Optimization using Calculus

Kuhn-Tucker Conditions
Introduction

- Optimization with multiple decision variables and equality constraint: Lagrange Multipliers.
- Optimization with multiple decision variables and inequality constraint: Kuhn-Tucker (KT) conditions.
- KT condition: Both necessary and sufficient if the objective function is concave and each constraint is linear or each constraint function is concave, i.e. the problems belongs to a class called the convex programming problems.
Consider the following optimization problem

Minimize \( f(X) \)

subject to

\[ g_j(X) \leq 0 \quad \text{for } j=1,2,\ldots,p \]

where the decision variable vector

\[ X=[x_1,x_2,\ldots,x_n] \]
Kuhn Tucker Conditions

Kuhn-Tucker conditions for $\mathbf{X}^* = [x_1^* \ x_2^* \ \ldots \ x_n^*]$ to be a local minimum are:

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \quad i = 1, 2, \ldots, n$$

$$\lambda_j g_j = 0 \quad j = 1, 2, \ldots, m$$

$$g_j \leq 0 \quad j = 1, 2, \ldots, m$$

$$\lambda_j \geq 0 \quad j = 1, 2, \ldots, m$$
Kuhn Tucker Conditions …contd.

- In case of minimization problems, if the constraints are of the form \( g_j(X) \geq 0 \), then \( \lambda_j \) have to be non-positive.
- On the other hand, if the problem is one of maximization with the constraints in the form \( g_j(X) \geq 0 \), then \( \lambda_j \) have to be nonnegative.
Example (1)

Minimize \[ f = x_1^2 + 2x_2^2 + 3x_3^2 \]

subject to

\[ g_1 = x_1 - x_2 - 2x_3 \leq 12 \]
\[ g_2 = x_1 + 2x_2 - 3x_3 \leq 8 \]
Example (1) …contd.

Kuhn – Tucker Conditions

\[ \frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0 \]

2. \[ 2x_1 + \lambda_1 + \lambda_2 = 0 \]
3. \[ 4x_2 - \lambda_1 + 2\lambda_2 = 0 \]
4. \[ 6x_3 - 2\lambda_1 - 3\lambda_2 = 0 \]

\[ \lambda_j g_j = 0 \]
5. \[ \lambda_1 (x_1 - x_2 - 2x_3 - 12) = 0 \]
6. \[ \lambda_2 (x_1 + 2x_2 - 3x_3 - 8) = 0 \]

\[ g_j \leq 0 \]
7. \[ x_1 - x_2 - 2x_3 - 12 \leq 0 \]
8. \[ x_1 + 2x_2 - 3x_3 - 8 \leq 0 \]

\[ \lambda_j \geq 0 \]
9. \[ \lambda_1 \geq 0 \]
10. \[ \lambda_2 \geq 0 \]
Example (1) …contd.

From (5) either \( \lambda_1 = 0 \) or \( x_1 - x_2 - 2x_3 - 12 = 0 \)

**Case 1**

- From (2), (3) and (4) we have \( x_1 = x_2 = -\lambda_2 / 2 \) and \( x_3 = \lambda_2 / 2 \)
- Using these in (6) we get \( \lambda_2^2 + 8\lambda_2 = 0 \), \( \therefore \lambda_2 = 0 \) or \(-8\)
- From (10), \( \lambda_2 \geq 0 \), therefore, \( \lambda_2 = 0 \),
- Therefore, \( \mathbf{X}^* = [0, 0, 0] \)

This solution set satisfies all of (6) to (9)
Example (1) …contd.

Case 2: \( x_1 - x_2 - 2x_3 - 12 = 0 \)

- Using (2), (3) and (4), we have \( \frac{-\lambda_1 - \lambda_2}{2} - \frac{\lambda_1 - 2\lambda_2}{4} - \frac{2\lambda_1 + 3\lambda_2}{3} - 12 = 0 \)
  or \( 17\lambda_1 + 12\lambda_2 = -144 \)

- But conditions (9) and (10) give us \( \lambda_1 \geq 0 \) and \( \lambda_2 \geq 0 \) simultaneously, which cannot be possible with \( 17\lambda_1 + 12\lambda_2 = -144 \)

Hence the solution set for this optimization problem is \( X^* = [0 \ 0 \ 0 \ 0] \)
Example (2)

Minimize \[ f = x_1^2 + x_2^2 + 60x_1 \]
subject to

\[ g_1 = x_1 - 80 \geq 0 \]
\[ g_2 = x_1 + x_2 - 120 \geq 0 \]
Example (2) ...contd.

Kuhn – Tucker Conditions

\[
\begin{align*}
\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} &= 0 \\
2x_1 + 60 + \lambda_1 + \lambda_2 &= 0 \\
2x_2 + \lambda_2 &= 0 \\
\lambda_1 (x_1 - 80) &= 0 \\
\lambda_2 (x_1 + x_2 - 120) &= 0 \\
x_1 - 80 &\geq 0 \\
x_1 + x_2 + 120 &\geq 0 \\
\lambda_1 &\leq 0 \\
\lambda_2 &\leq 0
\end{align*}
\]
Example (2) …contd.

From (13) either $\lambda_1 = 0$ or $(x_1 - 80) = 0$,

Case 1

- From (11) and (12) we have $x_1 = -\frac{\lambda_2}{2} - 30$ and $x_2 = -\frac{\lambda_2}{2}$
- Using these in (14) we get $\lambda_2 \left(\lambda_2 - 150\right) = 0$
  \[\therefore \lambda_2 = 0 \text{ or } -150\]
- Considering $\lambda_2 = 0$, $X^* = [30, 0]$. But this solution set violates (15) and (16)
- For $\lambda_2 = -150$, $X^* = [45, 75]$. But this solution set violates (15)
Example (2) …contd.

Case 2: \((x_1 - 80) = 0\)

- Using \(x_1 = 80\) in (11) and (12), we have

  \[
  \lambda_2 = -2x_2 \\
  \lambda_1 = 2x_2 - 220
  \]

(19)

- Substitute (19) in (14), we have \(-2x_2(x_2 - 40) = 0\)

- For this to be true, either \(x_2 = 0\) or \(x_2 - 40 = 0\)
Example (2) …contd.

- For $x_2 = 0$, $\lambda_1 = -220$
- This solution set violates (15) and (16)
- For $x_2 - 40 = 0$, $\lambda_1 = -140$ and $\lambda_2 = -80$
- This solution set is satisfying all equations from (15) to (19) and hence the desired
- Thus, the solution set for this optimization problem is $X^* = [80 \ 40]$. 
Thank you