Module 15

Redistribution of Moments

Lesson

38

Redistribution of Moments – Theory and Numerical Problems

Instructional Objectives:

At the end of this lesson, the student should be able to:

- explain the concept of redistribution of moments in the design of statically indeterminate reinforced concrete structures,
- explain the behaviour of statically indeterminate reinforced concrete structures with increasing loads till the collapse of the structures,
- state the advantages of redistributing the moments in statically indeterminate reinforced concrete structures,
- identify structures when such redistributions are to be made based on the analysis of structures,
- mention the reasons why full redistribution is not allowed in reinforced concrete structures,
- specify the stipulations of IS Codes about the redistribution separately when the structure is designed by working stress method and by limit state method,
- apply the theories in solving numerical problems of statically indeterminate beams of one or more spans as per the stipulations of IS Code.

15.38.1 Introduction

Statically indeterminate structures made of reinforced concrete like fixed ended one span beams, continuous beams and frames are designed considering internal forces like bending moment, shear force and axial thrust obtained from structural analysis. Either one or several sections of these structures may have peak values of the internal forces, which are designated as critical sections. These sections are dimensioned and reinforced accordingly. Flexural members, however, do not collapse immediately as soon as the loads at a particular section cause bending moment exceeding the maximum resisting moment capacity of that section. Instead, that section starts rotating at almost constant moment. This is known as formation of plastic hinge at that section reaching its maximum resisting moment capacity. The section then transfers loads to other sections if the applied loads are further increased. This process continues till the structures have plastic binges at sufficient sections to form a failure mechanism when it actually collapses. However, significant transfer of loads has occurred before the collapse of the structure. This transfer of loads after the formation of first plastic hinge at section having the highest bending moment till the collapse of the structure is known as redistribution of moments. By this process, therefore, the structure continues to accommodate higher loads before it collapses.

The elastic bending moment diagram prior to the formation of first plastic hinge and the final bending moment diagram just before the collapse are far different. The ratio of the negative to positive elastic bending moments is no more valid. The development of plastic hinges depends on the available plastic moment capacity at critical sections. It is worth mentioning that the redistribution of moment is possible if the section forming the plastic hinge has the ability to rotate at constant moment, which depends on the amount of reinforcement actually provided at that section. The section must be under-reinforced and should have sufficient ductility.

This phenomenon is well known in steel structures. However, the redistribution of moment has also been confirmed in reinforced concrete structure by experimental investigations. It is also a fact that reinforced concrete structures have comparatively lower capacity to rotate than steel structures. Yet, this phenomenon is drawing the attention of the designers. Presently, design codes of most of the countries allow the redistribution up to a maximum limit because of the following advantages:

1) It gives a more realistic picture of the actual load carrying capacity of the indeterminate structure.

2) Structures designed considering the redistribution of moment (though limited) would result in economy as the actual load capacity is higher than that we determine from any elastic analysis.

3) The designer enjoys the freedom of modifying the design bending moments within limits. These adjustments are sometimes helpful in reducing the reinforcing bars, which are crowded, especially at locations of high bending moment.

15.38.2 Two Span Beam

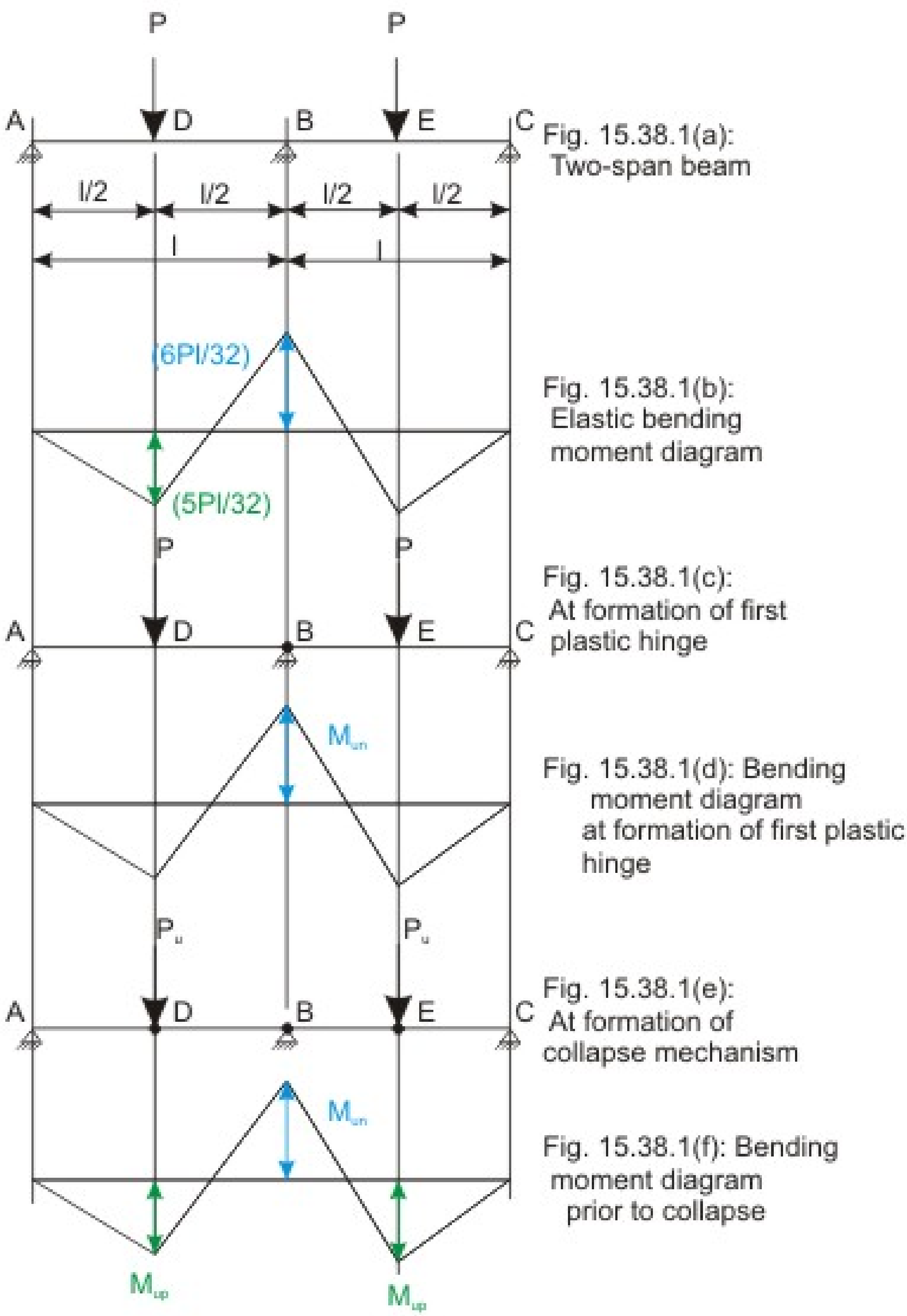


Fig. 15.38.1: Two-span continuous beam

Let us take up a two-span beam of uniform cross-section, as shown in Fig. 15.38.1a, with the following assumptions:

a) The ultimate moments of resistance of sections at B and D are $-M_{un}$ and $+M_{up}$, respectively.

b) There will not be any premature shear failure so that the sections can attain the respective ultimate moment capacity.

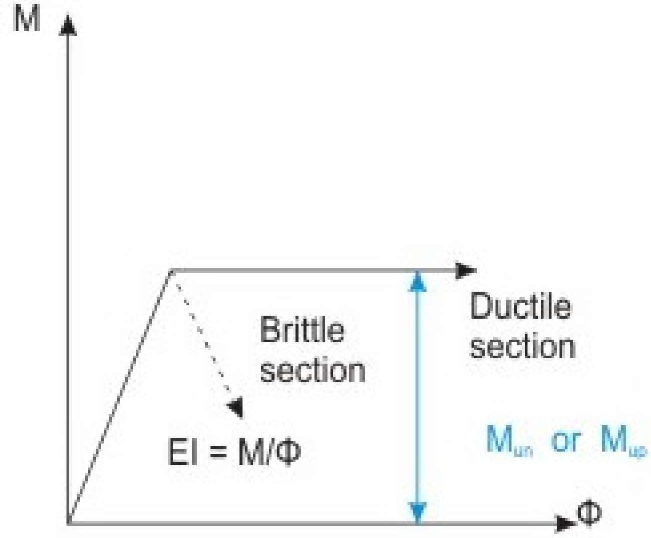


Fig. 15.38.2: Idealised moment-curvature relationship

c) The moment-curvature relationship for the ductile sections is the idealised bilinear as shown in Fig. 15.38.2. It may be noted that for all practical purposes only a limited rotation will take place.

d) All sections of the beam have the same constant flexural rigidity up to the ultimate moment and moment remains constant at the ultimate moment with increasing curvature.

e) The self-weight of the beam is neglected.

The beam is subjected to two point loads of magnitude P each and acting at distances of $l/2$ from the two supports A and C, as shown in Fig. 15.38.1a. The elastic bending moment diagram is shown in Fig. 15.38.1b. On increasing the loads P, the ultimate moment of resistance M_{un} of cross-section at B will be reached at the support first before reaching at other sections, which is shown in Fig. 15.38.1d. The plastic hinge will be formed at the cross-section B (Fig. 15.38.1c). The two point loads P can still be increased as long as the plastic hinge at B will rotate sufficiently. If the cross-section at B is brittle, the load will decrease fast (see Fig. 15.38.2) and the beam will fail suddenly. Hence, the beam will not carry any additional load. The cross-section at B has to be ductile so that it undergoes rotation at the constant moment of resistance, which will enable the beam to carry additional loads. This increase of load will continue until the maximum positive moments in the span (at D and E) reach M_{up}, as shown in Fig. 15.38.1f, when plastic hinges will form at D and E also. The three plastic hinges at B, D and E will form the collapse mechanism (Fig. 15.38.1e). The structure will fail at this stage carrying much higher loads. It is important to note that the requirement of equilibrium will be satisfied at all stages, i.e., M_p and M_n , during the elastic phase, will follow the equation:

$$M_p + 0.5\, M_n = Pl/4 \qquad (15.1)$$

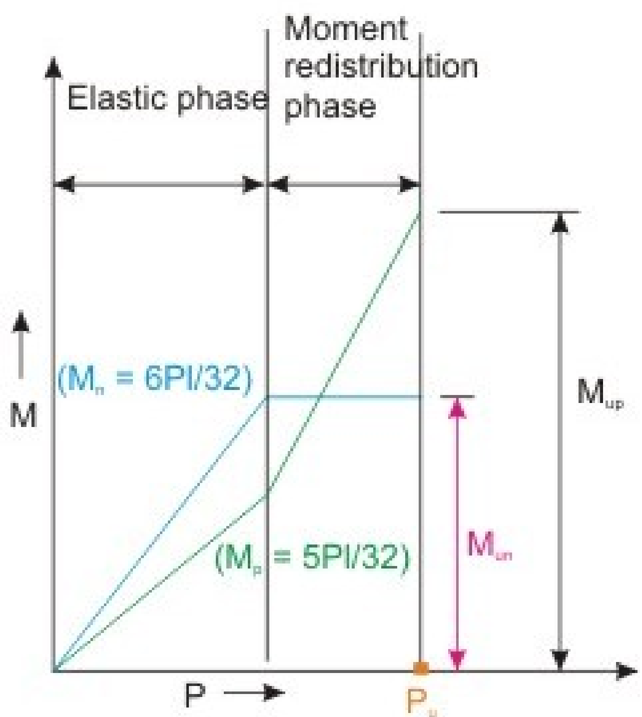


Fig. 15.38.3: Change of bending moment with load

From the structural analysis of the beam of Fig. 15.38.1a, we know that:

$$M_n = 6\, Pl / 32 \quad \text{and} \quad M_p = 5\, Pl / 32.$$

Substituting these values in Eq. 15.1, we find that the equation is satisfied where $Pl/4$ is the maximum positive moment of a simply supported beam having a load P at the centre of the beam. From the above values of M_n and M_p (= 6 Pl/32 and 5 Pl/32, respectively), we also note that:

$$M_n / M_p = 1.2 \qquad (15.2)$$

As the loads are increased till the formation of the first plastic hinge at B, this ratio of M_n / M_p as 1.2 will be maintained, as shown in Fig. 15.38.3. This phase is known as the elastic phase when the bending moments increases with increasing loads maintaining the ratio of M_n / M_p as 1.2.

With further increase of the loads, the plastic hinge at B will rotate at the constant moment M_{un} and positive moments at D and E will increase as shown in Fig. 15.38.3. This phase is known as moment redistribution phase as the loads are now transferred to sections, which have less moment. However, the cross-section at B must have the ability to sustain the required plastic rotation at this stage with increasing loads. As the value of M_p is now increasing when M_n is remaining constant, the ratio of M_n / M_p will not be 1.2 any more. Thus, in the redistribution phase, the additional moments at higher loads are to be redistributed to the support and mid-span in such a manner that we get a similar equation like Eq. 15.1, i.e.

$$M_{up} + 0.5\, M_{un} = Pl/4 \qquad (15.3)$$

which means that, after the redistribution

$$M_{un} = M_n - M \qquad (15.4)$$

where M is some amount of moment by which negative moment at the support is reduced. From Eqs. 15.3 and 15.4, we have: $M_{up} = Pl/4 - 0.5\, M_{un} = Pl/4 - 0.5\,(M_n - M) = (Pl/4 - 0.5\, M_n) + 0.5\, M = M_p + 0.5\, M$, which we get by substituting $M_p = Pl/4 - 0.5\, M_n$ from Eq. 15.1. Therefore, we have:

$$M_{up} = M_p + 0.5\, M \qquad (15.5)$$

Thus, in the moment redistribution, if we reduce some amount M from the negative moment M_{un} at support as in Eq. 15.4, we have to add 0.5 M to the span moment M_p to get the M_{up} as in Eq. 15.5. The redistributed moments M_{un} and M_{up} satisfy the equilibrium condition of Eq. 15.3. Similar equilibrium condition is also satisfied at the elastic stage by the M_n and M_p as in Eq. 15.1.

The amount of moment M which will be reduced from the negative support moment depends on the rotational capacity as per the actual reinforcement provided in the cross-section. Furthermore, the deformation of the support should be within acceptable limits under service loads.

In the extreme case, if the negative moment at the support M_{un} is reduced to zero, i.e., $M = M_n$, we have from Eq. 15.5:

$$M_{up} = M_p + 0.5\, M_n = Pl/4 \qquad (15.6)$$

That is, the unreinforced central support will crack and the continuous beam will behave as two simply supported beams.

Thus, the choice of the bending moment diagram after the redistribution should satisfy the equilibrium of internal forces and external loads. Moreover, it must ensure the following:

1. The plastic rotations required at the critical sections should not exceed the amount the sections can sustain.

2. The extent of cracking or the amount of deformation should not make the performance unsatisfactory under service loads.

The redistribution of moments is permitted if the analysis of forces and moments is done following linear elastic behaviour. Analysis by linear elastic behaviour is a logical procedure in the working stress method of design where the design concept is based on the assumptions of linear elastic behaviour of materials up to the level of recommended safe stresses. However, analysis by linear elastic theory and design by the limit state of collapse appear to be somewhat contradictory. At the stress levels of 0.87 f_y for steel and 0.67 f_{ck} for concrete, the strains are not linearly related. Unfortunately, till now there is no such analysis, which takes into account the complex behaviour of reinforced concrete just before the collapse. Accordingly, the elastic theory of analysis of forces and moments cannot be avoided at present except for the slabs.

Moreover, unlike steel structures, cross-section forming the first plastic hinge in reinforced concrete members undergoes limited rotations, which is insufficient for other sections to attain the desired M_{up}. Therefore, full redistribution has been restricted in the design of reinforced concrete members. These restrictions and stipulations of IS Codes are taken up in the next section.

15.38.3 Recommendations of IS 456

IS 456 recommends the redistribution of moments during the analysis provided the analysis is done by linear elastic theory. Such redistribution of moments is not permitted when either simplified analysis is done or coefficients of cl.22.5 of IS 456 are used to determine the moments.

In the working stress method, cl. B-1.2 of IS 456 stipulates that the moments over the supports for any assumed arrangement of loading, including the dead load moments may each be increased or decreased by not more than 15 per cent, provided that these modified moments over the supports are used for the calculation of the corresponding moments in the spans.

In the limit state of collapse method, cl.37.1 of IS 456 recommends the redistribution of the calculated moment in continuous beams and frames satisfying the following conditions:

1. Equilibrium between the internal forces and the external loads should be maintained.

2. The ultimate moment of resistance provided at any cross-section of a member after the redistribution should not be less than 70 per cent of the moment at that cross-section obtained from an elastic maximum moment diagram covering all appropriate combinations of loads.

3. The elastic moment at any cross-section in a member due to a particular combination of loads shall not be reduced by more than 30 per cent of the numerically largest moment given anywhere by the elastic maximum moment diagram for the particular member, covering all appropriate combinations of loads.

4. Cross-sections having moment capacity after redistribution less than that of the elastic maximum moment shall satisfy the relationship:

$$(x_u / d) + (\delta M / 100) \le 0.6 \qquad (15.7)$$

where

x_u = depth of the neutral axis,
d = effective depth, and
δM = percentage reduction in moment.

5. Thirty per cent reduction in moment permitted in 3 above shall be restricted to ten per cent for structures in which the structural frames provide lateral stability.

15.38.4 Explanations of the Conditions Stipulated in IS Code

The first condition of maintaining the equilibrium between internal forces and external loads is explained in sec.15.38.2 taking up a two-span continuous beam. For the other conditions, let us use the following notations for the sake of brevity:

M_{ew} = elastic bending moment under working loads,
M_{eu} = elastic bending moment under design loads i.e., the factored loads considering partial safety factor of loads γ_f,
M_{du} = design factored moment after redistribution, and

δM = percentage reduction in M_{eu}.

Since γ_f = 1.5, we can write

$$M_{eu} = 1.5\, M_{ew} \qquad (15.8)$$

The second condition of cl.37.1 of IS 456, as given in sec. 15.38.3 of this lesson, is expressed as

$$M_{du} \geq 0.7\, M_{eu} \qquad (15.9)$$

Beeby has reported that the redistribution as given in Eq. 15.8 is possible before the lower bound rotation capacity is reached (Beeby, A.W, "The analysis of beams in plane frames according to CP 110" Cement and Concrete Association, U.K. Development Report No. 1, October 1978). Thus, Eq. 15.9 ensures the first criterion of the redistribution of moments in reinforced concrete structures as given in sec. 15.38.2. Equation 15.9 also satisfies other requirements as explained below.

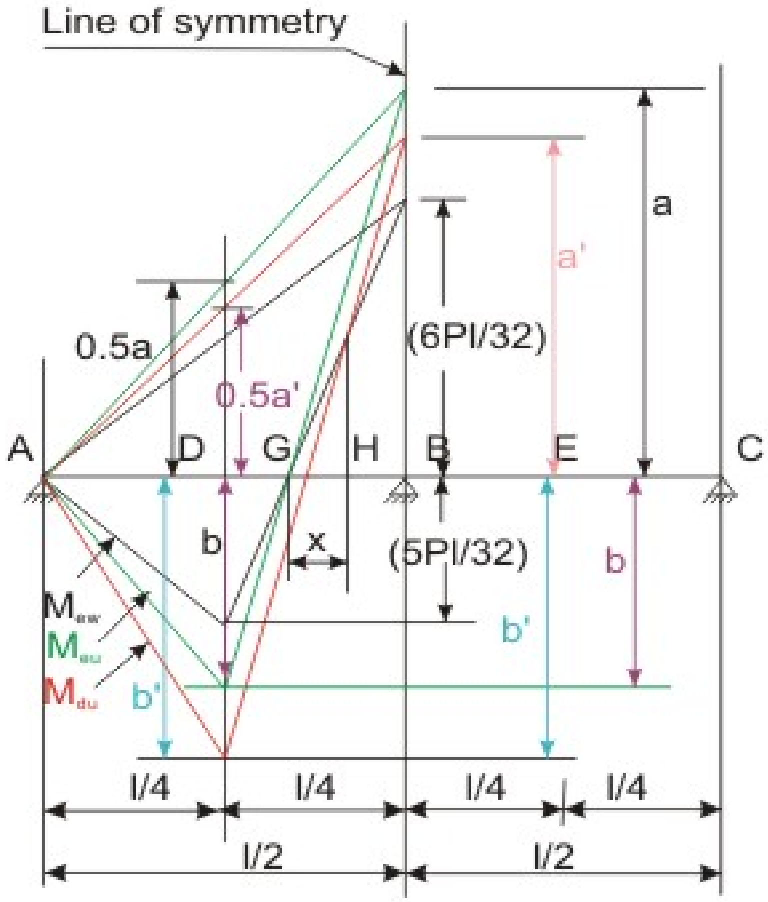


Fig. 15.38.4: Elastic working load, elastic ultimate load and design ultimate load bending moment diagrams

Using the value of M_{eu} from Eq. 15.8 into Eq. 15.9, we have

$$M_{du} \geq 0.7\ (1.5\ M_{ew}) \geq 1.05\ M_{ew} \tag{15.10}$$

This ensures that the design factored moments are greater than the elastic moments everywhere in the structures. Figure 15.38.4 presents three bending moment diagrams: (i) elastic moment under working loads (M_{ew}), (ii) elastic moment under ultimate loads (M_{eu}) and (iii) design moment under ultimate loads (M_{du}) of the two-span beam of Fig. 15.38.1a. The moment diagram after the redistribution (M_{du}) satisfies the condition of Eq. 15.1 such that

$$b + 0.5\ a = b' + 0.5\ a' \tag{15.11}$$

However, M_{du} diagram is not satisfying the requirement of Eq. 15.10 as in the zone GH of length x (Fig.15.38.4), the negative moment after redistribution is less than the negative moment under working loads. To satisfy Eq. 15.10, the bending moment in the zone should be as per M_{ew}.

We now take up the fourth condition of cl.37.1 of IS 456 and given in sec. 15.38.3 of this lesson, which is expressed in Eq. 15.7 as

$$(x_u/d) + (\delta M / 100) \leq 0.6 \qquad (15.7)$$

This equation ensures that the cross-section of the member is under-reinforced, as explained below.

Allowing the maximum redistribution of thirty per cent as per the third condition of sec. 15.38.3, we have $\delta M/100 = 0.3$. Substituting this in Eq. 15.7, we get $x_u/d \leq 0.3$. For ($\delta M/100$) less than thirty per cent, the corresponding x_u/d can be determined from Eq. 15.7. The values of $x_{u,max}/d$ for the three grades of steel are given in Table 3.2 of Lesson 5. It is easy to verify that permitting the redistribution of thirty per cent, the cross-sections remain under-reinforced. This ensures low amount of steel reinforcement to have smaller value of x_u which will give higher value of the rotation ϕ as the rotation $\phi = 0.0035/x_u$.

It is evident, therefore, that for the analysis of determining the values of bending moments after the redistribution, bending moment envelopes are to be drawn satisfying the requirements discussed above. This is explained with the help of illustrative examples in the next section.

15.38.5 Illustrative Examples

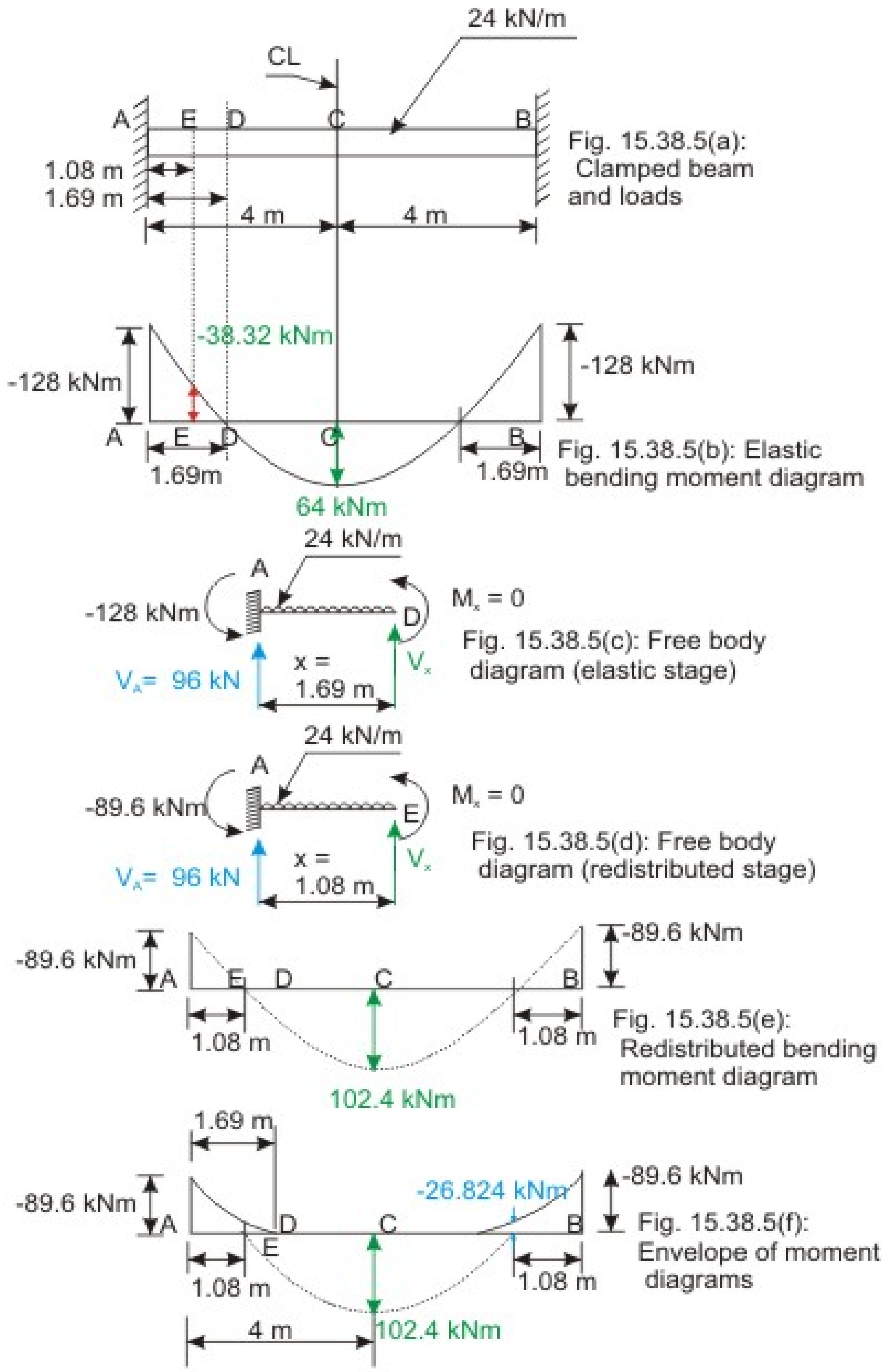


Fig. 15.38.5: Problem 1

Problem 1. Draw the design bending moment diagram of the beam of Fig. 15.38.5a, clamped at both ends and carrying ultimate uniformly distributed load of 24 kN/m with full redistribution of 30 per cent as per IS 456.

Solution 1.

Step 1: **Elastic bending moment diagram**

The beam (Fig.15.38.5a) is statically indeterminate. The elastic bending moment diagram is shown in Fig. 15.38.5b. The negative and positive moments at the supports (A and B) and mid-span (C), respectively, are:

$M_A = M_B = - w\, l^2 / 12 = - 24\ (8)\ (8) / 12 = - 128$ kNm
$M_C = + w\, l^2/24 = + 24\ (8)\ (8) / 24 = + 64$ kNm

The distance x where the bending moment is zero is obtained by taking moments of forces of the free body diagram (Fig. 15.38.5c) about D after determining the vertical reaction at A.

Vertical reaction at A, $V_A = 24\ (8) / 2 = 96$ kN. At section D, $M_x = 96x - 128 - 12x^2$. So, the value of x when $M_x = 0$ is obtained from $96x - 128 - 12x^2 = 0$ or, $3x^2 - 24x + 32 = 0$. The solution of the equation is x =1.69 m and 6.31 m.

Step 2. **Redistributed bending moment diagram**

From Fig.15.38.5e, the maximum negative moment at A after the full redistribution of 30 per cent = 0.7 (128) = 89.6 kNm. So, the maximum positive moment at the mid-span C = $w\, l^2 / 8 - 89.6 = 102.4$ kNm. The vertical reaction at A is: $V_A = 96$ kN.

The point of inflection, i.e., the point where bending moment is zero, is obtained by taking moment of forces about E of the free body diagram shown in Fig. 15.38.5d. Thus, we have: $96x - 12x^2 - 89.6 = 0$, which gives $x = 1.08$ m and 6.92 m. The bending moment diagram after the redistribution is shown in Fig. 15.38.5e.

Step 3. **Envelope of bending moment diagrams**

At x =1.08 m, the elastic bending moment is $M_x = 96x - 128 - 12x^2 = - 38.32$ kNm, which becomes 0.7 (- 38.32) = - 26.824 kNm after the redistribution. The envelope of the bending moment diagram is shown in Fig. 15.38.5f.

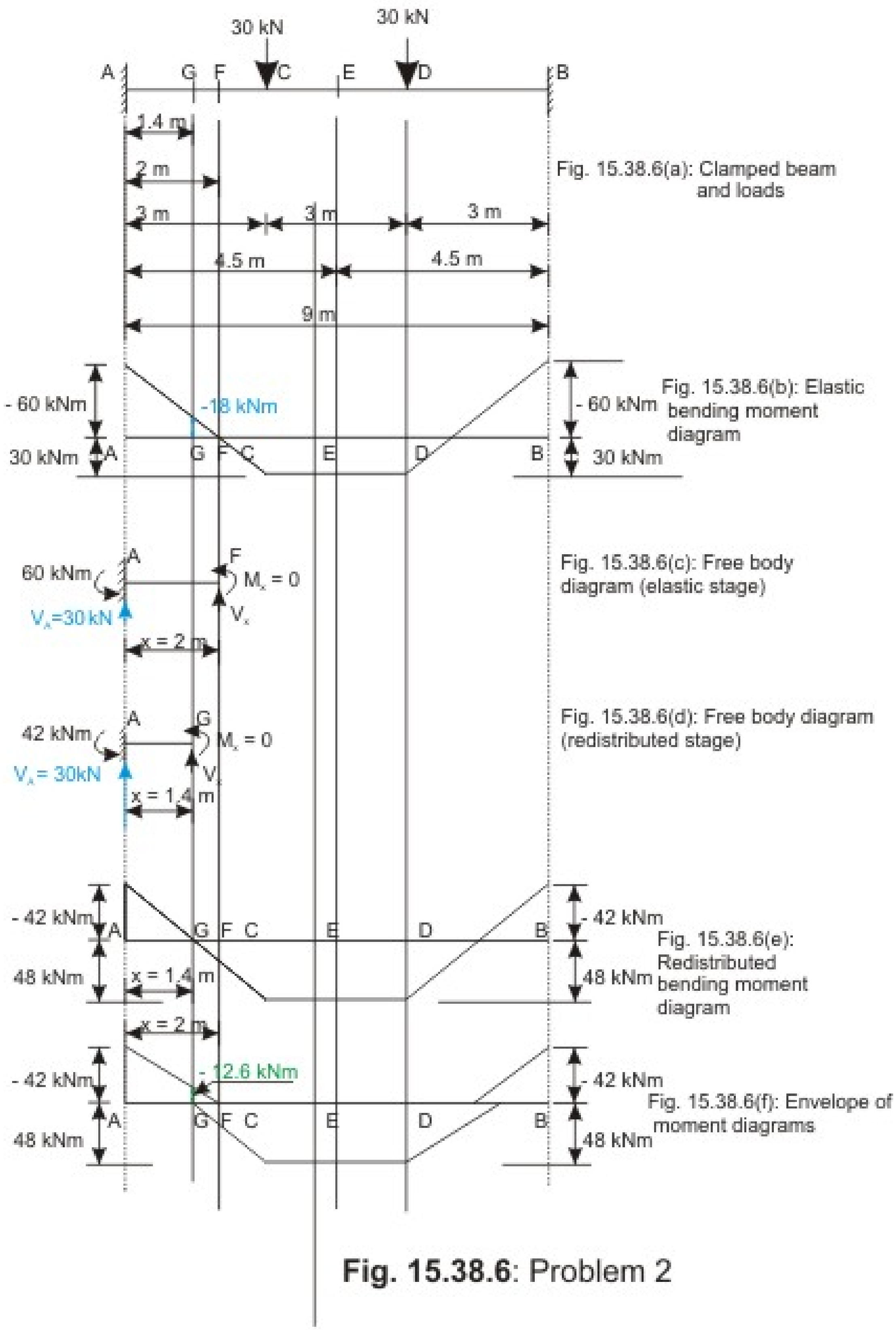


Fig. 15.38.6: Problem 2

Problem 2. Draw the design bending moment diagram of the beam of Fig. 15.38.6a, clamped at both ends and carrying two point loads of 30 kN each at distances of 3 m from the supports. Assume full redistribution of 30 per cent as per IS 456.

Solution 2.

Step 1. **Elastic bending moment diagram**

The beam (Fig.15.38.6a) is statically indeterminate. The elastic bending moment diagram is shown in Fig. 15.38.6b. The negative and positive moments at the supports (A and B) and at mid-span (E), respectively, are:

$M_A = M_B = - 2\ P\ l/9 = -2\ (30)\ (9)/\ 9 = - 60$ kNm
$M_E = 90 – 60 = 30$ kNm.

Vertical reaction at A = $V_A = 30$ kN

Taking moment of all the forces about F at a distance of x ($0 \le x \le 3$ m) from A of the free body diagram shown in Fig. 15.38.6c, we have $30\ x – 60 = 0$, which gives $x = 2$ m, where the bending moment is zero.

Step 2. **Redistributed bending moment diagram**

Redistributed negative moment at A = 0.7 (60) = 42 kNm
Redistributed positive moment at E = 90 - 42 = 48 kNm

The point of zero moment (point G) is obtained by taking moments of forces about G of the free body diagram shown in Fig. 15.38.6d, when $x \le 0 \le 3$ m; gives $30x – 42 = 0$. Therefore, $x = 1.4$ m.

The redistributed moment diagram is shown in Fig. 15.38.6e.

Step 3. **Envelope of bending moment diagram**

At G when $x = 1.4$ m, elastic moment = 30(1.4) – 60 = - 18 kNm. The redistributed moment at G = 0.7 (-18) = -12.6 kNm. The envelope of the bending moment diagrams is shown in Fig. 15.38.6f.

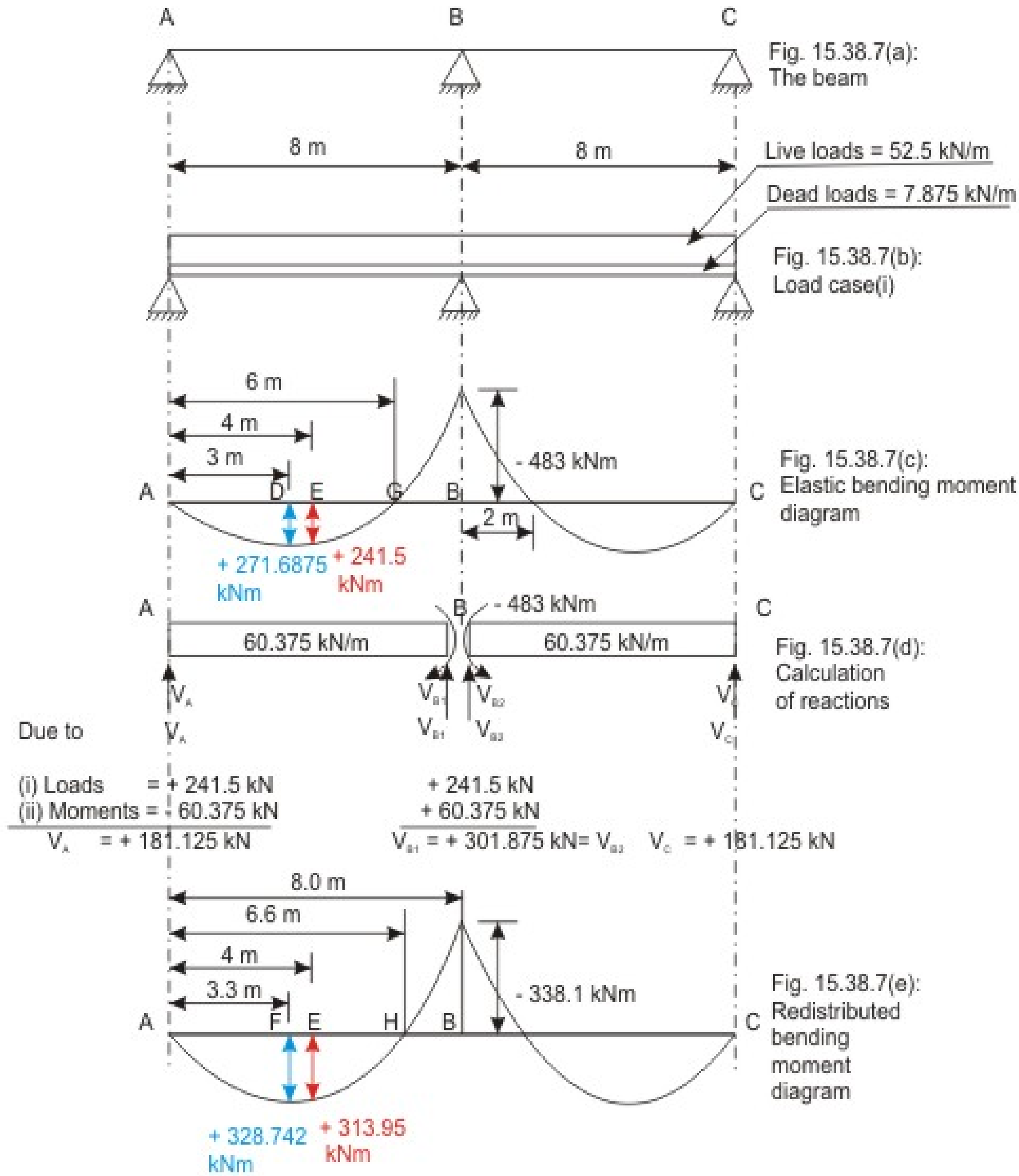


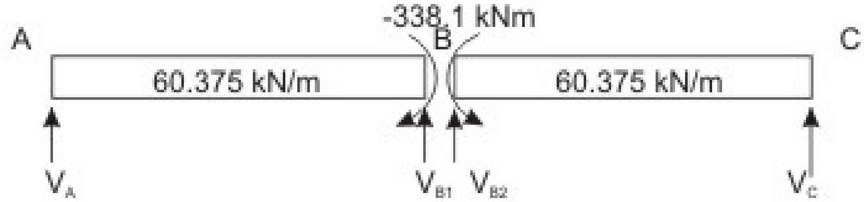
Fig. 15.38.7(a): The beam

Fig. 15.38.7(b): Load case(i)

Fig. 15.38.7(c): Elastic bending moment diagram

Fig. 15.38.7(d): Calculation of reactions

Fig. 15.38.7(e): Redistributed bending moment diagram



Due to	V_A	V_{B1} V_{B2}	V_C
(i) Loads	= + 241.5 kN	+ 241.5 kN	
(ii) Moments	= - 42.2625 kN	+ 42.2625 kN	
	V_A = + 199.2375 kN	V_{B1} = + 283.7625 kN = V_{b2}	V_C = + 199.2375 kN

Fig. 15.38.7(f): Calculation of reactions after redistribution

Fig. 15.38.7: Problem 3 - load case(i)

Problem 3. Draw envelope of the design moments of the two-span continuous beam shown in Fig. 15.38.7a, carrying characteristic live load of 35 kN/m in addition to its characteristic self-weight. The cross-section of the beam is 300 mm × 700 mm.

Solution 3. The two-span continuous beam of Fig. 15.38.7a is statically indeterminate and the method of moment distribution is employed for the analysis. Clause 22.4.1 of IS 456 stipulates the arrangements of live loads, which are:

1) Design dead load on all spans with full design imposed loads on two adjacent spans to get the maximum negative moment over the support, and

2) Design dead load on all spans with full design imposed loads on alternate spans to get the maximum span moment (positive moment).

In this case of two-span continuous beam, therefore, we have the following three load cases:

(i) For the maximum negative moment at the support B, both the spans are loaded with self-weight and live loads,

(ii) For the maximum positive moment in span AB, only the span AB is loaded with live load and self-weight of the beam in both the spans, and

(iii) For the maximum positive moment in span BC, only the span BC is loaded with live loads and self-weight of the beam in both the spans.

However, we are considering only cases (ii) and (iii) as they are the mirror image of each other.

Step 1. Evaluation of design loads

Characteristic dead load of the beam = (0.3) (0.7) (25) = 5.25 kN/m. Therefore, the design dead load with γ_f = 1.5 is (1.5) (5.25) = 7.875 kN/m. The design live load = 1.5 (35) = 52.5 kN/m.

Step 2. Load case (i): Elastic bending moment diagram

Figure 15.38.7b shows the load case (i) of the beam. The calculations of the moment distribution are presented in Table 15.1. Figure 15.38.7c presents the elastic bending moment diagram and Fig. 15.38.7d presents the free body diagram and the calculations of determining the reactions. The final values of the vertical reactions are: V_A = + 181.125 kN, V_{B1} = + 301.875 kN, V_{B2} **=** + 301.875 kN and V_C = +181.125 kN (+ means upward). Shear force is zero at D (x = 3 m), where the moment is 271.6875 kNm, obtained from the following equations:

(a) 181.125 – 60.375 x = 0 gives x = 3 m, and
(b) M_x (at x = 3.0 m) = 181.125 (3) – 60.375 (3) (3) /2 = 271.6875 kNm.

At point E (mid-span), M_x = 181.125 (4) – 60.373 (4) (4)/2 = 241.5 kNm.
At point G, where x = 6 m, the bending moment is zero as obtained from equation 181.125x – 30.1875x^2 = 0.

For the span BC, the bending moment diagram is the mirror image of the bending moment diagram of span AB due to symmetrical beams and loadings.

Table 15.1 Moment Distribution for Load Case (i)

Joint	**A**	**B**		**C**
Member	AB	BA	BC	CB

D.F	1.0	0.5	0.5	1.0
F.E.M	+ 322.0	- 322.0	+ 322.0	- 322.0
Balance moment	- 322.0	-	-	+ 322.0
Carry over moment	-	- 161.0	+ 161.0	-
Total balanced moment	0	- 483.0	+ 483.0	0

Note: D.F. = Distribution Factors; F.E.M = Fixed End Moments

Step 3. Load Case (i): Redistributed bending moment diagram

Redistributed moment at support B = 0.7 (483) = 338.1 kNm. Vertical reaction at A = 199.2375 kN (Fig. 15.38.7f).

Shear is zero at F where x = 3.3 m as obtained from: 199.2375 – 60.375 x = 0.

Maximum bending moment at F (x = 3.3 m) is: 199.2375 (3.3) - 60.375 (3.3) (3.3) /2 = 328.742 kNm.

At the mid-span E (x = 4 m), the moment is 60.373 (8) (8) / 8 – 338.1 /2 = 313.95 kNm.

Moment is zero at H where x = 6.6 m is obtained from: 199.2375 x – 60.373 x^2/2 = 0.

Calculations for the span BC are not performed due to symmetrical loadings. The redistributed bending moment diagram is shown in Fig. 15.38.7e.

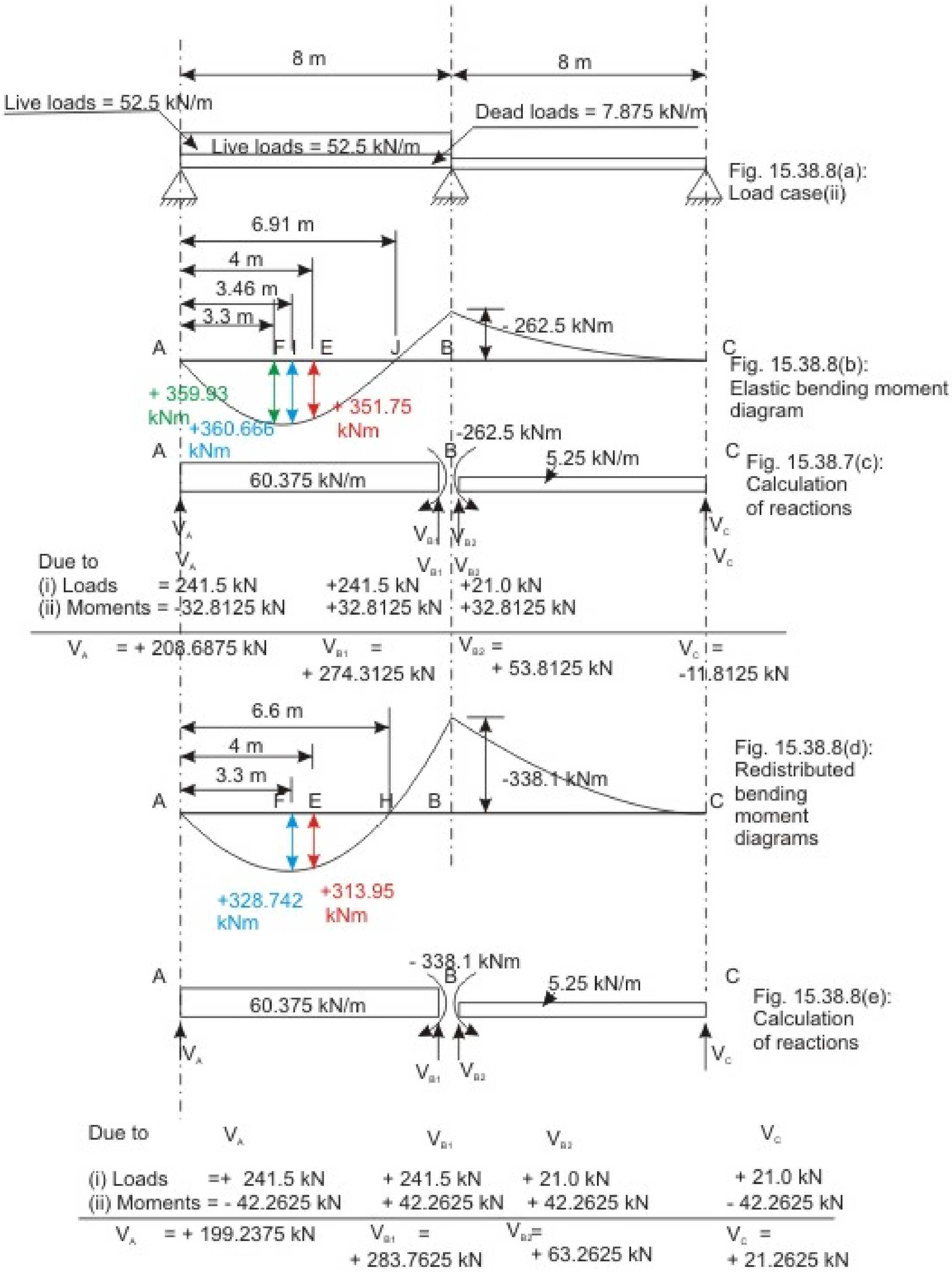


Fig. 15.38.8: Problem 3 - load case(ii)

Step 4. **Load Case (ii): Elastic bending moment diagram**

This load case is for the maximum positive moment in span AB. The design loads consist of factored dead load and factored live load = 1.5 (5.25 + 35) = 60.375 kN/m for the span AB. For the span BC, we consider only the dead load with a load factor of one only. So, the load in span BC is 5.25 kN/m. The loads are shown in Fig. 15.38.8a. Bending moments are determined by moment distribution method and presented in Table 15.2. Bending moment diagram is shown in Fig. 15.38.8b and the vertical reactions are shown in Fig. 15.38.8c.

Table 15.2 Moment Distribution for Load Case (ii)

Joint	**A**	**B**		**C**
Member	AB	BA	BC	CB
D.F	1.0	0.5	0.5	1.0
F.E.M	+ 322.0	- 322.0	+ 28.0	- 28.0
Balanced Moment	- 322.0	+ 147.0	+ 147.0	+ 28.0
Carry over Moment	-	- 161.0	+ 14.0	-
Balanced Moment	-	+ 73.5	+ 73.5	-
Total Moment	0.0	- 262.5	+ 262.5	0.0

Note: D.F. = Distribution Factors, F.E.M = Fixed End Moments

Final values of the reactions are: V_A = + 208.6875 kN, V_{B1} = + 274.3125 kN, V_{B2} = + 53.8125 kN and V_C = - 11.8125 kN (+ means upward).

In the span AB, shear force is zero at I where x = 3.46 m and the maximum moment is 360.666 kNm, obtained from the following equations:

(a) 208.6875 – 60.375 x = 0, gives x = 3.46 m and
(b) M_x (at x = 3.46 m) = 208.6875 (3.46) – 60.375 (3.46) (3.46)/2 = 360.666 kNm.

At point E (mid-span), M_x = 208.6875 (4) – 60.375(4) (4)/(2) = 351.75 kNm.
The bending moment is zero at J (x = 6.91 m), obtained from the equation 208.6875 x – 60.375 (x^2/2) = 0 where $0 \le x \le 8$ m in the span AB. In the span BC, the bending moment is zero only at C (x = 8 m from B), as obtained from the equation: - 262.5 – 5.25 (x^2/2) + 53.8125 x = 0, where $0 \le x \le 8$ m in the span BC. The other value

of x where the negative moment is zero is x = 12.5 m. This point is outside the span 8m of BC. Thus, the bending moment is negative in the whole span BC.

Step 5. Load case (ii): Redistributed bending moment diagram

While redistributing the moments for the load case (ii), we have to take care of the static equilibrium of span AB regarding the redistributed moments. Figure 15.38.7d shows that the redistributed positive moment is the maximum at F (x = 3.3 m) and the value is 328.742 kNm. The maximum redistributed negative moment is 338.1 kNm at B. So, let us reduce the elastic bending moment at F (x = 3.3 m) to 328.742 kNm for the load case (ii) also. It should be checked if this reduction is within the maximum limit of 30 per cent or not.

Elastic bending moment at F (x = 3.3 m) for the load case (ii) = $V_A x - wx^2/2$ = 208.6875 (3.3) – 60.375 (3.3) (3.3)/2 = 359.93 kNm (Fig. 15.38.8b). Hence, the reduction of 359.93 kNm to 328.742 kNm is (100) (359.93 – 328.742) / 359.93 = 8.665 per cent, which is within the limit of 30 per cent.

To have the redistributed moment at F = 328.742 kNm, the magnitude of V_A is determined from the equation:

V_A (3.3) – w (3.3) (3.3)/2 = 328.742 kNm, which gives V_A = 199.237 kN.

At F where x = 3.3 m, the shear force = 199.237 – 60.375 (3.3) = 0. Therefore, the moment at F is the maximum positive redistributed moment of magnitude = 328.742 kNm.

At B where x = 8 m, the redistributed moment for the load case (ii) is obtained from: (199.237) (8) – 60.375 (8) (8) / 2 = - 338.1 kNm, which is the same moment at B in the load case (i).

At E, (mid-span) where x = 4 m, the moment is: (199.237) 4 – 60.375 (4) (4)/2 = 313.95 kNm. The value of x where the redistributed bending moment = 0, in the span AB, is obtained from: 199.37 x – 60.375 (x^2/2) = 0, which gives x = 6.6 m.

For the span BC, the redistributed negative moment is – 338.1 kNm, as obtained in the span AB for this load case. The vertical reaction V_{B2} in the span BC is (5.25) (4) + 338.1/8 = 63.2625 kN. The location of the point where the moment is zero in the span BC is obtained from the equation: $– M_B - wx^2/2 + V_{B2}\, x = 0$ or – 338.1 –

5.25 (x^2/2) + 63.2625 x = 0, which gives x = 8 m. The other value of x where moment is zero is 16.1 m (outside the span BC). Therefore, the redistributed moment is negative in the entire span BC. The redistributed bending moment diagram is shown in Fig. 15.38.8d and calculations of reaction are shown in Fig. 15.38.8e.

As mentioned earlier, the load case (iii) is not separately taken up since it is the mirror image of load case (ii).

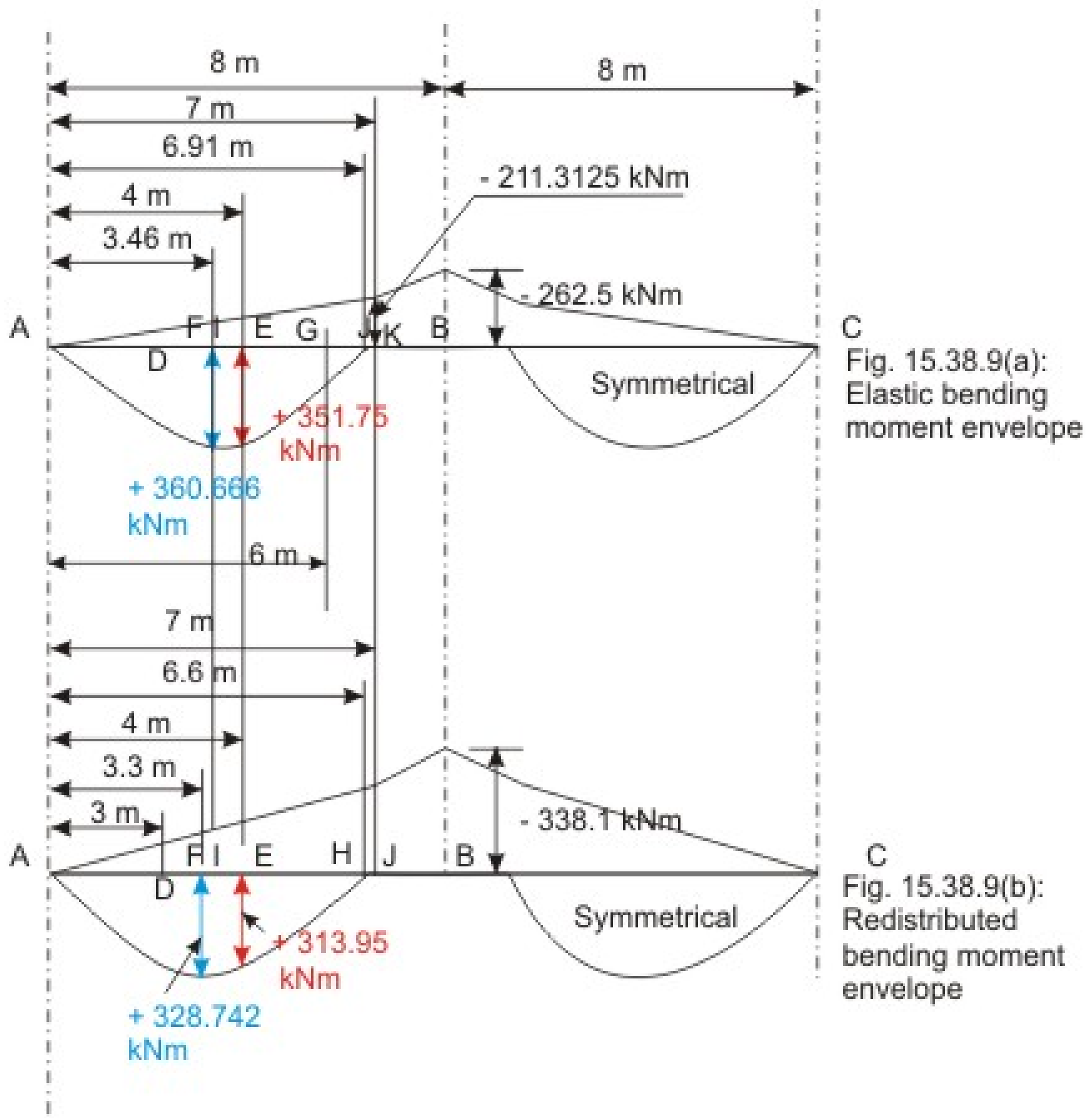


Fig. 15.38.9: Problem 3 - Envelopes of bending moment

Step 6. Superimposition of the elastic bending moment diagrams

The envelope of the positive and negative bending moments of elastic analyses is to be prepared considering all three load cases from Figs. 15.38.7c and 15.38.8b. The bending moment diagram for the third load case is the mirror image of Fig. 15.38.8b. The positive moment is to be taken from the diagram of span AB of Fig. 15.38.8b for both the spans as that is more than the values of Fig. 15.38.7c.

For the negative moment in span BC, the diagram for the load case (i) will intersect that of the load case (ii). Similarly, for the negative moment in span AB, the diagram of load case (i) will intersect that of load case (iii). Due to the mirror image, the point of intersection is determined for span BC only.

For the load case (i) in span BC, the bending moment at a distance of x from the support B is given by:

$$M_x = -483 + 301.875\,x - 60.375\,(x^2/2)$$

Similarly, for the load case (ii) in span BC, the bending moment at a distance of x from the support B is given by:

$$M_x = -262.5 + 53.8125\,x - 5.25\,(x^2/2)$$

We can find the value of x, where these two negative moments are the same by equating the two expressions:

$$-483 + 301.875x - 60.375\,(x^2/2) = -262.5 + 53.8125x - 5.25\,(x^2/2)$$

or $27.5625\,x^2 - 248.0625\,x + 220.5 = 0$

The values of x are 1.0 m and 8.0 m. So, at a distance of 1.0 m from the support B, the two negative moments intersect. The value of the moment is obtained from either of the two equations, which comes out to be 211.3125 kNm. Accordingly, Fig 15.38.9a presents the envelope of the elastic bending moment diagrams making use of symmetry of the load in the two spans.

Step 7. Superposition of redistributed bending moment diagrams

The procedure is exactly the same as in Step 6. The positive bending moment diagram is taken from that of span AB of load case (ii) and the negative moment envelope is obtained combining the negative bending moments of span BC for the two load cases. In this redistributed negative moments, there is no intersection of the diagram as in Step 6. Accordingly, Fig. 15.38.9b presents the envelope of the redistributed bending moments using symmetry of the loads in the two spans.

15.38.6 Practice Questions and Problems with Answers

Q.1: Explain the concept of redistribution of moments in statically indeterminate reinforced concrete structures.

A.1: Paragraphs 1 and 2 of sec. 15.38.1

Q.2: Mention three advantages of considering the redistribution of moments for the design of statically indeterminate reinforced concrete structures.

A.2: Paragraph 3 of sec. 15.38.1

Q.3: State the assumptions of considering the redistribution of moments in the design of statically indeterminate reinforced concrete structures.

A.3: Paragraph 1 of sec. 15.38.2

Q.4: What are the recommendations of IS 456 regarding the redistribution of moment in the design of statically indeterminate structures employing working stress and limit state methods?

A.4: Section 15.38.3 is the full answer.

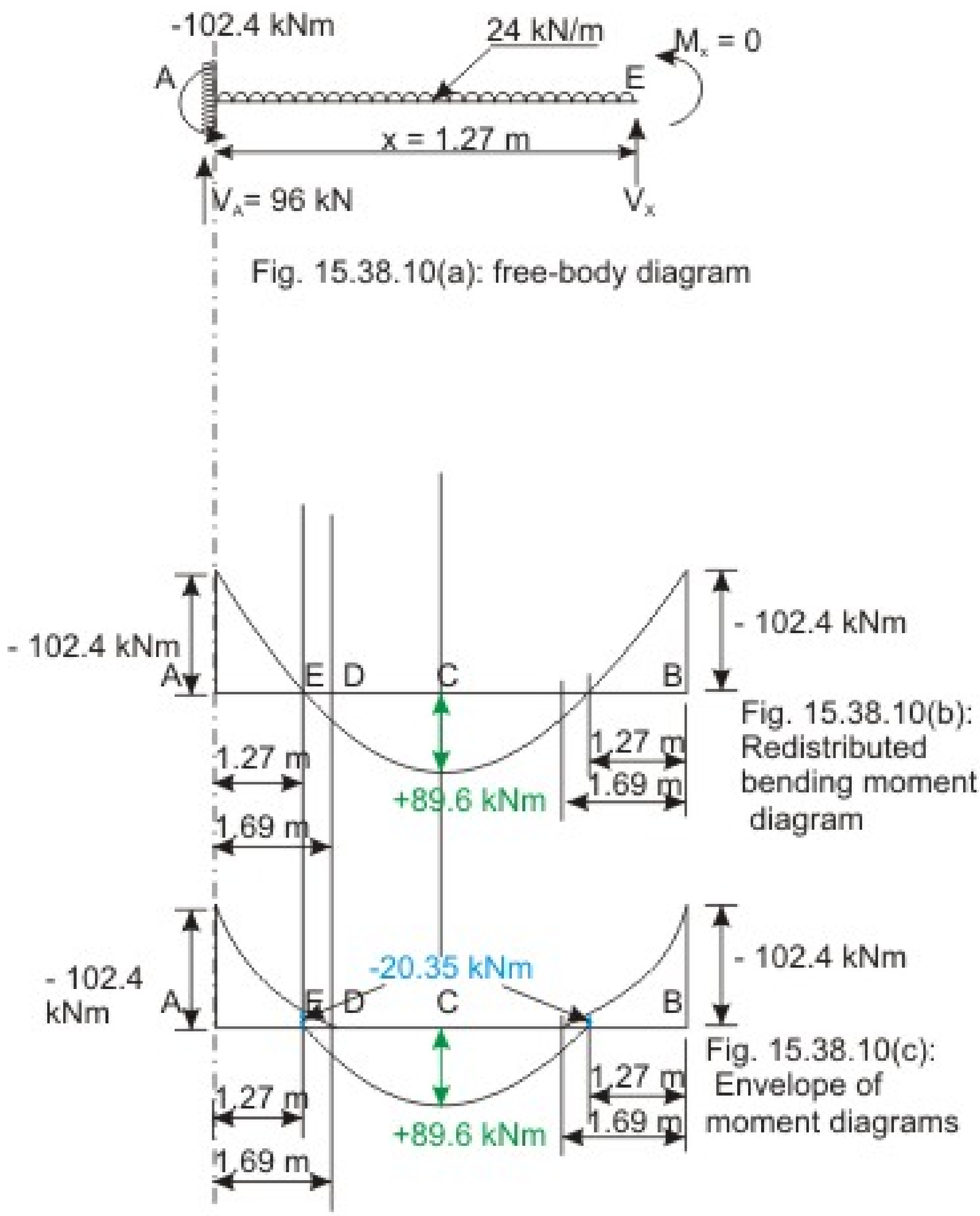


Fig. 15.38.10: Q.5

Q.5: Solve Problem 1 of sec. 15.38.5 (Fig. 15.38.5a) when the redistribution is limited to 20 per cent.

A.5: **Step 1** is the same as that of Problem 1 of sec. 15.38.5. Please refer to Figs. 15.38.5b and c for the elastic bending moment diagram and free body diagram, respectively.

Step 2. Redistributed bending moment diagram and envelope of moment diagrams.

Maximum negative moment at A after the redistribution of 20 per cent = 0.8 (-128) = - 102.4 kNm. So, the maximum positive moment at the mid-span C = 192-102.4 = 89.6 kNm (Please refer to Fig. 15.38.10b). V_A = 96 kN.

The point of inflection is obtained from: $96\,x - 12x^2 - 102.4 = 0$, which gives $x = 1.27$ m (Fig. 15.38.10a). At $x = 1.27$ m, elastic moment = $96x - 12x^2 - 128$ = - 25.435 kNm. The redistributed moment at E ($x = 1.27$ m) = 0.8 (- 25.435) = - 20.35 kNm.

The redistributed bending moment diagram is shown in Fig. 15.38.10b and the envelope of the moment diagram is shown in Fig. 15.38.10c.

15.38.7 References

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15.38.8 Test 38 with Solutions

Maximum Marks = 50 Maximum Time = 30 minutes

TQ.1: Mention three advantages of considering the redistribution of moments for the design of statically indeterminate reinforced concrete structures.

(10 marks)

A.TQ.1: Paragraph 3 of sec. 15.38.1

TQ.2: What are the recommendations of IS 456 regarding the redistribution of moment in the design of statically indeterminate structures employing working stress and limit state methods?

(10 marks)

A.TQ.2: Section 15.38.3 is the full answer

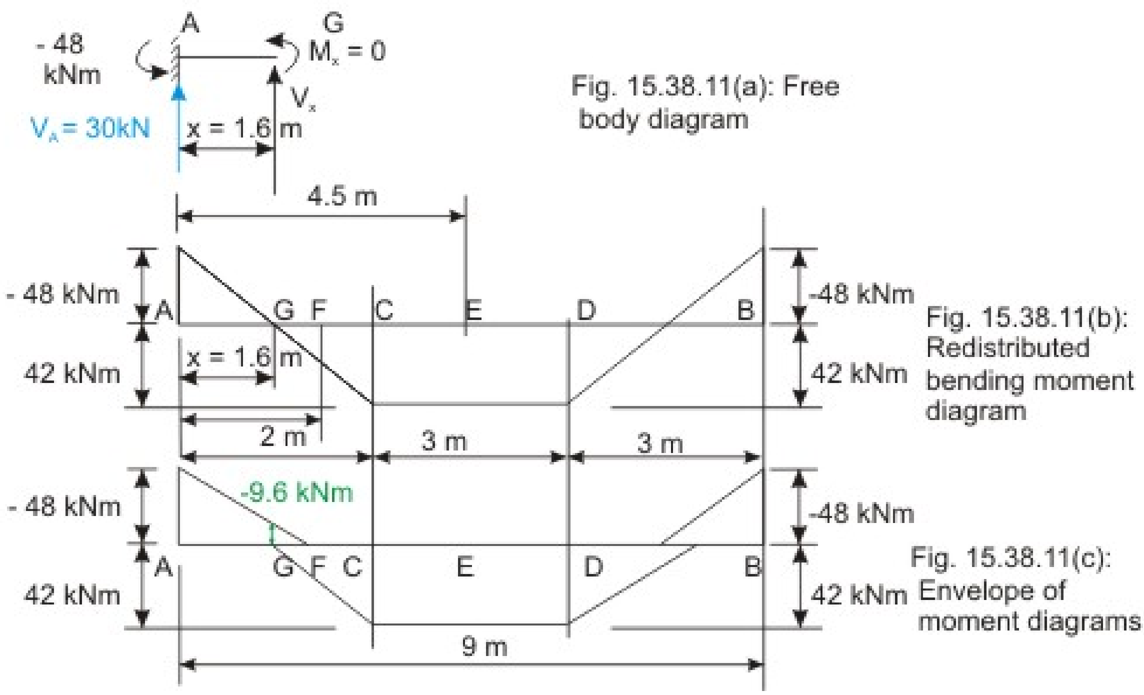

Fig. 15.38.11: TQ. 3 (Redistribution = 20 per cent)

TQ.3: Solve Problem 2 of sec. 15.38.5 (Fig. 15.38.6a) considering the redistribution up to (a) 20 per cent and (b) 10 per cent, separately.

(30 marks)

A.TQ.3: **Step 1** is the same as that of Problem 2 of sec. 15.38.5. Please refer to Figs. 15.38.6b and c for the elastic bending moment diagram and free body diagram, respectively.

Step 2. Redistributed bending moment diagram (20 per cent redistribution) and envelope of the moment diagrams

Redistributed negative moment at A = 0.8 (- 60) = - 48 kNm.
Redistributed positive moment at E = 90 – 48 = 42 kNm

The point of zero moment (point G) is obtained by taking moment of forces of the free body diagram about G, shown in Fig. 15.38.11a, when $0 \le x \le 3$ m, gives $30x - 48 = 0$ or $x = 1.6$ m. The redistributed moment diagram is shown in Fig. 15.38.11b.

At G where $x = 1.6$ m, the elastic moment = 30 (1.6) – 60 = - 12kNm. Therefore, the redistributed moment at G = 0.8 (-12) = - 9.6 kNm. The envelope of the moment diagrams is shown in Fig. 15.38.11c.

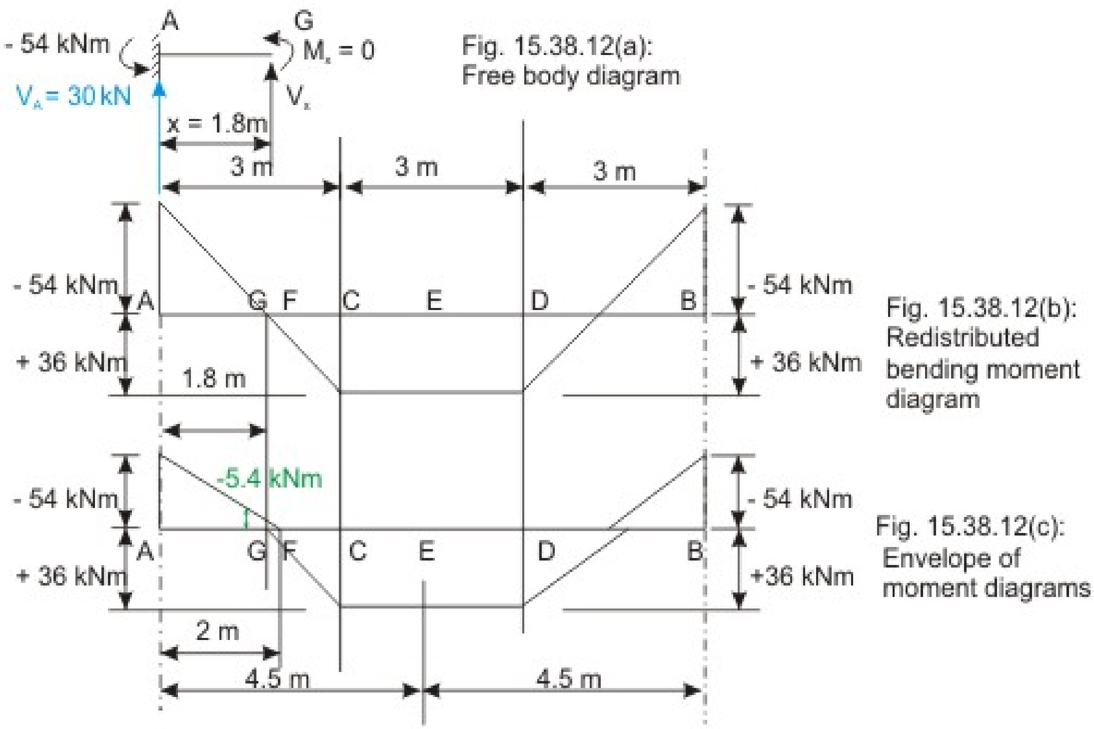

Fig. 15.38.12: TQ.3 (Redistribution = 10 per cent)

Step 3. Redistributed bending moment diagram (10 per cent redistribution) and envelope of the moment diagrams.

Redistributed negative moment at A = 0.9 (- 60) = - 54 kNm.
Redistributed positive moment at E = 90 - 54 = 36 kNm.

The point of zero moment (point G) is obtained by taking moment of forces about G of free body diagram shown in Fig. 15.38.12a, when $0 \le x \le 3$ m, gives $30x - 54 = 0$, or $x = 1.8$ m. The redistributed moment diagram is shown in Fig. 15.38.12b.

At G, where $x = 1.8$ m, the elastic moment = 30 (1.8) – 60 = - 6 kNm. Therefore, the redistributed moment at G = 0.9 (- 6) = - 5.4 kNm. The envelope of the moment diagrams is shown in Fig. 15.38.12c.

15.38.9 Summary of this Lesson

This lesson explains the concept of redistribution of moments in statically indeterminate reinforced concrete structures designed either by the working stress or by the limit state methods. The behaviour of such structures with increasing loads after reaching the respective design loads is important to understand the inclusion of redistribution of moments. The advantages of such inclusion are mentioned in this lesson. The formations of first plastic hinge and subsequently other plastic hinges leading to the formation of collapse mechanism will not only increase the load carrying capacity of the structure but also help the designer to have flexibility in reducing or increasing moments to a limited extent. By this, the detailing of reinforcement helps to avoid congestion of the bars at sections having high moments. Due to the limited rotation capacities of reinforced concrete cross-sections, the reductions / additions of moments are somewhat restricted, which are also explained as stipulated in IS 456. Several numerical problems are solved for the purpose of illustration considering full or partial redistribution in clamped or two-span continuous beams taking up different combinations of dead and live loads. Understanding the illustrative examples and solving the practice and test problems will help in utilising the redistribution to get the advantages in designing statically indeterminate reinforced concrete structures.