THERMO-MECHANICAL FATIGUE ANALYSIS USING GENERALIZED CONTINUUM DAMAGE MECHANICS AND THE FINITE ELEMENT METHOD

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SUMMARY

In this paper, a thermo-mechanical constitutive model for the predictions of fatigue in structures using the finite element method is formulated. The model is based on the damage mechanics of the continuous medium and allows the treatment, in a unified way, of coupled phenomena such as fatigue with damage, plasticity, viscosity and temperature effects.

A formulation based on the theories of damage and plasticity is developed. The necessary modifications in these theories are outlined in order to include the fatigue phenomena. Details of the finite element implementation are given.

Finally, results of the performance of the proposed model are shown through the fatigue analysis of an aluminium engine alternator support is presented.

KEY WORDS: fatigue; continuum damage mechanics; coupled thermo-mechanical behaviour

1. INTRODUCTION

Time varying cyclic loads produce failure of structural parts for values of stress lower than those obtained in static tests. This phenomenon is called fatigue and it is defined more generally in the ASTM code \cite{12} as: "the process of permanent, progressive and localized structural change which occurs to a material point subjected to strains and stresses of variable amplitudes which produces cracks which lead to total failure after a certain number of cycles".

Fatigue is the main cause of failure of machine parts in service, in mechanisms and structural elements functioning in aeronautics, naval and the automotive industry as well as in civil engineering structures such as bridges, buildings, etc.

The study of fatigue starts from the basic fact that it is not a phenomenon associated to the classic concept of plasticity and/or damage and that failure occurs under load conditions well below the strength limit of the material. Typically, a progressive loss of strength occurs depending on the number of stress cycles that induces local plasticity and/or damage effects. In addition, all these phenomena are usually coupled with thermal effects.

Collapses by fatigue are especially dangerous because they are unpredictable, giving no prior notification of the imminent failure: they occur suddenly and show no exterior plastic deformations. They are fragile failures which display two well separated zones: a dark polished zone showing obvious ductile cleavage which happened...
smoothly and a rough shinier zone where the final break is localized after surpassing the fatigue-reduced material strength.

It must be noted that fatigue itself always occurs with accompanying mechanical actions, which do not seem critical by themselves but, at the end, they are decisive for the life of the material.

In a part subjected to cyclic loading, several stages may be distinguished during a fatigue process (see Figure 1):

- **Region I**, where the first micro-structural changes occur, micro-cracks form, the density of dislocations grow and later irreversible damage zones localize.
- **Region II**, where the macro-cracks start to coalesce from micro-cracks into dominant direction patterns.
- **Region III**, where unstable propagation of damage occurs, leading rapidly to total collapse.

The time length of each of the above stages may vary considerably depending on the type of material, load, topology, temperature, etc. Often it is difficult to distinguish the boundaries between these stages.

**2. DIFFERENT FATIGUE ANALYSIS APPROACHES**

The first study on fatigue was done on metallic materials around 1829 in Germany, after observing collapses of mine scaffoldings subjected to loads usually considered safe. That generated interest in the study of parts subjected to cyclic loading, which later was further fueled by the developing railway industry.

Towards 1860, A. Wöhler, a Bavarian railway engineer, gave a decisive push to the knowledge of the fatigue phenomena by performing numerous tests under diverse loading conditions in order to determine the reason of premature collapse of railway
axles (Wöhler [13]). Results of his studies are the well-known Stress-N°cycles curves (S-N) as well as the concept of fatigue limit strength or endurance.

Wöhler's S-N curves (see Figure 2) are obtained experimentally by subjecting identical specimens to cyclic harmonic stresses and establishing their life span measured in number of cycles. The curves depend on the level of the maximum applied stress and the ratio between the lowest and the highest stresses \( R = S_{\text{min}} / S_{\text{max}} \).

![Figure 2a. Stress evolution at a single point.](image1.png)  
![Figure 2b. S-N Wöhler's Curves](image2.png)

Years later, around 1900, the studies set out for understanding the hardening and softening cycles of metals discovered the hysteretic strain and life curves in fatigue processes (Bairstow [15]). This was accomplished using constant strain amplitude cycles, thus avoiding uncontrolled stress increases which induce complex combined mechanical and fatigue phenomena.

In the 1950s, Manson [16] and Coffin [17] formulated that the plastic strain is one of the main causes of cyclic damage in metals, and proposed an experimentally-derived expression relating the number of cycles to the inelastic strain.

Another point of view for studying fatigue was based on fracture mechanics. Irwin [18] related fatigue phenomena with fracture via stress intensity factors. Along the same line, years later Forrest [23] characterized fatigue crack growth by means of the stress intensity factor. He was the first to suggest that crack length increase depends on the fluctuation range of this stress intensity factor. Later, Paris [19] found a now notorious mathematical relation between the number of cycles and crack length, so fulfilling the original idea. This last approach is valid only for simple stress states and geometries. Moreover, prediction of coupled phenomena effects present great inconveniences when cyclic actions are superposed with uncyclic thermal or mechanical loads.

In recent years a fundamental change in focus has occurred. Chaboche [20] included fatigue into the general damage theory of continuum mechanics. This study was based on the hypothesis that fatigue damage is essentially of the same nature as mechanical damage and can be described via an internal variable allowing the adequate treatment of the accumulation and localization of dislocations. This internal variable relates damage to the number of cycles.

All these approaches have been formulated for periodic loads, where a dominant period may be clearly established. Little has been done in the more general case of non-periodic actions, Miner [21].
The problem becomes more complicated when plastic effects occur due to high-level loads combined with cyclic effects including additional plastic effects. Even in the absence of detectable plastic effects, fatigue generates non-linear behaviour at micro-structural level in metals (Suresh [26], Osgood [27]).

The previous description suggests that the theoretical structure of continuum mechanics, such as plasticity and damage (Lubliner [3,4], Malvern [7]) are suitable for the study of non linear fatigue problems. From another point of view, it can be stated that the mechanical effect known as fatigue produces a loss of material strength as a function of the number of cycles, reversion index, load amplitude, etc. This loss of strength induce the material to inelastic behavior, which may be interpreted as micro-cracking followed by crack coalescence leading to the final collapse of structural parts.

This paper extend the original work of Chaboche [20] by formulating a new fatigue prediction model based on a continuum mechanics formulation accounting for coupled thermo-mechanical effects. The rest of the paper layout is as follows. In next sections the basis of the continuum mechanics formulation is presented. Next the fatigue life prediction methodology proposed is described. Details of the finite element formulations are also given. Finally the efficiency and accuracy of the proposed model is shown in a practical example.

3. A CONTINUUM MECHANICS MODEL FOR FATIGUE ANALYSIS

3.1. Introduction

A fatigue model formulated within a continuum mechanics framework overcomes most of drawbacks associated to the standard fracture mechanics based procedures:

a. The classical models proposed for fatigue only forecast the life of a part as a function of the number of cycles of periodical loads (Paris [19]). The proposed model permits the introduction of the loss of strength of metals due to combined effects of fatigue, fracture damage, plasticity, viscoelasticity, temperature, etc. This means that complex phenomena that occur in materials may be forecast for parts in service to study their safety at a given moment of their life, prior to total collapse.

b. Extending the previous idea, it must be recalled that experimental tests constitute a good tool for specific cases and for obtaining parameters, but they cannot be extended to situations more complex than those studied in the laboratory. Fracture mechanics does not offer a true solution to this problem, thus being extremely complicated to quantify the effect due to complex load conditions. Approaching the problem in a more general way, using continuum mechanics theory, makes it possible to take into account the combined effects of other factors such as thermal loads, mean stress, multi-axial stress states, plastic damage, etc.

c. The introduction of a new internal variable related to fatigue allows treating accumulative damage without the need to formulate a complementary constitutive rule (Suero [28]). In this manner, the model proposed here is able to account for non-linear damage accumulation problems that occur when a structural part is subjected to cycles of different load amplitudes.
3.2. Thermal and thermo-mechanical fatigue

Thermal fatigue can occur due to the alone action of cyclic variation in temperature without the influence of other external loads (Forrest [22]), meanwhile thermo-mechanical fatigue can be identify with fatigue due to cyclic loads at high (or low) temperatures. Usually both phenomena produce a joint action:

a. Acting like a load whose mechanical effects are of more complexity that those that are achieved under the own mechanical loads.

b. Producing an alteration in the properties of the material, driving to an activation or attenuation of the fatigue effects.

Micro-structural changes take place under thermal alterations of the solid. In this work a macroscopic behavior interpretation at phenomenological level is adopted. This allows studying the problem at macroscopic level by means of continuous mechanics theory. Particularly, an approach through the non-linear theories based on internal variables is adopted. More specifically, a theoretical thermo mechanical framework is developed based on the general theory of plasticity and damage that it is sensitive to the number of cycles of the thermo mechanical action (Salomón [23]).

3.3. Elasto-plastic damage model. Thermo mechanical formulation

3.3.a. Introduction

The inelastic theories of plasticity and/or damage solve the problem of material behavior beyond the elastic range and both theories allow the study of the change in strength that suffers a material point by inelastic effects, however they are not sensitive to cyclic load effects. In this work the standard inelastic theories are modified to account for fatigue effect coupled with non-fatigue behaviour.

It is assumed that each point of the solid follows a thermo-damage-elasto-plastic constitutive law (stiffness hardening/softening) (Lubliner [5], Luccioni [6], Oller [9]) with the stress evolution depending on the free strain variable and plastic and damage internal variables. The theory here presented studies the phenomena of stiffness degradation and irreversible strain accumulation through the combined effect of damage and plasticity.

3.3.b. Thermo Plastic damage model

It is accepted the additivity of the elastic $\Psi^e$ and plastic $\Psi^p$ parts of the free energy written in the reference configuration for a given entropy $\eta$ and temperature $\theta$ field and elastic Green strains $E^e_{ij} = E_{ij} - E^p_{ij} - E^\theta_{ij}$; the two last variables operate as free field variables (Green [1], Lubliner [5], Luccioni [6], Oller [9]). The free energy is thus written as

$$\Psi = \Psi^e(E^e_{ij}, d, \theta) + \Psi^p(\alpha^p, \theta) = (1 - d) \left[ \frac{1}{2m} \sum_{ijkl} E^e_{ij} C^e_{ijkl} (\theta) E^e_{kl} \right] + \Psi^p(\alpha^p, \theta) - \theta \eta$$

Considering the second thermodynamic law (Clasius-Duhem inequality), the thermo mechanical dissipation can be obtained as
\[ \mathbf{\Xi} = \frac{S_{ij} \cdot \varepsilon_{ij}^p}{m^0} - \frac{\partial \Psi}{\partial \alpha^p} \alpha - \frac{\partial \Psi}{\partial d} \dot{d} - \frac{J}{\theta m^0} q_i \nabla \theta \geq 0 \quad , \] (2)

The dissipation condition allows to define the constitutive laws for the stress and the entropy as:

\[ S_{ij} = m^o \frac{\partial \Psi^c}{\partial E_{ij}^c} = (1 - d) C_{ijkl}^0 (\theta) \left( E_{kl}^c \right) \quad , \quad \eta = \frac{\partial \Psi}{\partial \theta} \] (3a)

Also, from the last expressions, other thermo-mechanical variables can be obtained as:

Constitutive tensor :
\[ C_{ijkl}^S (d, \theta) = \frac{\partial S_{ij}}{\partial E_{ij}^c} = m^o \frac{\partial^2 \Psi^c}{\partial E_{ij}^c \partial E_{kl}^c} \]

Conjugate thermal expansion coefficient :
\[ \beta_{ij} (d, \theta) = -\frac{\partial S_{ij}}{\partial \theta} = -m^o \frac{\partial^2 \Psi^c}{\partial \theta \partial E_{ij}^c} \] (3b)

Specific Heat :
\[ c_k = \theta \frac{\partial \eta}{\partial \theta} = -m^o \theta \frac{\partial^2 \Psi}{\partial \theta} \]

where \( m^o \) is the material density, \( E_{ij}^c, E_{ij}^p, E_{ij}^0 \) are the elastic, total, plastic and thermal strain tensors, \( d^{ini} \leq d \leq 1 \) is the internal mechanical damage variable where \( d^{ini} \) is the initial value, \( C_{ijkl}^0 \) and \( C_{ijkl}^S \) are the constitutive tensors for the original and secant material and, \( S_{ij} \) is the stress tensor for a single material point.

### 3.3.c. Yield and potential plastic functions

The yield \( F \) and potential \( G \) plastic functions take into account the influence of the current stress state, the internal plastic variables, the temperature and other variables such as the number of cycles. The general forms of these functions are

\[ F(S_{ij}, \alpha^p, \theta) = f(S_{ij}) - K(S_{ij}, \alpha^p, N, \theta) \]
\[ G(S_{ij}, \alpha^p) = g(S_{ij}) = cte. \] (4)

where \( f(S_{ij}) \) and \( g(S_{ij}) \) are the uniaxial equivalent stress functions, \( K(S_{ij}, \alpha^p, N, \theta) \) is the strength threshold, \( \alpha^p = \int \dot{\alpha}^p dt \) is a plastic internal variable at current time \( t \). (Lubliner [5], Luccioni [6], Oller [9]), \( N \) is the number of cycles and \( \theta \) is the temperature.

The evolution laws for the plastic strain and the internal plastic variables are defined as,

\[ \dot{\varepsilon}_{ij}^p = \lambda \frac{\partial G}{\partial S_{ij}} \]
\[ \dot{\alpha}^p = \lambda H(S_{ij}, \alpha^p) = \lambda (h_p)_{ij} \frac{\partial G}{\partial S_{ij}} \] (5)
where \( \lambda \) is the consistency plastic factor and \( H \) is a scalar function with tensorial arguments (Lubliner [5], Luccioni [6], Oller [9]) that describes the evolution of each internal variable.

### 3.3.d. Threshold damage function

Onset of damage depends on the current stress state, the internal damage variable, the temperature and other variables as the number of cycles. The thresholds damage function is defined as

\[
G^D (S_{ij}, d, \Theta) = \tilde{S}(S_{ij}) - F^D (S_{ij}, d, N, \Theta)
\]

(6)

where \( \tilde{S}(S_{ij}) \) is the equivalent stress function in the undamaged space, \( F^D (S_{ij}, d, N, \Theta) \) is the strength threshold and \( d = \int_0^\infty \dot{d} \, dt \) the damage internal variable (Lubliner [5], Luccioni [6], Oller [9]).

The evolution of the damage variable is defined as,

\[
\dot{d}_{\text{mec}} = \mu \frac{\partial G^D}{\partial \tilde{S}}
\]

(7)

where \( \mu \) is the consistency damage factor and \( G^D \) the threshold damage function defined above.

### 3.3.e. Tangent constitutive law

From the consistency conditions for the plastic (\( \dot{F} = 0 \)) and damage (\( \dot{G} = 0 \)) problems, the stress rate can be obtained as:

\[
\dot{S}_{ij} = \frac{2}{\partial} [(1 - d) C^0_{ijkl}(\Theta) E^e_{kl}] = C^e_{ijkl} \dot{E}_{kl} - C^p_{ijkl} \dot{\dot{E}}_{kl} - C^0_{ijkl} \dot{E}^0_{kl}, \text{ with: } E_{ij} = E^e_{ij} + E^p_{ij} + E^0_{ij}
\]

(8.a)

where:

\[
C^e_{ijkl} = (1 - d) C^0_{ijkl}
\]

\[
C^e_{ijkl} = C^s_{ijkl} - \frac{1}{(1 - d)} \frac{\partial G^D}{\partial \tilde{S}} \left[ \left( \frac{\partial \tilde{S}}{\partial S^s_{ij}} \right) C^s_{ijkl} \right] S_{kl}
\]

Eq (8.a) can be written in the more standard form, after some algebraic manipulation, as

\[
\dot{S}_{ij} = C^p_{ijkl} \dddot{E}_{kl} - C^0_{ijkl} \dot{E}^0_{kl}
\]

(8.b)

with,

\[
C^p_{ijkl} = C^e_{ijkl} - \left( \frac{\partial G}{\partial S^s_{ij}} \frac{\partial F}{\partial \tilde{S}} \right) C^e_{ijkl}
\]

\[
C^0_{ijkl} = C^e_{ijkl} - \left( \frac{\partial G}{\partial S^s_{ij}} \frac{\partial F}{\partial \tilde{S}} \right) C^e_{ijkl}
\]
In the last expressions and in the successive presentation, the material properties are assumed to be dependent on the temperature state.

### 3.4. Thermo-mechanical coupling

The heat balance equation for the coupled thermo-mechanical problem is deduced from the first thermodynamic law and Fourier’s conduction equation (Lubliner [3,4], Malvern [7]). The energy equation for thermo-mechanical problem can be written in the classical form as,

\[ Q + J \text{div} \left( k \frac{\partial \theta}{\partial x_j} \right) - \theta \beta_{ij} \dot{E}_{ij} + D^\theta - c_k m^\theta \dot{\theta} = 0 \]  

(9)

where \( Q \) represent the caloric power, \( \theta \beta_{ij} \dot{E}_{ij} \) the thermo-elastic coupled term, \( \beta_{ij} \) is the conjugate of thermal expansion coefficient, \( D^\theta \) the thermo-plastic coupled term and \( J \) is the determinant of the Jacobian matrix. Equation (9) with the properly imposed boundary conditions can simulate the evolution of temperature influenced by the mechanical variables. Next the form in which the phenomenon of thermal fatigue is included within this theoretical frame is shown.

### 3.5. Thermo-elasto-plastic-damage model for fatigue analysis. – Macroscopic approach

We will define next a fatigue function that modifies the discontinuity threshold (yield or damage). This produces an “implicit evolution” of the internal plastic and/or damage variables of the inelastic model. In addition, thermal effects in the fatigue life evolution are introduced as in the mechanical formulation, throughout an implicit internal damage variable. This approach differs from that introduced by Chaboche [20], where the evolution of the internal damage variable in terms of the number of cycles is defined in an explicit form.

The effect of the number of cycles on the plastic and/or damage consistency conditions is introduced as follows,

\[
\int \left( f(S_{ij}) - \frac{K(S_{ij} , \alpha^\theta) \cdot \bar{f}_{\text{red}}(N,S_{\text{med}},R,\theta)}{f(N,S_{\text{med}},R,\theta)} \right) = 0 \Rightarrow \left( \frac{f(S_{ij})}{f(N,S_{\text{med}},R,\theta)} \right) - \frac{\bar{K}(S_{ij} , \alpha^\theta)}{\bar{f}(S_{ij},N,R,\theta)} = 0
\]

\[
\int \left( \bar{S}(S_{ij}) - \frac{F^D(S_{ij},d) \cdot \bar{f}_{\text{red}}(N,S_{\text{med}},R,\theta)}{F^D(N,S_{\text{med}},R,\theta)} \right) = 0 \Rightarrow \left( \frac{\bar{S}(S_{ij})}{\bar{f}(N,S_{\text{med}},R,\theta)} \right) - F^D(S_{ij},d) = 0
\]

(10)

The reduction function \( \bar{f}_{\text{red}}(N,S_{\text{max}},R,\theta) = f(N,S_{\text{max}} R) \cdot f_\theta(\theta) \) makes the plasticity and damage models dependent on the phenomenon of thermal fatigue. In eq.(10), \( N \) is the current number of cycles, \( R = \frac{S_{\text{min}}}{S_{\text{max}}} \) it is the stress reversion factor, \( S_{\text{max}} \) the maximum...
applied stress (see Figure 2a), \( f_N(N, S_{\text{max}}, R) \) is the reduction function influenced by the number of the cycles \( N \) and \( f_\theta(\theta) \) is the thermal reduction function.

The constitutive law for the thermo-mechanical fatigue problem can be rewritten as equations (8.a) and (8.b), simply changing the \( f \) and \( S \) functions by the normalized ones \( f^* = f / (f_N f_\theta) \) and \( S^* = S / (f_N f_\theta) \).

It is also necessary to define a unique global hardening internal variable, for the considered plastic and damage effects, based on the normalized dissipation energy as

\[
\dot{q} = \Xi_m \cdot \mathcal{R}(S_{\gamma}) = \left( \Xi^p + \Xi^d \right) \left[ \frac{r(S_{\gamma})}{g_f} + \frac{(1 - r(S_{\gamma}))}{g_e} \right], \quad r(S_{\gamma}) = \begin{cases} 1 & \text{for pure tension} \\ 0 & \text{for pure compression} \end{cases}
\]  

In this equation \( \Xi^p, \Xi^d \) are the plastic and damage energy dissipations and \( g_f, g_e \) are the maximum energy limits that can be dissipated at a point at the end of the inelastic process. It must be observed that this definition enforces the simultaneous fulfillment of the plastic and damage consistency conditions. The mechanical process described previously allows the coupling of rate dependent or independent phenomena with the number of cycles. Therefore, a given strength results from the combination of two phenomena defined in independent spaces (see Figure 3).

\[
q = q(\alpha^p, d, \theta)
\]

Figure 3. Schematic view of the hyper yield-damage surface

### 4. THE STRESS LIFE APPROACH PROPOSED

As it was mentioned in Section 2, Wöhler or “\( S-N \)” curves are obtained from constant amplitude tests of smooth specimens cyclically loaded between a maximum \( (S_{\text{max}}) \) and a minimum stress \( (S_{\text{min}}) \) levels until failure. Typical “\( S-N \)” curves for metallic materials look like as those in Figure 2b, where it can be seen that fatigue life (for the same \( S_{\text{max}} \) applied stress) changes for different values of the ratio \( R \) between \( S_{\text{max}} \) and \( S_{\text{min}} \).
4.1. Influence of ratio \( R = S_{\text{min}} / S_{\text{max}} \) on “S-N” curves

Usually, “S-N” curves are obtained for fully reversed stress (\( R = S_{\text{min}} / S_{\text{max}} = -1.0 \)) by rotating bending fatigue tests. However, this zero mean stress is not typical of real industrial components working under cyclic loads. Let’s see all possible situations:

The maximum possible tension stress at a material point is \( S_{\text{max}} = S_u \) (\( S_u = P_{\text{max}} / A_0 \), ultimate stress). Let’s hold \( S_{\text{max}} = S_u \) while \( S_{\text{min}} \) changes its value from \( S_{\text{min}} = S_u \) (\( R = +1.0 \)) to \( S_{\text{min}} = 0.0 \) (\( R = 0.0 \)), where the point is under pure tension stress. Now, let’s \( S_{\text{min}} \) being a compression stress, changing from \( S_{\text{min}} = 0.0 \) (\( R = 0.0 \)) to \( S_{\text{min}} = -S_u \) (\( R = -1.0 \)), where the point is under a combined tension-compression stress (See the upper part of Figure 4). If \( S_{\text{min}} \) holds its last value, \( S_{\text{min}} = -S_u \), and \( S_{\text{max}} \) goes down from \( S_{\text{max}} = S_u \) (\( R = -1.0 \)) to \( S_{\text{max}} = 0.0 \) (\( R = \infty \)), and the stress at the point is still under combined tension-compression. The possible values of \( S_{\text{max}} \) could go even lower, from \( S_{\text{max}} = 0.0 \) (\( R = \infty \)) to \( S_{\text{max}} = -S_u \) (\( R = +1 \)), being the point under a compression-compression stress (See the lower part of Figure 4).

![Figure 4. Possible values for R=S_{\text{min}}/S_{\text{max}} in case of harmonic constant amplitude function load.](image-url)
4.2. S-N Curves function proposed

Based on the actual value of the R ratio and a basic value of the endurance stress $S_e$ (for $R = -1$) the model proposed here postulates a threshold stress $S_{th}$. The meaning of $S_{th}$ is that of an endurance stress limit for a given value of $R=S_{min}/S_{max}$.

If the actual value of $R$ is $R= -1$ then, $S_{th}=S_e$. The effect of the number of load cycles ($N_{cycles}$) on the ultimate stress $S_u$ for a given value of $R$ is taken into account by an exponential function,

$$S_{(R,N_{cycles})} = S_{th(R)} + (S_u-S_{th(R)}) \times \exp(-ALFAT(R) \times \log_{10}(N_{cycles})^{BETAF})$$  \hspace{1cm} (13)

The value of $ALFAT(R)$ is given by the function,

$$ALFAT(R) = \begin{cases} ALFAF + (0.5 + 0.5 \times R) \times AUXR1 \\
ALFAF - (0.5 + 0.5/R) \times AUXR2 \end{cases} \hspace{1cm} (14)$$

$ALFAF$, $BETAF$, $STHR1$, $STHR2$, $AUXR1$ and $AUXR2$ are material parameters that need to be adjusted according to experimental tests. Figure 5 shows an example of application of these functions.

4.3. Reduction Strength Function

The proposed S-N curves (equations 12-14) are fatigue life estimators for a material point with a fixed maximum stress and a given ratio $R$. If, after a number of cycles lower than the cycles to failure, the constant amplitude cyclic loads giving that maximum stress $S_{max}$ (and ratio $R$) are removed, some change in $S_u$ is expected due to accumulation of fatigue cycles.

In order to describe that variation of $S_u$ the following function is proposed:

$$f_{red(R,N_{cycles})} = \exp(-B0 \times \log_{10}(N_{cycles}^{BETAF \times BETAF})$$  \hspace{1cm} (15)

$BETAF$ is one of the parameters of eq.(13) and $B0$ is obtained as a function of the ratio $S_{max}/S_u$ and the number of cycles to failure $N_F$, by

$$B0 = -\log_{10}(S_{max}/S_u) / (\log_{10}(N_F))^{BETAF \times BETAF}$$  \hspace{1cm} (16)

The value of $N_F$ can be obtained from (13),

$$N_F=10^{((-\log_{10}(S_{max}-S_{th(R)}) / (S_u-S_{th(R)}))) / (ALFAT(R)) / BETAF})$$  \hspace{1cm} (17)
Figure 5. Proposed S-N curves for different values of R (S_{min}/S_{max}). Parameters were chosen as follow: Se=0.5*Su, ALFAF=0.0068, BETAF=3.35, STHR1=0.7, STHR2=0.5, AUXR1=0.0133 and AUXR2=0.0068
Figure 6 shows a S-N curve and the reduction strength function $f_{red}$ using the same material parameters of Figure 5, a ratio $R = -1$ and a $S_{max} = 0.7 \times S_u$. $S_u \times f_{red}$ intersect S-N curve at a $S_{max}$ stress level.

Figure 6. Proposed S-N and $f_{red}$ curves for $R = -1$ and $S_{max} = 0.7 \times S_u$. Parameters were chosen as follow: $S_e = 0.5 \times S_{\infty}$, $ALFAF = 0.0068$, $BETAF = 3.35$, $STHR1 = 0.7$, $STHR2 = 0.5$, $AUXRI = 0.0133$ and $AUXR2 = 0.0068$

5. Finite Element Equations

Following the virtual work principle and the first thermodynamic law, the mechanical and thermal equilibrium equations in the referential configuration are obtained,

\[
\begin{align*}
\text{mechanical:} & \quad \int (u_i \cdot m^\alpha \cdot \dot{u}_i + S_{ij} \cdot \nabla_j \dot{u}_i) dV - \int m^\alpha b_i \dot{u}_i dV - \oint t_i \dot{u}_i dS = 0 \\
\text{thermal:} & \quad \int \theta \text{div}(q_i) dV + \int q_i \nabla \theta dV - \oint \theta q_i n_i dS_b = 0
\end{align*}
\]

substituting eq.(11) in the mechanical equation and eq.(9) in the thermal one, and approximating the displacements $u(x)$ and the temperature $\theta(x)$ by the standard finite element procedure (Zienkiewicz and Taylor [29]), the following equilibrium equations are obtained,

\[
\begin{align*}
\begin{cases}
M_u \cdot \dot{U} + f_{\text{int}}^u + M_{u-\theta} \cdot \dot{\Theta} - f_{\text{ext}}^u = 0 \\
C_\theta \cdot \dot{\Theta} + M_{\theta-u} \cdot \dot{U} + K_\theta \cdot \Theta + D_\theta - f_\theta = 0
\end{cases}
\end{align*}
\]

where $N^u(x)$ and $N^\theta(x)$ are the displacement and temperature shape functions, $U$ and $\Theta$ are the nodal values of the displacements and the temperature, $M_u$ is the mass matrix, $f_{\text{int}}^u = (f_{\text{ext}}^u)^{\text{int}} = \int S_{ij} \nabla_i N^u_{jk} \, dV$ is the mechanical internal forces vector, $M_{\theta-u}$ is the thermal stiffness matrix, $f_{\text{ext}}^\theta$ is the nodal force vector due to external loads, $C_\theta$ is the caloric capacity matrix, $M_{u-\theta}$ is the thermoelastic coupling matrix, $K_\theta$
is the conductivity matrix, $D^p$ is the mechanical dispassion and $f_0$ is the thermal load vector. The detailed definitions of these matrices is given in [31, 32]. The solution of eqs.(18) follows a structured staggered implicit time integration scheme. Details can be found in [31, 32].

5.1. Time advancing strategy

A great advantage of the methodology before presented consists in the way the loading is applied. In a mechanical problem each load is applied in two intervals, in the following order (see Figure 7),

a. **Tracing load**, (described by “$a_i$” periods on Figure 7). It is used to obtain the stress ratio $R_{min}/S_{max}$ at each integration point, following the load path during several cycles until the $R$ relationship tends to a constant value. This occurs when the following norm is satisfied,

$$
\eta = \sum_{GP} \left\| R_{GP}^{i+1} - R_{GP}^{i} \right\| \to 0
$$

(19)

where $R_{GP}^{i} = S_{min}/S_{max}$ is computed at each Gauss interpolation point “$GP$” for the load increment “$i$”.

b. **Enveloping load**, (described by “$b_i$” periods on Figure 7). After the first “tracing load interval” (a), the number of cycles $N$ is increased at each Gauss interpolation point keeping constant the maximum applied load (thick line in Figure 7) and the $R$ stress ratio. In this new load interval, the variable is not the level of the load (kept constant), but the number of cycles.

These two-stages strategy allows a very fast advance in the time loading. A new interval with the two stages explained should be added for each change in the loading level.

![Figure 7: Schematic time advancing loads representation](image-url)
6. INDUSTRIAL VALIDATION OF THE PROPOSED FATIGUE MODEL

The analytical model proposed in the previous section has been implemented into the general-purpose thermo-mechanical finite element code COMET [25]. The fatigue model and its numerical implementation had been validated using real industrial components. In this section the analysis of an aluminium engine alternator support is presented. The alternator support, Figure 8, was selected by Fonderie 2A among the components of their actual production and experimentally tested by Politecnico de Torino (POLITO) in the framework of the DARCAST Project [26].

![Figure 8. Engine alternator support and equipment used for the fatigue experimental testing](image)

Two batches were provided: one batch of non-porous components (i.e. components with an acceptable level of porosity) and one batch of porous components; the distinction was done by the manufacturer based on radioscopy examination. All the about 50 components submitted to fatigue testing were previously examined through radioscopic and radiographic inspection in order to classify their actual porosity level. Few components were also scanned through computed tomography to gain a more detailed analysis of the defects. The higher resolution of this last technique permitted to visualise that a scattered porosity was also present in the “non-porous” sample. At the end of the fatigue test, the fracture surfaces were analysed both through metallographic and scanning electron microscopy to evaluate the nature and dimensions of the defects (that include pores as well as inclusions as oxide films). The fatigue testing showed that for both samples (porous and non-porous) four different regions of failure could be identified. At regions of failure not highly stressed, pores of significant dimensions, oxide films and defects in the microstructure were observed on the fracture surfaces. The difference in the fatigue limit as found for the porous and non porous samples (the classification was made by Fonderie 2A on the basis of their standard radiosopic inspection) was observed to be statistically non significant.

6.1. Material characterization

Three batches of specimens were experimentally tested, corresponding to porosity ranges 0, 2, 4 (category A) according to ASTM E505 standard. The porosity level was assessed through X-ray examination.
In order to calibrate the elasto-plastic constitutive model (J2) to the material (GD-ALSi8.5Cu3.5Fe -UNI 5075) the support was made of, a simple tension numerical test was carried out for material with porosity A0 and A2. Figure 9 show the numerical results (Stress-Strain curves) in comparison with those from experimental tests for material with porosity A2.

![Figure 9. Stress-Strain curve for the aluminium alloy. Experimental and numerical results](image)

The fatigue life prediction model was also calibrated using experimental results from POLITO. Rotating bending tests (R=Smin/Smax= -1) were run on the aluminium specimens. Tests performed on the three batches of aluminium alloy specimens showed a great influence of porosity on the fatigue strength of the material. A significant reduction in the fatigue strength at 2 million cycles (high cycle fatigue) was found when the degree of porosity is increased from A0 to A2. The decrease was distinctly lower when the degree of porosity was further increased from A2 to A4.

Experimental results, Stress-N° of cycles curves, for aluminium alloy with porosity A2 are shows in Figure 10 and the corresponding numerical approach appears in Figure 11. For the fatigue characterisation of the aluminium alloy, the material parameters used for the numerical test were:

\[
S_{th(R)} = S_c + (S_u - S_c)^{0.5 + 0.5 R}^{3.5} ; \quad S_c=0.4 \quad S_u=220 \text{ MPa}
\]

\[
S_{th(R,N_{cycles})} = S_{th(R)} + (S_u - S_{th(R)}) \times \exp(-0.00065 \times \log_{10}(N_{cycles})^{1.55})
\]
6.2. Die Cast Aluminium Support Analysis

Once the material was characterized for quasi-static and cyclic loading, a numerical simulation of the aluminium alternator support (Figure 8) was done and results of the analysis were compared with those from experimental testing.
The experimental tests were done applying a minimum load of 0.5 kN and a maximum load between 10 and 11.5 kN. For the numerical analysis these values were a minimum of 0.5 kN and a maximum of 10.5 kN, given a mean load of 5.5 kN and alternating component of 5.0 kN. This cycling force was applied as a surface load as indicated in Figure 12 on a three-dimensional model on two time intervals. Throughout the first time interval the time steps were fixed as 1/8 of the period cycle. During the second interval the fatigue analysis was carried out using the maximum stresses ($S_{\text{max}}$) and ratios ($R=S_{\text{min}}/S_{\text{max}}$) obtained on the previous interval. In this way time steps can be far larger than the used during the first interval reducing the computing time. Results presented here were obtained using a time step of 1.e+5 cycles.

Figure 12. Geometry, loading, boundary conditions and displacement results

Figure 13 compares experimental regions of failure with stress distribution obtained from the numerical simulation. Regions of failure 2, 3 and 4 are in coincidence with extended areas highly stressed.

Figure 14 demonstrates the correlation between the damage areas due to fatigue and experimental regions of failure. Damage is observed in concordance with regions of failure 3 and 4. Region of failure 2 (upper part) correspond to a highly compressed area what dismiss the fatigue effects. For region of failure 1 the numerical analysis only found a small area of stress concentration due to local geometry conditions.

Stress and damage index evolution during the time loading, in semi-logarithmic scale, are showed in figures 15 and 16 for a Gauss point whit one of the highest damage index. Life prediction of the aluminium support according with these curves (material A2) agree with experimental tests: Failure around 900,000 cycles for a maximum load of 10.5 kN (50% probability of survival, porous and no-porous components. No porous components indicate ‘acceptable’ level of porosity).
Figure 13. Stress distribution from numerical simulation and experimental regions of failure

Figure 14. Fatigue damage distribution from numerical simulation and experimental regions of failure
It is well known that the results from fatigue tests show a wide scattering. In the case of the aluminium support analysed, the fatigue testing identified four different regions of failure. At regions of failure not highly stressed, pores of significant dimensions, oxide films and defects in the microstructure were observed on the fracture surfaces (remark from POLITO). The uncertainty about location of these defects, as well as errors in the estimation of material properties (variability from one specimen to another) and its effects on the probability of failure could be solved using an statistical approach. The authors are currently working on such type of fatigue analysis.

7. CONCLUDING REMARKS

A powerful tool, based in continuum mechanic and the finite element method, that allows accurate predictions on fatigue behaviour of metallic components has been presented. General expressions for the elasto-plastic-damage constitutive equations sensitive for cyclic loads have been derived and stress-cycles and reduction strength curves functions have been proposed. The $S-N$ functions proposed depend on and are
capable of dealing with any value of the ratio between minimum and maximum stress. A time advancing strategy has been formulated. This two-stages strategy (tracing load and enveloping load intervals) permits a very fast advance in the time loading. An aluminium industrial component has been studied to show the potential of the methodology developed which can be applied to analyse components and structures made of different materials (material parameters must be characterised for each one) and under complex cyclic loads. Comparing results from numerical mechanical analysis with those from experimental test on the industrial component, excellent relationship between highly stressed zones (as well as plastic strain zones) and regions of failure have been found. Damaged areas according the numerical analysis were observed in concordance with experimental regions of failure. Lifetime prediction of the aluminium support agrees with experimental tests.

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