**

LPP is transformed to its standard form

Maximize \[ Z = 4x_1 - x_2 + 2x_3 \]

subject to \[ 2x_1 + x_2 + 2x_3 + x_4 = 6 \]
\[ x_1 - 4x_2 + 2x_3 + x_5 = 0 \]
\[ 5x_1 - 2x_2 - 2x_3 + x_6 = 4 \]
\[ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \]

Note that \( x_4, x_5 \) and \( x_6 \) are slack variables
**Simplex Algorithm ... contd.**

Set of equations, including the objective function is transformed to canonical form

\[-4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6 + Z = 0\]
\[2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 = 6\]
\[x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 = 0\]
\[5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 = 4\]

Basic feasible solution of above canonical form is

\[x_4 = 6, \ x_5 = 0, \ x_6 = 4, \ x_1 = x_2 = x_3 = 0 \text{ and } Z = 0\]

\[x_4, \ x_5, \ x_6 : \text{Basic Variables}; \ x_1, \ x_2, \ x_3 : \text{Nonbasic Variables}\]
Simplex Algorithm … contd.

Symbolized form (for ease of discussion)

\[
\begin{align*}
(Z) & \quad c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_5 + Z = b \\
(x_4) & \quad c_{41}x_1 + c_{42}x_2 + c_{43}x_3 + c_{44}x_4 + c_{45}x_5 + c_{46}x_5 = b_4 \\
(x_5) & \quad c_{51}x_1 + c_{52}x_2 + c_{53}x_3 + c_{54}x_4 + c_{55}x_5 + c_{56}x_5 = b_5 \\
(x_6) & \quad c_{61}x_1 + c_{62}x_2 + c_{63}x_3 + c_{64}x_4 + c_{65}x_5 + c_{66}x_5 = b_6
\end{align*}
\]

- The left-most column is known as **basis** as this is consisting of basic variables
- The coefficients in the first row \((c_1, \ldots, c_6)\) are known as **cost coefficients**.
- Other subscript notations are self explanatory
Simplex Algorithm … contd.

This completes first step of algorithm. After completing each step (iteration) of algorithm, following three points are to be examined:

1. Is there any possibility of further improvement?
2. Which nonbasic variable is to be entered into the basis?
3. Which basic variable is to be exited from the basis?
Is there any possibility of further improvement?
If any one of the cost coefficients is negative further improvement is possible.

Which nonbasic variable is to be entered?
Entering variable is decided such that the unit change of this variable should have maximum effect on the objective function. Thus the variable having the coefficient which is minimum among all cost coefficients is to be entered, i.e., \( x_s \) is to be entered if cost coefficient \( c_s \) is minimum.
Simplex Algorithm … contd.

Which basic variable is to be exited?

After deciding the entering variable $x_s$, $x_r$ (from the set of basic variables) is decided to be the exiting variable if \( \frac{b_r}{c_{rs}} \) is minimum for all possible $r$, provided $c_{rs}$ is positive.

$c_{rs}$ is considered as pivotal element to obtain the next canonical form.
**Simplex Algorithm ... contd.**

**Entering variable**
c₁ is minimum (-4), thus, x₁ is the entering variable for the next step of calculation.

**Exiting variable**
r may take any value from 4, 5 and 6. It is found that \( \frac{b₉}{c₄₁} = \frac{6}{2} = 3, \frac{b₅}{c₅₁} = \frac{0}{1} = 0 \)

\( \frac{b₆}{c₆₁} = \frac{4}{5} = 0.8 \). As \( \frac{b₅}{c₅₁} \) is minimum, r is 5. Thus, x₅ is to be exited.

\( c₅₁ (= 1) \) is considered as pivotal element and x₅ is replaced by x₁ in the basis.

Thus a new canonical form is obtained through pivotal operation, which was explained in first class.
Simplex Algorithm … contd.

Pivotal operation as a refresher

- Pivotal row is transformed by dividing it with the pivotal element. In this case, pivotal element is 1.

- For other rows: Let the coefficient of the element in the pivotal column of a particular row be “l”. Let the pivotal element be “m”. Then the pivotal row is multiplied by ‘l/m’ and then subtracted from that row to be transformed. This operation ensures that the coefficients of the element in the pivotal column of that row becomes zero, e.g., Z row: \( l = -4 \), \( m = 1 \). So, pivotal row is multiplied by \( l/m = -4/1 = -4 \), obtaining \(-4x_1 + 16x_2 - 8x_3 + 0x_4 - 4x_5 + 0x_6 = 0\). This is subtracted from Z row, obtaining,

\[
\begin{align*}
0x_1 - 15x_2 + 6x_3 + 0x_4 + 4x_5 + 0x_6 + Z &= 0
\end{align*}
\]

The other two rows are also suitably transformed.
After the pivotal operation, the canonical form obtained as follows

\[
\begin{align*}
(Z) & : 0x_1 - 15x_2 + 6x_3 + 0x_4 + 4x_5 + 0x_6 + Z = 0 \\
(x_4) & : 0x_1 + 9x_2 - 2x_3 + 1x_4 - 2x_5 + 0x_6 = 6 \\
(x_1) & : 1x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 = 0 \\
(x_6) & : 0x_1 + 18x_2 - 12x_3 - 0x_4 - 5x_5 + 1x_6 = 4
\end{align*}
\]

The basic solution of above canonical form is \(x_1 = 0, x_4 = 6, x_6 = 4, x_2 = x_3 = x_5 = 0\) and \(Z = 0\).

Note that cost coefficient \(c_2\) is negative. Thus optimum solution is not yet achieved. Further improvement is possible.
**Simplex Algorithm … contd.**

**Entering variable**

$c_2$ is minimum (-15), thus, $x_2$ is the entering variable for the next step of calculation.

**Exiting variable**

$r$ may take any value from 4, 1 and 6. However, $c_{12}$ is negative (-4). Thus, $r$ may be either 4 or 6. It is found that $\frac{b_4}{c_{42}} = \frac{6}{9} = 0.667$ and $\frac{b_6}{c_{62}} = \frac{4}{18} = 0.222$. As $\frac{b_6}{c_{62}}$ is minimum, $r$ is 6. Thus $x_6$ is to be exited. $c_{62}$ (=18) is considered as pivotal element and $x_6$ is to be replaced by $x_2$ in the basis.

Thus another canonical form is obtained.
**Simplex Algorithm ... contd.**

The canonical form obtained after third iteration.

\[
egin{align*}
(Z) & : 0x_1 + 0x_2 - 4x_3 + 0x_4 - \frac{1}{6}x_5 + \frac{5}{6}x_6 + Z = \frac{10}{3} \\
(x_4) & : 0x_1 + 0x_2 + 4x_3 + 1x_4 + \frac{1}{2}x_5 - \frac{1}{2}x_6 = 4 \\
(x_1) & : 1x_1 + 0x_2 - \frac{2}{3}x_3 + 0x_4 - \frac{1}{9}x_5 + \frac{2}{9}x_6 = \frac{8}{9} \\
(x_2) & : 0x_1 + 1x_2 - \frac{2}{3}x_3 + 0x_4 - \frac{5}{18}x_5 + \frac{1}{18}x_6 = \frac{2}{9}
\end{align*}
\]

The basic solution of above canonical form is

\[x_1 = \frac{8}{9}, x_2 = \frac{2}{9}, x_4 = 4, x_3 = x_5 = x_6 = 0\] and \(Z = \frac{10}{3}\).
Simplex Algorithm … contd.

Note that cost coefficient $c_3$ is negative. Thus optimum solution is not yet achieved. Further improvement is possible.

**Entering variable** (Following the similar procedure)

$x_3$ is the entering variable for the next step of calculation.

**Exiting variable** (Following the similar procedure)

$x_4$ is the exiting variable. Thus $c_{43} (= 4)$ is the pivotal element and $x_4$ is to be replaced by $x_3$ in the basis.

Thus another canonical form is obtained.
**Simplex Algorithm … contd.**

The canonical form obtained after fourth iteration

\[
\begin{align*}
(Z) & \quad 0x_1 + 0x_2 + 0x_3 + 1x_4 + \frac{1}{3}x_5 + \frac{1}{3}x_6 + Z = \frac{22}{3} \\
(x_3) & \quad 0x_1 + 0x_2 + 1x_3 + \frac{1}{4}x_4 + \frac{1}{8}x_5 - \frac{1}{8}x_6 = 1 \\
(x_1) & \quad 1x_1 + 0x_2 + 0x_3 + \frac{1}{6}x_4 - \frac{1}{36}x_5 + \frac{5}{36}x_6 = \frac{14}{9} \\
(x_2) & \quad 0x_1 + 1x_2 + 0x_3 + \frac{1}{6}x_4 - \frac{7}{36}x_5 - \frac{1}{36}x_6 = \frac{8}{9}
\end{align*}
\]

The basic solution of above canonical form is

\[x_1 = 14/9, \ x_2 = 8/9, \ x_3 = 1, \ x_4 = x_5 = x_6 = 0 \text{ and } Z = 22/3.\]
Simplex Algorithm … contd.

Note that all the cost coefficients are nonnegative. Thus the optimum solution is achieved.

Optimum solution is

\[ Z = \frac{22}{3} = 7.333 \]

\[ x_1 = \frac{14}{9} = 1.556 \]

\[ x_2 = \frac{8}{9} = 0.889 \]

\[ x_3 = 1 \]
Construction of Simplex Tableau: General notes

Calculations shown till now can be presented in a tabular form, known as simplex tableau.

After preparing the canonical form of the given LPP, first simplex tableau is constructed.

After completing each simplex tableau (iteration), few steps (somewhat mechanical and easy to remember) are followed.

Logically, these steps are exactly similar to the procedure described earlier.
Construction of *Simplex Tableau: Basic steps*

**Check for optimum solution:**

1. Investigate whether all the elements (coefficients of the variables headed by that column) in the first row (i.e., $Z$ row) are nonnegative or not. If all such coefficients are nonnegative, optimum solution is obtained and no need of further iterations. If any element in this row is negative follow next steps to obtain the simplex tableau for next iteration.
Construction of Simplex Tableau: Basic steps

Operations to obtain next simplex tableau:

2. Identify the entering variable (described earlier) and mark that column as *Pivotal Column*.

3. Identify the exiting variable from the basis as described earlier and mark that row as *Pivotal Row*.

4. Mark the coefficient at the intersection of *Pivotal Row* and *Pivotal Column* as *Pivotal Element*.
Construction of *Simplex Tableau*: *Basic steps*

Operations to obtained simplex tableau...contd.:

5. In the basis, replace the exiting variable by entering variable.
6. Divide all the elements in the pivotal row by pivotal element.
7. For any other row, identify the elementary operation such that the coefficient in the pivotal column, in that row, becomes zero. Apply the same operation for all other elements in that row and change the coefficients.

Follow similar procedure for all other rows.
Consider the same problem discussed before. Canonical form of this LPP is

\[-4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6 + Z = 0\]
\[2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 = 6\]
\[x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 = 0\]
\[5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 = 4\]
## Construction of Simplex Tableau: example

### Corresponding simplex tableau

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basis</th>
<th>Z</th>
<th>Variables</th>
<th>b_r</th>
<th>b_r/c_{rs}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td></td>
<td>Z</td>
<td>-4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>x_4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>x_5</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>x_6</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

Pivotal Row

Pivotal Column

Pivotal Element
Construction of *Simplex Tableau*: example

Successive iterations

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basis</th>
<th>Z</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$b_r$</th>
<th>$\frac{b_r}{c_{rs}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td>Z</td>
<td>1</td>
<td>0</td>
<td>-15</td>
<td>6</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x_4$</td>
<td>0</td>
<td>0</td>
<td>9</td>
<td>-2</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>-4</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x_6$</td>
<td>0</td>
<td>0</td>
<td>18</td>
<td>-12</td>
<td>0</td>
<td>-5</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
Construction of *Simplex Tableau: example*

Successive iterations…contd.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basis</th>
<th>Z</th>
<th>Variables</th>
<th>$b_r$</th>
<th>$\frac{b_r}{c_{rs}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>4</td>
<td>$x_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-2/3</td>
</tr>
<tr>
<td></td>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2/3</td>
</tr>
</tbody>
</table>
### Final Tableau

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basis</th>
<th>$Z$</th>
<th>Variables</th>
<th>$b_r$</th>
<th>$c_{rs}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

All the coefficients are nonnegative. Thus optimum solution is achieved.

Value of $x_3$, $x_1$, $x_2$
All the elements in the first row (i.e., Z row), at iteration 4, are nonnegative. Thus, optimum solution is achieved.

Optimum solution is

\[
Z = \frac{22}{3} = 7.333
\]

\[
x_1 = \frac{14}{9} = 1.556
\]

\[
x_2 = \frac{8}{9} = 0.889
\]

\[
x_3 = 1
\]
Construction of Simplex Tableau: A note

It can be noted that at any iteration the following two points must be satisfied:

1. All the basic variables (other than Z) have a coefficient of zero in the Z row.
2. Coefficients of basic variables in other rows constitute a unit matrix.

Violation of any of these points at any iteration indicates a wrong calculation. However, reverse is not true.
Thank You