A model of continuum damage mechanics for fatigue failure

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Received 15 December 1989; Accepted in revised form 1 October 1990

Abstract. This paper describes the development of a generalized model of continuum damage mechanics for fatigue fracture. With the introduction of a new damage effect tensor, the necessary constitutive equations of elasticity and plasticity coupled with damage are for the first instance derived. This is followed by the formulations of fatigue damage dissipative potential function and a fatigue damage criterion which are required for the development of a fatigue damage evolution equation. The fatigue evolution model is based on the hypothesis that the overall fatigue damage is induced by the summation of 'elastic' and 'plastic' damages.

The validity of the damage model proposed is verified by comparing the predicted and measured number of cycles to failure for ten tensile specimens each applied with different load ranges and excellent agreement has been achieved.

1. Introduction

A great deal of work has been devoted in the past to the material characterization of fatigue crack propagation \cite{1} which provides some understanding and insight into material behavior against fatigue failure. Unfortunately, much of the work is concerned with methods of testing and data collection and correlation. Their empirical nature is confined to a rather narrow region of material characterization. Unless a satisfactory theory with sound physical reasoning is established, no confidence can be placed on the safe design of complex engineering structures against fatigue simply based on semi-empirical formulations.

During the past two decades various damage models derived from the theory of continuum damage mechanics have been developed \cite{2–14}. This area of solid mechanics centered on micro void/crack development provides a better understanding of the mechanics of fracture in structures by means of damage variables which represent the deterioration of a material element. However, only a few papers have so far dealt with fatigue, primarily due to the fact that fatigue damage is much more complex than other types of damage, thus difficult to handle with classical continuum mechanics. Damage models known as NLCD (nonlinear continuous damage) advocated by Lemaitre \cite{3} and Chaboche \cite{15} based on the concept of damage mechanics attempted to describe the governing features of fatigue damage with some degree of success under various loading conditions. However, as a scalar damage variable was employed, these models may yield questionable conclusions under complex loading conditions. In addition, the models involved the use of the maximum and mean stresses. Under multiaxial loading conditions, two fundamental problems are inevitably confronted: (1) the definition of the multiaxial stress parameters associated with the maximum and mean values, (2) the applicability of the models to complex loading conditions including non-proportional loading for complex engineering structures. It is obvious that the development of a comprehensive theory capable

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of characterizing fatigue damage of engineering materials appropriate for practical design analysis is called for.

This paper presents a practical damage model for fatigue life prediction of engineering structures. Based on the concepts of continuum damage mechanics, a generalized damage evolution equation under multiaxial loading condition is developed. Finally, verification of the fatigue damage model is performed by comparing the predicted and measured number of cycles from tensile specimens made of aluminum alloy 2024-T3.

2. Constitutive equations coupled with damage

The present model employs a thermodynamic theory of irreversible process by means of internal state variables for which the effective Cauchy stress tensor may be expressed in a generalized form as

\[ \tilde{\sigma} = \mathcal{M}(D) : \sigma \]  

(2.1)

where \( \sigma \) is the Cauchy stress tensor and \( \mathcal{M}(D) \), the damage effect tensor. The material considered is assumed to be isotropic due to the distributed nature of micro-voids and defects. This implies that, although the values of elastic modulus and Poisson’s ratio of engineering materials may vary during a loading process, these mechanical properties remain isotropic. Based on this assumption, the formation of the damage effect tensor \( \mathcal{M} \) may be written in a generalized form as

\[
\begin{bmatrix}
1 & \mu & \mu & 0 & 0 & 0 \\
\mu & 1 & \mu & 0 & 0 & 0 \\
\mu & \mu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 - \mu & 0 & 0 \\
0 & 0 & 0 & 0 & 1 - \mu & 0 \\
0 & 0 & 0 & 0 & 0 & 1 - \mu
\end{bmatrix}
\]

(2.2)

where \( D \) and \( \mu \) are internal state variables to be measured experimentally.

Obviously, for \( \mu = 0 \) (2.2) is reduced to

\[
\begin{bmatrix}
1 & \mu & \mu & 0 & 0 & 0 \\
\mu & 1 & \mu & 0 & 0 & 0 \\
\mu & \mu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(2.3)

where \( I \) is an identity as unit matrix. This is the typical form frequently chosen to describe the isotropic damage phenomenon. As recent experimental observations [9] revealed that the values of effective Poisson’s ratio of the materials vary with the increase of load which cannot be accounted for by (2.3), the generalized formulation of \( \mathcal{M}(D) \) in (2.2) is proposed to take into effect varying Poisson’s ratio on the damage model.
According to the hypothesis of strain energy equivalence [16], the elastic energy for damaged material can be written as
\[
W^e(\sigma, D) = \frac{1}{2} \dot{\sigma}^T : C^{-1} : \dot{\sigma}
\]
\[
= \frac{1}{2} \dot{\sigma}^T : \mathcal{M}^T (D) : C^{-1} : \mathcal{M}(D) : \sigma,
\]
where \( C \) is the elastic tensor of undamaged material.

The constitutive relation of damaged material is correspondingly deduced as:
\[
\varepsilon^e = \frac{\partial W^e(\sigma, D)}{\partial \sigma} = (\mathcal{M}^T : C^{-1} : \mathcal{M}) : \sigma
\]
\[
= \tilde{C}^{-1} : \sigma,
\] (2.4)
where \( \tilde{C} \) is the effective elastic tensor and expressed as
\[
\tilde{C}^{-1} = \mathcal{M}^T : C^{-1} : \mathcal{M}
\]
\[
= \frac{1 - 4\nu + 2(1 - \nu)\mu^2}{E(1 - D)^2} \begin{bmatrix}
1 - \lambda & \lambda & 0 & 0 & 0 \\
-\lambda & 1 - \lambda & 0 & 0 & 0 \\
-\lambda & -\lambda & 1 & 0 & 0 \\
0 & 0 & 0 & 2(1 + \lambda) & 0 \\
0 & 0 & 0 & 0 & 2(1 + \lambda)
\end{bmatrix}
\] (2.5)
\[\tilde{\lambda} = \nu - 2\mu(1 - \nu) - \mu^2(1 - 3\nu) \]
\[
= \frac{1 - 4\nu + 2(1 - \nu)\mu^2}{1 - 4\mu + 2(1 - \nu)\mu^2}.
\] (2.6)

It is worth noting that the effective Young's modulus \( \tilde{E} \) shown in (2.5) is based on energy equivalence and is different from the conventional definition of \( \tilde{E} = E(1 - D) \). The latter is derived based on the hypothesis of stress or strain equivalence which has been found to yield anomaly in the establishment of constitutive equations of damaged materials. The damage strain energy release rate is expressed as [8, 9]
\[
Y = -\frac{\partial W^e(\sigma, D)}{\partial D} = -\frac{1}{2} \dot{\sigma}^T : \frac{\partial \tilde{C}^{-1}}{\partial D} : \sigma
\]
\[
= -\frac{1}{2} \dot{\sigma}^T \left( \frac{\partial \mathcal{M}^T}{\partial D} : C^{-1} : \mathcal{M} + \mathcal{M}^T C^{-1} \frac{\partial \mathcal{M}}{\partial D} \right) : \sigma.
\] (2.7)
If we define

\[ \sigma^T = [\sigma_1 \sigma_2 \sigma_3 \sigma_{23} \sigma_{31} \sigma_{12}], \]

\[ D^T = [D \mu], \]

\[ Y^T = [Y_1 Y_2], \]

(2.7) can be presented alternatively as

\[
\begin{align*}
Y_1 &= -\frac{1}{1-D} \sigma^T : \tilde{C}^{-1} : \sigma \\
Y_2 &= -\frac{1}{1-D} \sigma^T : \tilde{A}^{-1} : \sigma
\end{align*}
\]

(2.8)

where \( \tilde{A}^{-1} \) is expressed in matrix form as

\[
[A_{ij}]^{-1} = \frac{1}{E(1 - D)} \begin{bmatrix}
2\mu(1 - \nu) - 2\nu & (1 + \mu)(1 - \nu) - 2\mu\nu & (1 + \mu)(1 - \nu) - 2\mu\nu \\
(1 + \mu)(1 - \nu) - 2\mu\nu & 2\mu(1 - \nu) - 2\nu & (1 + \mu)(1 - \nu) - 2\mu\nu \\
(1 + \mu)(1 - \nu) - 2\mu\nu & (1 + \mu)(1 - \nu) - 2\mu\nu & 2\mu(1 - \nu) - 2\nu \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-2(1 + \nu)(1 - \mu) & 0 & 0 \\
0 & -2(1 + \nu)(1 - \mu) & 0 \\
0 & 0 & -2(1 + \nu)(1 - \mu)
\end{bmatrix}.
\]

(2.9)

For damaged materials, following von Mises theorem, the thermodynamic potential function may be defined as

\[
F_p(\sigma, D, R) = \gamma_p^{1/2} - \{R_0 + R(p)\},
\]

(2.10)

where

\[
\gamma_p = \frac{1}{2} \bar{\sigma}^T : \tilde{H} : \bar{\sigma} = \frac{1}{2} \sigma^T : \tilde{H} : \sigma
\]

\[
\tilde{H} = \tilde{M}^T : \tilde{H} : \tilde{M}.
\]

\( \tilde{H} \) is the positive semi-definite tensor, \( R_0 \) is the initial strain hardening threshold, and \( R(p) \) is
the increment of strain hardening threshold. For an isotropic material

\[
[H_{ij}] = \begin{bmatrix}
2 & -1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix}
\]  

Plastic strain rate and overall plastic strain rate may be deduced respectively from (2.10) as:

\[
\dot{\varepsilon}^p = \lambda_p \frac{\partial F_p}{\partial \sigma} = \frac{\lambda_p}{2\sigma_p^{1/2}} \mathbb{A} : \sigma,
\]

\[ (2.13) \]

\[
\dot{\varepsilon} = \lambda_p \frac{\partial F_p}{\partial (-R)} = \lambda_p,
\]

\[ (2.14) \]

where \( \lambda_p \) is a Lagrange multiplier which can be determined by means of the strain hardening criterion \( F_p = 0 \) as

\[
\lambda_p = \begin{cases} 
\frac{\left( \frac{\partial F_p}{\partial \sigma} \right)^T : \dot{\varepsilon} + \left( \frac{\partial F_p}{\partial \mathbf{D}} \right)^T : \dot{\mathbf{D}}}{\left( \frac{\partial F_p}{\partial R} \right)^2 \frac{\partial R}{\partial \sigma}} ; & \text{if } F_p = 0 \text{ and } \left( \frac{\partial F_p}{\partial \sigma} \right)^T : \dot{\varepsilon} + \left( \frac{\partial F_p}{\partial \mathbf{D}} \right)^T : \dot{\mathbf{D}} > 0 \\
0 ; & \text{if } F_p < 0 \text{ or } F_p = 0 \text{ and } \left( \frac{\partial F_p}{\partial \sigma} \right)^T : \dot{\varepsilon} + \left( \frac{\partial F_p}{\partial \mathbf{D}} \right)^T : \dot{\mathbf{D}} \leq 0
\end{cases}
\]

\[ (2.15) \]

From (2.10), we have

\[
\frac{\partial F_p}{\partial \sigma} = \frac{1}{2\sigma_p^{1/2}} \mathbb{A} : \sigma,
\]

\[
\frac{\partial F_p}{\partial \mathbf{D}} = \frac{1}{4\sigma_p^{1/2}} \sigma^{1/2} \frac{\partial \mathbb{H}}{\partial \mathbf{D}} : \sigma,
\]

\[
\frac{\partial F_p}{\partial R} = -1.
\]
Then the Lagrange multiplier $\lambda_p$ can be deduced as

$$
\lambda_p = \begin{cases} 
\frac{1}{2\sigma_p^{1/2}} \frac{\partial R}{\partial \dot{\epsilon}} \left[ \sigma^T \text{C} \sigma + \frac{1}{2} \sigma^T \left( \frac{\partial \text{H}}{\partial \sigma} \right)^T \sigma : \dot{D} \right] ; & \text{if } F_p = 0 \quad \text{and} \\
\sigma^T \text{C} \sigma + \frac{1}{2} \sigma^T \left( \frac{\partial \text{H}}{\partial \sigma} \right)^T \sigma : \dot{D} > 0 \\
0 ; & \text{if } F_p < 0 \quad \text{or } F_p = 0 \quad \text{and} \quad \sigma^T \text{C} \sigma + \frac{1}{2} \sigma^T \left( \frac{\partial \text{H}}{\partial \sigma} \right)^T \sigma : \dot{D} \leq 0
\end{cases}
$$

(2.16)

From (2.13), we can express the overall plastic strain rate in an alternative form:

$$
\dot{\sigma} = 2 \left[ \frac{1}{2} \dot{\epsilon}^T \text{C}^{-1} \dot{\epsilon} \right]^{1/2}.
$$

(2.17)

3. Damage evolution equation

An essential step in developing a damage model is to formulate an appropriate damage dissipative potential function and a fatigue damage criterion which may be respectively expressed as

$$
\Phi_d(Y, B) = Y_d^{1/2} - (B_0 + B(w))
$$

(3.1)

and

$$
F_d = Y_d^{3/4} - E^{1/2}(B_0 + B(w)) = 0,
$$

(3.2)

where

$$
Y_d = \frac{1}{2} Y^T : J : Y, \quad J = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix}.
$$

$B_0$ is the initial damage strain hardening threshold; $B(w)$, the increment of damage strain hardening threshold; and $\gamma$, the damage evolution coefficient.

Following the theory of irreversible thermodynamics with respect to different state variables, we have

$$
\dot{D} = -\dot{\lambda}_d \frac{\partial \Phi_d}{\partial Y} = -\frac{\dot{\lambda}_d}{2 Y_d^{1/2}} \dot{J} : Y,
$$

(3.3)

$$
\dot{\omega} = \dot{\lambda}_d \frac{\partial \Phi_d}{\partial (-B)} = \dot{\lambda}_d.
$$

(3.4)
where \( \lambda_d \) is the Lagrange multiplier which can be determined by damage criterion of (3.2). If \( F_d = 0 \) and \( (\partial F_d/\partial \ddot{Y})^T \dot{Y} > 0 \), from (3.2) and (3.4), we have

\[
\dot{F}_d(Y, B) = \left( \frac{\partial F_d}{\partial \ddot{Y}} \right)^T \dot{Y} + \left( \frac{\partial F_d}{\partial B} \right)^T \frac{\partial B}{\partial w} \dot{\lambda}_d = 0,
\]

\[
\dot{\lambda}_d = \frac{\ddot{Y}^T : \dot{Y}}{\frac{3}{6} E^{1/2} Y_d^{1/4} \frac{\partial B}{\partial w}}.
\]

(3.5)

For ease of damage analysis, the damage process during the loading cycles is hypothesized to be governed by two main events. One is defined as ‘elastic’ damage which corresponds to that portion of damage when the applied stress \( \sigma \) is less than the damage stress \( \sigma_d \), an intrinsic material property. Another is ‘plastic’ damage which corresponds to the damaging process when the applied stress \( \sigma \) exceeds \( \sigma_d \). For the case of low-cycle fatigue, the total damage is thus the summation of ‘elastic’ damage and ‘plastic’ damage. Based on this hypothesis, the Lagrange multiplier \( \dot{\lambda}_d \) for a loading process may be expressed as

\[
\dot{\lambda}_d = \begin{cases} 
\frac{\ddot{Y}^T : \dot{Y}}{\frac{3}{6} E^{1/2} Y_d^{1/4} \frac{\partial B}{\partial w}}; & \text{if } F_d = 0 \text{ and } \ddot{Y}^T : \dot{Y} > 0 \\
\frac{\ddot{Y}^T : \dot{Y}}{\frac{3}{6} E^{1/2} Y_d^{1/4} K(w)}; & \text{if } F_d < 0 \text{ and } \ddot{Y}^T : \dot{Y} > 0 \\
0; & \text{if } \ddot{Y}^T : \dot{Y} \leq 0,
\end{cases}
\]

(3.6)

where the first term corresponds to the ‘plastic’ and second term, is the ‘elastic’ damage. \( K(w) \) in (3.6) is a function to be determined experimentally and considered an intrinsic material property of the fatigue damage.

From (3.3), we have

\[
\dot{D}^T : \ddot{\gamma}^{-1} : \dot{D} = \frac{\dot{\lambda}_d^2}{4Y_d} \ddot{Y}^T : \dot{Y} = \frac{1}{2} \dot{\lambda}_d^2.
\]

Then the overall damage rate may be presented alternatively as:

\[
\dot{\omega} = \dot{\lambda}_d = 2 \left( \frac{1}{2} \dot{D}^T : \ddot{\gamma}^{-1} : \dot{D} \right)^{1/2}.
\]

(3.7)

Under uniaxial cyclical loading, we consider only the effect of the applied stress range on the ‘elastic’ damage cumulation without explicit reference to the mean stress effect which is however included during the ‘plastic’ fatigue damage. Then the damage rate for each
cycle is

\[
\frac{dD}{dN} = \begin{cases} 
- \int_{0}^{\sigma_{\text{max}} - \sigma_{\text{min}}} \frac{Y^T : \mathbf{J} : d\mathbf{Y}}{\frac{3}{2} E^{1/2} Y_d^{3/4} K(w)} \mathbf{J} : \mathbf{Y} ; & \text{if } \sigma_{\text{max}} \leq \sigma_d \\
- \int_{0}^{\sigma_d - \sigma_{\text{min}}} \frac{Y^T : \mathbf{J} : d\mathbf{Y}}{\frac{3}{2} E^{1/2} Y_d^{3/4} K(w)} \mathbf{J} : \mathbf{Y} - \int_{\sigma_d}^{\sigma_{\text{max}}} \frac{Y^T : \mathbf{J} : d\mathbf{Y}}{\frac{3}{2} E^{1/2} Y_d^{3/4} \frac{\partial B}{\partial w}} \mathbf{J} : \mathbf{Y} ; & \text{if } \sigma_{\text{max}} > \sigma_d
\end{cases}
\] (3.8)

and overall damage rate for each cycle is

\[
\frac{d\omega}{dN} = 2 \left[ \frac{1}{2} \left( \frac{d\mathbf{D}}{dN} \right)^T : \mathbf{J}^{-1} : \left( \frac{d\mathbf{D}}{dN} \right) \right]^{1/2}.
\] (3.9)

4. Determination of the damage variables

4.1. Damage variables $D$ and $\mu$

The damage variables $D$ and $\mu$ can be determined using the measured values of elastic modulus $E$ and Poisson's ratio $\nu$ of the test materials which have been described in [8, 9].

Under uniaxial tensile loading, the constitutive (2.4) can be written as

\[
\varepsilon_1 = \frac{1 - 4\mu\nu + 2(1 - \nu)\mu^2}{E(1 - D)^2} \sigma_1 = \frac{\sigma_1}{E},
\] (4.1)

\[
\varepsilon_2 = -\frac{\nu - 2\mu(1 - \nu) - \mu^2(1 - 3\nu)}{E(1 - D)^2} \sigma_1 = -\frac{\nu\sigma_1}{E}.
\] (4.2)

From (4.1) and (4.2), we have

\[
[2(1 - \nu)\tilde{\nu} + 1 - 3\nu]\mu^2 + 2(1 - \nu - 2\nu\tilde{\nu})\mu + \tilde{\nu} - \nu = 0.
\] (4.3)

Equation (4.1) can be presented in an alternative form

\[
(1 - D)^2 = \frac{E}{E} \left[ 1 - 4\mu\nu + 2(1 - \nu)\mu^2 \right].
\] (4.4)

Then the damage variables $\mu$ and $D$ can be readily evaluated using (4.3) and (4.4) from the experimentally determined values of $E$ and $\tilde{\nu}$. 
4.2. Damage characteristic tensor \( J \)

Under uniaxial tension, the damage strain energy release rate \( Y \) shown in (2.8) can be expressed as

\[
\begin{align*}
Y_1 &= -\frac{1 - 4\mu v + 2(1 - v)\mu^2}{E(1 - D)^3} \sigma_i^2 \\
Y_2 &= -\frac{2\mu(1 - v) - 2v}{E(1 - D)^2} \sigma_i^2
\end{align*}
\]

and the damage evolution equation (3.3) becomes

\[
\begin{pmatrix}
\dot{D} \\
\dot{\mu}
\end{pmatrix} = -\frac{\dot{\varepsilon}_d}{2Y_d^{3/2}} \begin{pmatrix}
Y_1 + \gamma Y_2 \\
\gamma Y_1 + Y_2
\end{pmatrix}.
\]

From (4.5) and (4.6), we have

\[
\frac{dD}{d\mu} = \frac{1 - 4\mu v + 2(1 - v)\mu^2 + \gamma(1 - D)}{\gamma[1 - 4\mu v + 2(1 - v)\mu^2] + 1 - D}.
\]

The solution of (4.7) is achieved using

\[
\left(\mu - \frac{v}{1 - v}\right)^2 + \frac{1 - v - 2v^2}{2(1 - v)^2} = c \left[\frac{x}{(x - \gamma)(e^{\gamma(1-x)} - 1)}\right]^{2(1 - \gamma^2)/\gamma^2},
\]

where

\[
x = f(\mu) + \gamma(1 - D), \quad f(\mu) = \frac{1 - 4\mu v + 2(1 - v)\mu^2}{2\mu(1 - v) - 2v}
\]

and \( c \) is a material constant which can be determined by applying initial or undamaged condition of the test material, as

\[ D = 0 \quad \text{and} \quad \mu = 0. \]

Substituting the initial values of \( D \) and \( \mu \) into (4.8), the material constant can be evaluated:

\[
c = \frac{1}{2(1 - v)} \left[\frac{1 - 2v\gamma}{(1 - \gamma^2) e^{\gamma(1 - 2v)(1 - \gamma)(1 + 2v)}}\right]^{2(1 - \gamma^2)/\gamma^2}.
\]

Substituting (4.9) into (4.8) enables the determination of the damage evolution coefficient \( \gamma \).
4.3. Fatigue damage function $K(w)$

For high-cycle fatigue, the concept of residual life is adopted to define damage. During uniaxial cyclic loading, the damage rate for $\sigma_{\text{max}} < \sigma_d$ is

$$\frac{dD}{dN} = -\int_{0}^{\sigma_{\text{max}} - \sigma_{\text{min}}} \frac{Y^T : J : dY}{\frac{3}{3} E^{1/2} Y_d^{3/4} K(w)} \, dY$$

(4.10)

and overall damage rate is

$$\frac{dw}{dN} = 2 \left[ \frac{1}{2} \left( \frac{dD}{dN} \right)^T : J^{-1} : \left( \frac{dD}{dN} \right) \right]^{1/2}.$$  

(4.11)

Integrating (4.11) between $w = 0$ and $w = w_c$ leads to

$$w_c = N_f \int_{0}^{w_c} 2 \left[ \frac{1}{2} \left( \frac{dD}{dN} \right)^T : J^{-1} : \left( \frac{dD}{dN} \right) \right]^{1/2} \, dN,$$

(4.12)

where $w_c$ is the critical value, an intrinsic material property and $N_f$ is the number of cycles to failure.

Although there are many possible formulations for the function $K(w)$, a simple yet effective form capable of describing the non-linear cumulative fatigue damage may be expressed as

$$K(w) = K_0 \left( 1 - \frac{w}{w_c} \right).$$

(4.13)

Substituting (4.13) into (4.12), we have

$$K_0 = \frac{\int_{0}^{N_f} 2 \left[ \frac{1}{2} \left( K_0 \frac{dD}{dN} \right)^T : J^{-1} : \left( K_0 \frac{dD}{dN} \right) \right]^{1/2} \, dN}{w_c}.$$  

(4.14)

4.4. Increment of strain hardening threshold $R(p)$

Under uniaxial tension, the strain hardening criterion (2.10) becomes

$$\frac{1 - \mu}{1 - D} \sigma_1 - [R_0 + R(p)] = 0,$$

(4.15)
where

\[ R_0 = \frac{1 - \mu_0}{1 - D_0} \sigma_y \]

\[ \begin{bmatrix} D_0 \\ \mu_0 \end{bmatrix} = -\int_0^{\sigma_y} \frac{\lambda_d}{2 Y_d^{1/\gamma}} \dot{\gamma} : \mathbf{Y} \]

and \( \sigma_y \) is the initial yield stress. The value of \( R \) is accordingly obtained as

\[ R(p) = \frac{1 - \mu}{1 - D} \sigma_1 - \frac{1 - \mu_0}{1 - D_0} \sigma_y \quad (4.16) \]

We choose the generalized inverse (or \( g \)-inverse) of the singular matrix \( \mathbf{H} \) as

\[
\mathbf{H}^{-1} = \frac{1}{9} \begin{bmatrix}
2 & -1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.5
\end{bmatrix}
\]

Then

\[ \tilde{\mathbf{f}}_1^{-1} = \frac{(1 - D)^2}{(1 - \mu)^2} \mathbf{H}^{-1}. \]

Equation (2.17) can be written as

\[ \dot{p} = -\frac{1 - D}{1 - \mu} \dot{\varepsilon}_1^p = \frac{1 - D}{1 - \mu} (\dot{\varepsilon}_1 - \dot{\varepsilon}_1^e), \]

which after integration becomes

\[ p = \int_{\varepsilon_1^p}^{\varepsilon_1} \frac{1 - D}{1 - \mu} d\varepsilon_1 - \int_{\varepsilon_1^e}^{\varepsilon_1} \frac{1 - D}{1 - \mu} d\varepsilon_1^e, \quad (4.17) \]

where

\[ \varepsilon_1^e = \frac{1 - 4\mu v + 2(1 - v)\mu^2}{E(1 - D)^2} \sigma_1 \]

and \( \varepsilon_1^e \) is the initial yield strain. The evaluation of \( R(p) \) can be readily obtained using (4.16) and (4.17).
4.5. Damage evolution function $B(w)$

Under uniaxial tension, we have

$$Y_d = \frac{1}{2} \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$= \frac{1}{2}(Y_1^2 + Y_2^2 + 2\gamma Y_1 Y_2)$$

$$= \frac{1}{2}(a_1^2 + a_2^2 + 2\gamma a_1 a_2)\sigma^4/E^2,$$

where

$$a_1 = \frac{1 - 4\mu v + 2(1 - v)\mu^2}{(1 - D)^3},$$

$$a_2 = \frac{2\mu(1 - v) - 2\nu}{(1 - D)^2}.$$

Then the damage criterion (3.2) becomes

$$B(w) = \left[ \frac{1}{2}(a_1^2 + a_2^2 + 2\gamma a_1 a_2)\sigma^4/E^2 \right]^{3/4} E^{-1/2} - B_0,$$  \hspace{1cm} (4.18)

where $B_0$ is the initial 'plastic' damage strengthening threshold.

$$B_0 = \left[ \frac{1}{2}(a_{1d}^2 + a_{2d}^2 + 2\gamma a_{1d} a_{2d})\sigma_{d}^4/E_{d}^2 \right]^{3/4} E_{d}^{-1/2},$$

$$a_{1d} = \frac{1 - 4\mu_d v + 2(1 - v)\mu_d^2}{(1 - D_d)^3},$$

$$a_{2d} = \frac{2\mu_d(1 - v) - 2\nu}{(1 - D_d)^2},$$

$$\begin{bmatrix} D_d \\ \mu_d \end{bmatrix} = -\int_0^{\sigma_d} \frac{Y^T : \mathbf{J} : dY}{8E_{d}^{1/2}Y_{d}^{3/4}K_0 \left( \frac{1 - w}{w_c} \right)} \mathbf{J} : \mathbf{Y}.$$

From (3.7) and (4.6), we have

$$\dot{w} = \left\{ \frac{2}{1 - \gamma^2} \left[ 1 + \left( \frac{\gamma a_1 + a_2}{a_1 + \gamma a_2} \right)^2 - \frac{2\gamma(\gamma a_1 + a_2)}{a_1 + \gamma a_2} \right] \right\}^{1/2} \dot{D}.$$  \hspace{1cm} (4.19)

Equations (4.18) and (4.19) enable the determination of the function $B(w)$. 
5. Damage model verification

In order to check the validity of the proposed fatigue damage model, various damage variables need to be, for the first instance, determined. This was achieved by evaluating the effective Young’s modulus $E$ and Poisson’s ratio $\nu$ from tensile specimens made of aluminum alloy 2024-T3, each marked with 1 mm divisions shown in Fig. 1. A method of evaluating these variables has been described in [9] to which interested readers may refer. The determination of $E$ and $\nu$ enabled the evaluation of the parameters $\mu$ and $D$ based on (4.3) and (4.4).

For the determination of the damage evolution coefficient $\gamma$, the relationship between $\varepsilon$ and $\gamma$ was established as shown in Fig. 2 by substituting the above computed damage variables of $\mu$ and $D$ into (4.8) and (4.9). It can be observed from the figure that $\gamma$ is, as expected, a material constant of 0.7.

Another intrinsic material property is the critical overall damage coefficient $w_c$, which was measured to be 0.245 from a tensile specimen using (4.19). Other measured mechanical
properties of the test material, Al 2024-T3, are:

\[ E = 74300 \text{ MPa}, \]
\[ \sigma_y = 330 \text{ MPa}, \]
\[ \nu = 0.33. \]

After the measurement of \( w_c \) and other variables, the fatigue material parameter \( K_0 \) in (4.13) can be evaluated by means of (4.10) and (4.12) in which the knowledge of \( N_f \) under cyclic loading is required. Based on the hypothesis that the \( K_0 \)-factor controls the 'elastic' damage induced by the applied load less than the 'plastic' damage stress threshold \( \sigma_y \), the maximum stress applied was chosen to be 300 MPa. Two levels of minimum stress were employed, namely 0 and 100 MPa. These two stress ranges were applied to the tensile specimen shown in Fig. 1 and yielded two experimentally determined values of \( N_f \). In conjunction with the \( Y_1 \)- and \( Y_2 \)-values evaluated based on (2.8), the fatigue material parameter \( K_0 \) was finally determined and found to be 685 MPa.

Before the proposed model can be used to predict fatigue failure, the relationship between the damage evolution function \( B(w) \) and overall damage \( w \) also needs to be measured. This relationship can be readily established via (4.18) and (4.19) and found to be

\[ B_0 = 2.34 \times 10^{-3} \text{ MPa}, \]
\[ B = 0.3w^{1.64} \text{ MPa}. \]

![Graph](image-url)  

*Fig. 3. Increment of damage threshold \( B \) vs overall damage \( w \).*
The predicted relationship between $B$ and $w$ is depicted in Fig. 3 together with those determined experimentally, while the predicted and measured values of $w$ and $\sigma$ are shown in Fig. 4. Both figures reveal satisfactory agreement between the predicted results based on the proposed damage model and the measurements.

The knowledge of all the damage parameters described above enables the prediction of the number of cycles to failure based on the proposed fatigue damage model. A total of ten tensile specimens shown in Fig. 1 under different applied load ranges described in Table 1 was employed. The measured number of cycles to failure for each load range is indicated in the sixth column of Table 1 together with the predicted cycles to failure based on (3.9) in the last column. Both the predicted and measured cycles to failure are also depicted in Fig. 5 from which satisfactory agreement can be observed.

![Fig. 4. Overall damage $w$ vs stress $\sigma$.](image)

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<th>No. of Specimen</th>
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<th>$\sigma_{\text{max}}$ MPa</th>
<th>$\sigma_{\text{mean}}$ MPa</th>
<th>$\Delta\sigma$ MPa</th>
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6. Conclusions

A generalized damage model for fatigue fracture is presented. Based on the concepts of the continuum damage mechanics, the constitutive equations of plasticity and damage evolution under multiaxial loading conditions were developed. This model was used to predict the number of cycles to failure of aluminum 2024-T3 specimens under tensile loading which were found to yield excellent agreement with those determined experimentally. Future studies on the extension of the proposed model to cover tension-compression cycle is in progress and will be the subject of subsequent publications.

Acknowledgement

The research described in this paper has been sponsored by the Croucher Foundation whose support is gratefully acknowledged.

References