Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Version 2 CE IIT, Kharagpur
Lesson

17

The Slope-Deflection Method: Frames with Sidesway

Version 2 CE IIT, Kharagpur
Instructional Objectives

After reading this chapter the student will be able to
1. Derive slope-deflection equations for the frames undergoing sidesway.
2. Analyse plane frames undergoing sidesway.
3. Draw shear force and bending moment diagrams.
4. Sketch deflected shape of the plane frame not restrained against sidesway.

17.1 Introduction

In this lesson, slope-deflection equations are applied to analyse statically indeterminate frames undergoing sidesway. As stated earlier, the axial deformation of beams and columns are small and are neglected in the analysis. In the previous lesson, it was observed that sidesway in a frame will not occur if

1. They are restrained against sidesway.
2. If the frame geometry and the loading are symmetrical.

In general loading will never be symmetrical. Hence one could not avoid sidesway in frames.

For example, consider the frame of Fig. 17.1. In this case the frame is symmetrical but not the loading. Due to unsymmetrical loading the beam end moments $M_{BC}$ and $M_{CB}$ are not equal. If $b$ is greater than $a$, then $M_{BC} > M_{CB}$.
such a case joint $B$ and $C$ are displaced toward right as shown in the figure by an unknown amount $\Delta$. Hence we have three unknown displacements in this frame: rotations $\theta_B$, $\theta_C$ and the linear displacement $\Delta$. The unknown joint rotations $\theta_B$ and $\theta_C$ are related to joint moments by the moment equilibrium equations. Similarly, when unknown linear displacement occurs, one needs to consider force-equilibrium equations. While applying slope-deflection equation to columns in the above frame, one must consider the column rotation $\psi = \frac{\Delta}{h}$ as unknowns. It is observed that in the column $AB$, the end $B$ undergoes a linear displacement $\Delta$ with respect to end $A$. Hence the slope-deflection equation for column $AB$ is similar to the one for beam undergoing support settlement. However, in this case $\Delta$ is unknown. For each of the members we can write the following slope-deflection equations.

$$
M_{AB} = M_{AB}^F + \frac{2EI}{h} \left[ 2\theta_A + \theta_B - 3\psi_{AB} \right] \\
\text{where } \psi_{AB} = -\frac{\Delta}{h}
$$

$\psi_{AB}$ is assumed to be negative as the chord to the elastic curve rotates in the clockwise directions.

$$
M_{BA} = M_{BA}^F + \frac{2EI}{h} \left[ 2\theta_B + \theta_A - 3\psi_{AB} \right] \\
M_{BC} = M_{BC}^F + \frac{2EI}{h} \left[ 2\theta_B + \theta_C \right] \\
M_{CB} = M_{CB}^F + \frac{2EI}{h} \left[ 2\theta_C + \theta_B \right] \\
M_{CD} = M_{CD}^F + \frac{2EI}{h} \left[ 2\theta_C + \theta_D - 3\psi_{CD} \right] \\
\psi_{CD} = -\frac{\Delta}{h} \\
M_{DC} = M_{DC}^F + \frac{2EI}{h} \left[ 2\theta_D + \theta_C - 3\psi_{CD} \right] \quad (17.1)
$$

As there are three unknowns ($\theta_B$, $\theta_C$ and $\Delta$), three equations are required to evaluate them. Two equations are obtained by considering the moment equilibrium of joint $B$ and $C$ respectively.

$$
\sum M_B = 0 \quad \Rightarrow \quad M_{BA} + M_{BC} = 0 \quad (17.2a)
$$

$$
\sum M_C = 0 \quad \Rightarrow \quad M_{CB} + M_{CD} = 0 \quad (17.2b)
$$

Now consider free body diagram of the frame as shown in Fig. 17.2. The horizontal shear force acting at $A$ and $B$ of the column $AB$ is given by

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Similarly for member CD, the shear force $H_3$ is given by

$$H_3 = \frac{M_{CD} + M_{DC}}{h} \quad (17.3b)$$

Now, the required third equation is obtained by considering the equilibrium of member $BC$,

$$\sum F_x = 0 \quad \Rightarrow \quad H_1 + H_3 = 0$$

$$\frac{M_{BA} + M_{AB}}{h} + \frac{M_{CD} + M_{DC}}{h} = 0 \quad (17.4)$$

Substituting the values of beam end moments from equation (17.1) in equations (17.2a), (17.2b) and (17.4), we get three simultaneous equations in three unknowns $\theta_a, \theta_c$ and $\Delta$, solving which joint rotations and translations are evaluated.
Knowing joint rotations and translations, beam end moments are calculated from slope-deflection equations. The complete procedure is explained with a few numerical examples.

Example 17.1

Analyse the rigid frame as shown in Fig. 17.3a. Assume $EI$ to be constant for all members. Draw bending moment diagram and sketch qualitative elastic curve.

![Figure 17.3 (a) Example 17.1](image)

Solution

In the given problem, joints $B$ and $C$ rotate and also translate by an amount $\Delta$. Hence, in this problem we have three unknown displacements (two rotations and one translation) to be evaluated. Considering the kinematically determinate structure, fixed end moments are evaluated. Thus,

$$
M_{AB}^F = 0 ; M_{BA}^F = 0 ; M_{BC}^F = +10kN.m ; M_{CB}^F = -10kN.m ; M_{CD}^F = 0 ; M_{DC}^F = 0. 
$$ (1)

The ends $A$ and $D$ are fixed. Hence, $\theta_A = \theta_D = 0$. Joints $B$ and $C$ translate by the same amount $\Delta$. Hence, chord to the elastic curve $AB'$ and $DC'$ rotates by an amount (see Fig. 17.3b)

$$
\psi_{AB} = \psi_{CD} = -\frac{\Delta}{3} \quad (2)
$$

Chords of the elastic curve $AB'$ and $DC'$ rotate in the clockwise direction; hence $\psi_{AB}$ and $\psi_{CD}$ are taken as negative.
Now, writing the slope-deflection equations for the six beam end moments,

\[ M_{AB} = M_{AB}^F + \frac{2EI}{3} \left[ 2\theta_A + \theta_B - 3\psi_{AB} \right] \]

\[ M_{AB}^F = 0 \quad \theta_A = 0 \quad \psi_{AB} = -\frac{\Delta}{3}. \]

\[ M_{AB} = \frac{2}{3} EI\theta_B + \frac{2}{3} EI\Delta \]

\[ M_{BA} = \frac{4}{3} EI\theta_B + \frac{2}{3} EI\Delta \]

\[ M_{BC} = 10 + EI\theta_B + \frac{1}{2} EI\theta_C \]

\[ M_{CB} = -10 + \frac{1}{2} EI\theta_B + EI\theta_C \]

\[ M_{CD} = \frac{4}{3} EI\theta_C + \frac{2}{3} EI\Delta \]
\[ M_{DC} = \frac{2}{3} EI \theta_c + \frac{2}{3} EI \Delta \]  \hspace{1cm} (3)

Now, consider the joint equilibrium of \( B \) and \( C \) (vide Fig. 17.3c).

\[ \sum M_B = 0 \Rightarrow M_{BA} + M_{BC} = 0 \]  \hspace{1cm} (4)

\[ \sum M_C = 0 \Rightarrow M_{CB} + M_{CD} = 0 \]  \hspace{1cm} (5)

Fig.17.3c Free - body diagram of joints B and C

The required third equation is written considering the horizontal equilibrium of the entire frame \( i.e. \) \( \sum F_x = 0 \) (vide Fig. 17.3d).

\[-H_1 + 10 - H_2 = 0\]

\[ \Rightarrow \quad H_1 + H_2 = 10. \]  \hspace{1cm} (6)
Considering the equilibrium of the column $AB$ and $CD$, yields

\[ H_1 = \frac{M_{BA} + M_{AB}}{3} \]

and

\[ H_2 = \frac{M_{CD} + M_{DC}}{3} \] (7)

The equation (6) may be written as,

\[ M_{BA} + M_{AB} + M_{CD} + M_{DC} = 30 \] (8)

Substituting the beam end moments from equation (3) in equations (4), (5) and (6)

\[ 2.333EI\theta_B + 0.5EI\theta_C + 0.667EI\Delta = -10 \] (9)

\[ 2.333EI\theta_C + 0.5EI\theta_B + 0.667EI\Delta = 10 \] (10)
\[ 2EI\theta_B + 2EI\theta_C + \frac{8}{3} EI\Delta = 30 \]  

Equations (9), (10) and (11) indicate symmetry and this fact may be noted. This may be used as the check in deriving these equations.

Solving equations (9), (10) and (11),

\[ EI\theta_B = -9.572 \quad ; \quad EI\theta_C = 1.355 \quad \text{and} \quad EI\Delta = 17.417 . \]

Substituting the values of \( EI\theta_B, EI\theta_C \) and \( EI\Delta \) in the slope-deflection equation (3), one could calculate beam end moments. Thus,

\[ M_{AB} = 5.23 \text{ kN.m (counterclockwise)} \]

\[ M_{BA} = -1.14 \text{ kN.m (clockwise)} \]

\[ M_{BC} = 1.130 \text{ kN.m} \]

\[ M_{CB} = -13.415 \text{ kN.m} \]

\[ M_{CD} = 13.406 \text{ kN.m} \]

\[ M_{DC} = 12.500 \text{ kN.m} . \]

The bending moment diagram for the frame is shown in Fig. 17.3 e. And the elastic curve is shown in Fig 17.3 f. the bending moment diagram is drawn on the compression side. Also note that the vertical hatching is used to represent bending moment diagram for the horizontal members (beams).
Fig. 17.3e Bending moment diagram
Example 17.2

Analyse the rigid frame as shown in Fig. 17.4a and draw the bending moment diagram. The moment of inertia for all the members is shown in the figure. Neglect axial deformations.
Solution:

In this problem rotations and translations at joints $B$ and $C$ need to be evaluated. Hence, in this problem we have three unknown displacements: two rotations and one translation. Fixed end moments are

$$M_{AB}^F = \frac{12 \times 3 \times 9}{36} = 9 \text{kN.m} ; M_{BA}^F = -9 \text{kN.m} ;$$
$$M_{BC}^F = 0 ; M_{CB}^F = 0 ; M_{CD}^F = 0 ; M_{DC}^F = 0.$$  \hspace{1cm} (1)

The joints $B$ and $C$ translate by the same amount $\Delta$. Hence, the chord to the elastic curve rotates in the clockwise direction as shown in Fig. 17.3b.

$$\psi_{AB} = -\frac{\Delta}{6}$$

and

$$\psi_{CD} = -\frac{\Delta}{3}$$  \hspace{1cm} (2)

Now, writing the slope-deflection equations for six beam end moments,

$$M_{AB} = 9 + \frac{2(2EI)}{6} \left[ \theta_B + \frac{\Delta}{2} \right]$$

$$M_{AB} = 9 + 0.667EI\theta_B + 0.333EI\Delta$$

$$M_{BA} = -9 + 1.333EI\theta_B + 0.333EI\Delta$$
\[ M_{BC} = EI\theta_B + 0.5EI\theta_C \]
\[ M_{CB} = 0.5EI\theta_B + EI\theta_C \]
\[ M_{CD} = 1.333EI\theta_C + 0.667EID \]
\[ M_{DC} = 0.667EI\theta_C + 0.667EID \]  \hspace{1cm} (3)

Now, consider the joint equilibrium of \( B \) and \( C \).

\[ \sum M_B = 0 \Rightarrow M_{BA} + M_{BC} = 0 \]  \hspace{1cm} (4)
\[ \sum M_C = 0 \Rightarrow M_{CB} + M_{CD} = 0 \]  \hspace{1cm} (5)

The required third equation is written considering the horizontal equilibrium of the entire frame. Considering the free body diagram of the member \( BC \) (vide Fig. 17.4c),

\[ H_1 + H_2 = 0 \]  \hspace{1cm} (6)
The forces $H_1$ and $H_2$ are calculated from the free body diagram of column $AB$ and $CD$. Thus,

$$H_1 = -6 + \frac{M_{BA} + M_{AB}}{6}$$

and

$$H_2 = \frac{M_{CD} + M_{DC}}{3}$$  \hspace{1cm} (7)

Substituting the values of $H_1$ and $H_2$ into equation (6) yields,

$$M_{BA} + M_{AB} + 2M_{CD} + 2M_{DC} = 36$$  \hspace{1cm} (8)

Substituting the beam end moments from equation (3) in equations (4), (5) and (8), yields

$$2.333EI\theta_B + 0.5EI\theta_C + 0.333EI\Delta = 9$$

$$2.333EI\theta_C + 0.5EI\theta_B + 0.667EI\Delta = 0$$

$$2EI\theta_B + 4EI\theta_C + 3.333EI\Delta = 36$$  \hspace{1cm} (9)

Solving equations (9), (10) and (11),

$$EI\theta_B = 2.76 \; ; \; EI\theta_C = -4.88 \; \text{ and } \; EI\Delta = 15.00.$$  

Substituting the values of $EI\theta_B$, $EI\theta_C$ and $EI\Delta$ in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 15.835 \text{ kN.m (counterclockwise)}$$

$$M_{BA} = -0.325 \text{ kN.m (clockwise)}$$

$$M_{BC} = 0.32 \text{ kN.m}$$

$$M_{CB} = -3.50 \text{ kN.m}$$

$$M_{CD} = 3.50 \text{ kN.m}$$

$$M_{DC} = 6.75 \text{ kN.m}.$$  

The bending moment diagram for the frame is shown in Fig. 17.4 d.
Fig. 17.4d Bending moment diagram
Example 17.3

Analyse the rigid frame shown in Fig. 17.5 a. Moment of inertia of all the members are shown in the figure. Draw bending moment diagram.

Under the action of external forces, the frame gets deformed as shown in Fig. 17.5b. In this figure, chord to the elastic curve are shown by dotted line. $BB'$ is perpendicular to $AB$ and $CC''$ is perpendicular to $DC$. The chords to the elastic
curve \( AB'' \) rotates by an angle \( \psi_{AB} \), \( B''C'' \) rotates by \( \psi_{BC} \) and \( DC \) rotates by \( \psi_{CD} \) as shown in figure. Due to symmetry, \( \psi_{CD} = \psi_{AB} \). From the geometry of the figure,

\[
\psi_{AB} = \frac{BB''}{L_{AB}} = -\frac{\Delta_1}{L_{AB}}
\]

But

\[
\Delta_1 = \frac{\Delta}{\cos \alpha}
\]

Thus,

\[
\psi_{AB} = -\frac{\Delta}{L_{AB} \cos \alpha} = -\frac{\Delta}{5}
\]

\[
\psi_{CD} = -\frac{\Delta}{5}
\]

\[
\psi_{BC} = \frac{\Delta_2}{2} = \frac{2\Delta \tan \alpha}{2} = \Delta \tan \alpha = \frac{\Delta}{5}
\]

We have three independent unknowns for this problem \( \theta_A, \theta_C \) and \( \Delta \). The ends \( A \) and \( D \) are fixed. Hence, \( \theta_A = \theta_D = 0 \). Fixed end moments are,

\[
M_{AB}^F = 0 ; M_{BA}^F = 0 ; M_{BC}^F = +2.50 \text{kN.m} ; M_{CB}^F = -2.50 \text{kN.m} ; M_{CD}^F = 0 ; M_{DC}^F = 0.
\]

Now, writing the slope-deflection equations for the six beam end moments,

\[
M_{AB} = \frac{2E(2l)}{5.1} \left[ \theta_A - 3\psi_{AB} \right]
\]

\[
M_{AB} = 0.784EI\theta_B + 0.471EI\Delta \\
M_{BA} = 1.568EI\theta_B + 0.471EI\Delta
\]

\[
M_{BC} = 2.5 + 2EI\theta_B + EI\theta_C - 0.6EI\Delta \\
M_{CB} = -2.5 + EI\theta_B + 2EI\theta_C - 0.6EI\Delta
\]

\[
M_{CD} = 1.568EI\theta_C + 0.471EI\Delta \\
M_{DC} = 0.784EI\theta_C + 0.471EI\Delta
\]

Now, considering the joint equilibrium of \( B \) and \( C \), yields
\[ \sum M_B = 0 \quad \Rightarrow \quad M_{BA} + M_{BC} = 0 \]

\[ 3.568EI\theta_B + EI\theta_C - 0.129EI\Delta = -2.5 \quad (3) \]

\[ \sum M_C = 0 \quad \Rightarrow \quad M_{CB} + M_{CD} = 0 \]

\[ 3.568EI\theta_C + EI\theta_B - 0.129EI\Delta = 2.5 \quad (4) \]

Shear equation for Column \( AB \)

\[ 5H_1 - M_{AB} - M_{BA} + (1)V_1 = 0 \quad (5) \]

Column \( CD \)

\[ 5H_2 - M_{CD} - M_{DC} + (1)V_2 = 0 \quad (6) \]

Beam \( BC \)

\[ \sum M_C = 0 \quad 2V_1 - M_{BC} - M_{CB} - 10 = 0 \quad (7) \]
\[
\sum F_x = 0 \quad \quad \quad \quad \quad H_1 + H_2 = 5 \quad \quad \quad \quad (8)
\]
\[
\sum F_y = 0 \quad \quad \quad \quad \quad V_1 - V_2 - 10 = 0 \quad \quad \quad \quad (9)
\]

From equation (7), \[ V_1 = \frac{M_{BC} + M_{CB} + 10}{2} \]

From equation (8), \[ H_1 = 5 - H_2 \]

From equation (9), \[ V_2 = V_1 - 10 = \frac{M_{BC} + M_{CB} + 10}{2} - 10 \]

Substituting the values of \( V_1, H_1 \) and \( V_2 \) in equations (5) and (6),

\[
60 - 10H_2 - 2M_{AB} - 2M_{BA} + M_{BC} + M_{CB} = 0 \quad \quad \quad \quad \quad (10)
\]
\[
-10 + 10H_2 - 2M_{CD} - 2M_{DC} + M_{BC} + M_{CB} = 0 \quad \quad \quad \quad \quad (11)
\]

Eliminating \( H_2 \) in equation (10) and (11),

\[ M_{AB} + M_{BA} + M_{CD} + M_{DC} - M_{BC} - M_{CB} = 25 \quad \quad \quad \quad \quad (12) \]

Substituting the values of \( M_{AB}, M_{BA}, M_{CD}, M_{DC} \) in (12) we get the required third equation. Thus,

\[
0.784 EI \theta_B + 0.471 EI \Delta + 1.568 EI \theta_B + 0.471 EI \Delta + 1.568 EI \theta_C + 0.471 EI \Delta + 0.784 EI \theta_C + 0.471 EI \Delta -(2.5 + 2EI \theta_B + EI \theta_C - 0.6EI \Delta) - (-2.5 + EI \theta_B + 2EI \theta_C - 0.6EI \Delta) = 25
\]

Simplifying,

\[ -0.648EI \theta_C - 0.648EI \theta_B + 3.084EI \Delta = 25 \quad \quad \quad \quad \quad (13) \]

Solving simultaneously equations (3) (4) and (13), yields

\[ EI \theta_B = -0.741 \quad ; \quad EI \theta_C = 1.205 \quad \text{and} \quad EI \Delta = 8.204 \]

Substituting the values of \( EI \theta_B, EI \theta_C \) and \( EI \Delta \) in the slope-deflection equation (3), one could calculate beam end moments. Thus,

\[ M_{AB} = 3.28 \text{ kN.m} \]
\[ M_{BA} = 2.70 \text{ kN.m} \]
\[ M_{BC} = -2.70 \text{ kN.m} \]
\[ M_{CB} = -5.75 \text{ kN.m} \]
\[ M_{CD} = 5.75 \text{ kN.m} \]
\[ M_{DC} = 4.81 \text{ kN.m}. \] (14)

The bending moment diagram for the frame is shown in Fig. 17.5 d.

**Summary**

In this lesson, slope-deflection equations are derived for the plane frame undergoing sidesway. Using these equations, plane frames with sidesway are analysed. The reactions are calculated from static equilibrium equations. A couple of problems are solved to make things clear. In each numerical example, the bending moment diagram is drawn and deflected shape is sketched for the plane frame.