MODULE - 8 LECTURE NOTES – 3

GROUNDWATER SYSTEMS

INTRODUCTION
Groundwater management deals with planning, implementation, development and operation of water resources containing groundwater. Numerical-simulation models have been used extensively for understanding the flow characteristics of aquifers and evaluate the groundwater resource. While simulation models attempt various scenarios to find the best objective, optimization models directly consider the management objective taking care of all the constraints. In this lecture, we will discuss about the governing equations in these modes and various management models.

GROUNDWATER HYDROLOGY
Subsurface water is stored underneath in subsurface formations called aquifers. In an unconfined aquifer, the upper surface is the water table itself. On the other hand, a confined aquifer is confined under pressure greater than the atmospheric. A confined aquifer may be confined between two impermeable layers. An aquifer serves two functions: storage and transmission.

Storage function is exhibited through porosity $\Phi$, specific yield $S_y$ and storage coefficient $S$. Transmission function is exhibited through the permeability property (coefficient of permeability $K$). Porosity is the measure of the amount of water an aquifer can hold. Specific yield is the water drained from a saturated sample of unit volume. Specific retention is the water retained in the unit volume. Porosity is the sum of specific yield and specific retention. Storage coefficient is the volume of water an aquifer releases or stores per unit surface area per unit decline of head.

Permeability is a measure of the ease of movement of water through aquifers. The coefficient of permeability or hydraulic conductivity is the rate of flow of water through a unit cross-sectional area under a unit hydraulic gradient. Transmissivity, $T$ is the rate of flow of water through a vertical strip of unit width extending the saturated thickness of the aquifer under a unit hydraulic gradient. Therefore,

\[ T = Kb \]  \hspace{1cm} \text{for a confined aquifer where } b \text{ is the saturated thickness of aquifer}

\[ T = Kh \]  \hspace{1cm} \text{for a confined aquifer where } h \text{ is the head (saturated thickness)}
Darcy’s law
The flow thorough an aquifer is expressed by Darcy’s law which states that flow rate through a porous media is proportional to the head loss and inversely proportional to the length of flow path. It can be expressed as

\[ v = -K \frac{\partial h}{\partial l} \quad (1) \]

where \( v \) is the velocity or specific discharge, \( l \) is the length of flow along the average direction and \( \frac{\partial h}{\partial l} \) is the rate of headloss per unit length. Then, the total discharge, \( q \) is

\[ q = Av = -KA \frac{\partial h}{\partial l} \quad (2) \]

SIMULATION OF GROUNDWATER SYSTEMS
Governing equations:
Darcy’s law in terms of transmissivity is

\[ v = -\frac{T}{b} \frac{\partial h}{\partial l} \quad \text{for confined aquifers} \quad (3) \]
\[ v = -\frac{T}{h} \frac{\partial h}{\partial l} \quad \text{for unconfined aquifers} \quad (4) \]

Considering a two-dimensional horizontal flow as shown by a rectangular control volume element in Figure 1, the general equations of flow can be expressed as:

\[ q_i = -T_i \Delta y \left( \frac{\partial h}{\partial x} \right)_i \quad (5) \]
\[
q_2 = -T_x \Delta y \left( \frac{\partial h}{\partial x} \right)_2
\]  
(6)

\[
q_3 = -T_y \Delta x \left( \frac{\partial h}{\partial y} \right)_3
\]  
(7)

\[
q_4 = -T_y \Delta x \left( \frac{\partial h}{\partial y} \right)_4
\]  
(8)

where \( A = \Delta x \cdot h \) for unconfined case or \( A = \Delta x \cdot b \) for confined case and assuming constant transmissivities along the \( x \) and \( y \) directions. \( \left( \frac{\partial h}{\partial x} \right)_1, \left( \frac{\partial h}{\partial x} \right)_2 \ldots \) are the hydraulic gradients at sides 1, 2, ..., respectively.

The rate at which water is stored or released in the element is

\[
q_5 = S \Delta x \Delta y \left( \frac{\partial h}{\partial t} \right)_4
\]  
(9)

where \( S \) is the storage coefficient of the element.

The flow rate for constant net withdrawal or recharge for time \( \Delta t \)

\[
q_6 = q_t
\]  
(10)

Applying continuity law,

\[
q_1 - q_2 + q_3 + q_4 = q_5 + q_6
\]  
(11)

Substituting eqns. 5 – 10 in above eqn, and dividing by \( \Delta x \Delta y \) and simplifying, the final form of eqn. 11 will be

\[
T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} = S \frac{\partial h}{\partial t} + W
\]  
(12)

where \( W = q / \Delta x \Delta y \).

These equations can be written in finite difference form and solved for each rectangular element. The partial derivatives in eqns. 5-9 can be expressed in finite difference form as,
These can be substituted in eqn. 12 and solved using finite element methods.

OPTIMIZATION MODEL
Optimization models for hydraulic management for groundwater have been developed based on three approaches: embedding approach, optimal control approach and unit response matrix approach. In embedding approach, the equations from a simulation model are incorporated into an optimization model directly. In optimal control approach, the simulation model solves the governing equations, for each iteration of the optimization. It works on the concept of optimal control theory. In response matrix approach, a unit response matrix is generated by running the simulation model several times with unit pumpage at a single node. Total drawdowns are then determined by superpositions. We will discuss only about the embedding approach for steady-state one-dimensional confined and unconfined aquifers.

Steady state one-dimensional confined aquifer
Consider a confined aquifer with penetrating wells and flow in one-dimension as shown in Figure 2.
From eqn. 12, the governing equation is
\[ \frac{\partial^2 h}{\partial x^2} = \frac{W}{T_x} \]
where \( \frac{\partial h}{\partial t} = 0 \) (14)

This can be expressed in finite difference form as (using central difference scheme)
\[ \frac{h_{i+1} - 2h_i + h_{i-1}}{\Delta x^2} = \frac{W_i}{T_x} \] (15)

The optimization problem can be stated as
\[ \text{Maximize } Z = \sum_i h_i \] (16)

where \( i \) is the number of wells.

Subject to
\[ \sum_i W_i \geq W_{\text{min}} \]
\[ h_i \geq 0 \]
\[ W_i \geq 0 \] (17)

where \( W_{\text{min}} \) is the minimum production rate for each well. The pumpage can be finally determined from the relation \( q_i = W \Delta x_i^2 \).

**Example:**
Formulate an LP model for the above confined aquifer for maximum heads. The boundaries have a constant head \( h_0 \) and \( h_5 \). The distance between the wells is \( \Delta x \).

**Solution:**
Objective function: Maximize \( Z = h_1 + h_2 + h_3 + h_4 \)
Subject to:

Acc. to eqn. 15

\[-2h_1 + h_2 - \frac{\Delta x^2}{T} W_i = -h_0\]
\[h_1 - 2h_2 + h_3 - \frac{\Delta x^2}{T} W_2 = 0\]
\[h_2 - 2h_3 + h_4 - \frac{\Delta x^2}{T} W_3 = 0\]
\[h_3 - 2h_4 - \frac{\Delta x^2}{T} W_4 = -h_5\]

And acc. to eqn. 17

\[W_1 + W_2 + W_3 + W_4 \geq W_{\text{min}}\]
\[h_i \geq 0 \quad i = 1, ..., 4\]
\[W_i \geq 0 \quad i = 1, ..., 4\]

The unknowns are \(h_1, h_2, h_3, h_4\) and \(W_1, W_2, W_3, W_4\).

**Steady state one-dimensional unconfined aquifer**

The governing equation can be written as

\[\frac{\partial}{\partial x} \left( T \frac{\partial h}{\partial x} \right) = W \]  
\[\frac{\partial^2 h^2}{\partial x^2} = \frac{2W}{K}\]

where \(T = Kh\). Substituting \(w=h^2\) and assuming \(K\) is constant, the finite difference form can be written as

\[\frac{\partial^2 w}{\partial x^2} = \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta x^2} = \frac{2W_i}{K}\]

The optimization problem is to

Maximize \(Z = \sum_i w_i\)

where \(i\) is the number of wells.

Subject to

\[\sum_i W_i \geq W_{\text{min}}\]
\[w_i \geq 0\]
\[W_i \geq 0\]
After solving the above problem, the heads \( h_i = \sqrt{w_i} \).

**Example**
Formulate a LP model to determine the steady state pumpages of an unconfined aquifer shown below.

**Solution**
The optimization problem is

Objective function: Maximize \( Z = w_1 + w_2 + w_3 + w_4 \)

Subject to:

Acc. to eqn. 19

\[
\begin{align*}
-2w_1 + w_2 - \frac{2}{K} \frac{x^2}{\Delta x} W_1 &= -w_0 \\
w_1 - 2w_2 + w_3 - \frac{2}{K} \frac{x^2}{\Delta x} W_2 &= 0 \\
w_2 - 2w_3 + w_4 - \frac{2}{K} \frac{x^2}{\Delta x} W_3 &= 0 \\
w_3 - 2w_4 - \frac{2}{K} \frac{x^2}{\Delta x} W_4 &= -w_5
\end{align*}
\]

And acc. to eqn. 21

\[
\begin{align*}
W_1 + W_2 + W_3 + W_4 &\geq W_{\text{min}} \\
w_i &\geq 0 \quad i = 1, \ldots, 4 \\
W_i &\geq 0 \quad i = 1, \ldots, 4
\end{align*}
\]

The unknowns are \( w_1, w_2, w_3, w_4 \) and \( W_1, W_2, W_3, W_4 \).